Optimal Nonlinear Taxes for Families

by

Craig Brett*
Department of Economics,
Mount Allison University,
144 Main Street, Sackville NB, E4L 1A7, Canada
(e-mail: `cbrett@mta.ca`)
(telephone: `+1 506 364 2532`)

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Abstract

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The problem faced by a taxation authority choosing a tax schedule for families is modeled as a multi–dimensional screening problem. A description of the possible constrained Pareto-efficient mechanisms is given. The implications of a standard redistributive assumption on the sign of marginal tax rates is explored. In contrast to unidimensional taxation models, the redistributive assumption does not imply that marginal tax rates are everywhere non-negative. The qualitative features of optimal tax schedules are discussed. It is concluded that taxation based solely on total family income is rarely optimal.

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1. Introduction

Ever since the work of Mirrlees (1971) it has been recognized that the design of income tax policy must take into account the asymmetry of information between agents and the government. When agents have private information about their characteristics upon which a taxation authority wishes to base its taxes, they may have an incentive to misreport these characteristics to the planner. The planner must design the taxation scheme to prevent this sort of misrepresentation. It is customary in the literature to assume that the hidden differences among agents can be summarized by a single parameter.[2] This assumption has been questioned as a complete description of individuals. It seems even more dubious when decision making units are comprised of more than one individual.

An example of a planner designing optimal policies for heterogeneous groups is the problem of family income taxation. Within a family, individuals may differ in labor productivity or they may have unequal say in family decisions. These differences influence labor supply behavior, which, in turn, has consequences for the design of a tax system. In this study, I trace the effects of family interactions on optimal tax schedules, paying particular attention to the role played by diversity within the family.

Some classic questions of public finance can be addressed by considering the nonlinear tax problem for families. For instance, the analysis of this problem can contribute to the debate over whether the base for income taxation ought to be family income or individual income. Indeed, the issue of whether all members of the same family should be taxed at the same rate is one of the central questions of this study. I show that taxing distinct members of a family at the same rate is not always optimal. This can also be viewed as a contribution to the debate over the desirability of a uniform flat tax.

I consider an economy inhabited by two–person families. Attention is restricted to the case of workers of two productivity types. This results in four possible family compositions. Each family member has preferences over leisure and a consumption good. Families are assumed to act so as to maximize a weighted sum of their members' utilities. The weights are assumed to be independent of incomes. In this way, the family decision process can be described by a single parameter. This parameter is known to the government.

The planner designs a tax schedule for these families. Families are then free to reallocate their after–tax incomes to maximize their objectives. Because decisions are made at the family level, self selection constraints are formulated in such a way that families have no incentive to misreport the types of their members. That is, families are viewed as decision making units that differ along two dimensions, namely the productivities of their members. Under the maintained hypothesis that there exists a family social welfare

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[2] Guesnerie and Seade (1982) provide a characterization of optimal tax schedules for a finite economy under this assumption. Weymark (1986a,b, 1987) shows how the problem can be decomposed into simpler subproblems when preferences are quasi–linear, and provides a more detailed description of optimal tax schedules.

function, there is no loss of generality in considering tax schedules that specify the total tax liability for a family as a function of the pair of before–tax incomes of its members. Hence, the model economy studied here has three goods in it. This contrasts with the work of Guesnerie and Seade (1982), who consider nonlinear taxation in a two–good economy.

Because there are three goods in the economy, family indifference surfaces are two dimensional. Thus, if the indifference surfaces of distinct families cross, their intersection usually occurs along a curve. In particular, it is unreasonable to expect a standard single–crossing property, such as the one posited by Guesnerie and Seade (1982, Assumption B, p.168) to hold. Nevertheless, the family objectives specified in this analysis posses some special geometric properties. When two families differ along exactly one dimension, certain projections of their indifference surfaces satisfy a single crossing property. Moreover, the indifference surfaces of any pair of such families intersect along a line parallel to one of the coordinate axes. I demonstrate that these two features of preferences have implications for the structure of optimal tax schedules.

A minimal amount of structure is placed on the objectives of the taxation authority. Only Pareto efficiency with respect to family objectives and a standard redistributive assumption are posited. Family objectives are assumed to be additively separable, but not quasi–linear. Even with this minimal amount of structure, it can be shown that individuals in the same family may face different marginal tax rates, so that using total family income as a tax base is not optimal. It is also shown that the redistributive assumption is not strong enough to rule out negative marginal tax rates for some individuals.

The formal analysis is an example of mechanism design with two dimensional uncertainty, which has many formal similarities to the work of Armstrong and Rochet (1999).[3] They consider an abstract screening problem with discrete types that differ along two dimensions. They characterize all the possible optimal mechanisms for the case of preferences that are quasi–linear in consumption. Mechanisms with the qualitative features they describe are possible in the present context. They interpret their results in the contexts of nonlinear pricing by a monopolist and regulation problems. Given the relatively unstructured environment considered here, some additional possibilities arise. Drawing attention to these possibilities demonstrates the strength of their quasi–linearity assumption.

Like nonlinear pricing and regulation problems, the taxation problem can be described as a choice among alternatives that satisfy self–selection criteria. However, nonlinear taxation problems are more than a mere reinterpretation of pricing and regulation problems. There is a natural group of consumers whom a monopolist wishes to identify and extract surplus from, those who have a greater taste for its product. Likewise, a regulator typically leaves information rents to regulated firms with low production costs. *A priori*,

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[3] Among the contributions to multi–dimensional income tax problems in the continuous case are Mirrlees (1976) and Seade (1979). Wilson (1993) and Basov (2005) provide overviews of multi–dimensional screening problems in continuous type spaces.

there is no group of workers that a planner wishes to tax more heavily than others. It is common to assume that the planner wishes to redistribute income from more able workers to less able workers. Nevertheless, it is important to recognize that such an assumption is a value judgment. Moreover, nonlinear pricing and regulation problems are usually presented in a partial equilibrium context, with the amount of total surplus extracted by the monopolist constrained by the voluntary participation of consumers. An economy–wide materials balance constraint limits the scope of the taxation authority.

Another important distinction between optimal taxation problems and monopoly pricing models is that the objective function of the taxation authority is usually assumed to be increasing in the welfare of each agent, whereas monopolists are modeled as being concerned about profits rather than the welfare of their customers.[4] As Brito et al. (1990) have shown, many qualitative features of optimal nonlinear tax schedules follow directly from the planner giving positive consideration to the utility of every agent. Their analysis is not restricted to the two good world, nor to the case of agents who differ in only one characteristic. Indeed, they do not use any parameterization of the differences among agents to derive their results. By exploiting the special structure of the family objectives in the present model, I can make some statements about the tax rates faced by specific families. These kinds of statements cannot be derived in the more general framework of Brito et al..

The remainder of the paper is organized as follows. Section 2 gives an outline of the economy. The implications of the self–selection constraints are presented in Section 3. Section 4 summarizes the implications of the Pareto–efficiency assumption. The consequences of adding the redistributive assumption are outlined in Section 5. Concluding remarks are found in Section 6. Technical lemmas and proofs are collected in an Appendix.

2. The Model

Individuals in the economy are assumed to differ according to their productivity. Specifically, there are two types of individuals, indexed by w_L and w_H with $w_L < w_H$. This index corresponds to the efficiency units of labor per unit of labor time supplied by the individual. I assume constant returns to scale in the production sector and a perfectly competitive labor market so that the before–tax income of an individual is given by $y_i := w_i l_i$, where l_i is the labor supplied by person i. There is a single consumption good, c. Individuals have preferences over the consumption good and labor supply given by

$$u_i(c_i, l_i) := U(c_i) - h(l_i). \tag{1}$$

The function $U(\cdot)$ is assumed to be continuously differentiable, increasing, and strictly concave. It is also assumed that $U'(c)$ tends to positive infinity as c tends to zero. The

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[4] Regulation design problems, like the one considered by Dana (1993), feature a planner with an objective function that takes both consumers' and producers' surplus into account.

function $h(\cdot)$ is assumed to be continuously differentiable, increasing, and strictly convex.

Families consist of two individuals. There are four types of families: LL, HL, HH, and LH. Let $\mathcal{F}$ denote the set of families. Family decisions are assumed to be consistent with the maximization of a weighted utilitarian household social welfare function

$$W(u_1, u_2) := u_1 + \gamma u_2, \quad \gamma > 0. \tag{2}$$

$W(\cdot)$ is not a symmetric function, so that a family of type HL is not identical to one of type LH. Let $x_1$and x_2 denote the after–tax incomes of the two individuals in the family and let $x := x_1 + x_2$. Then consumption decisions for a typical household arise as the solution to

$$\max_{c_1, c_2} \Big[U(c_1) - h\Big(\frac{y_1}{w_1}\Big)\Big] + \gamma \Big[U(c_2) - h\Big(\frac{y_2}{w_2}\Big)\Big] \quad \text{subject to} \quad c_1 + c_2 \leq x_1 + x_2. \tag{P}$$

I now turn to a description of the important features of the solutions to (P).[5]

Proposition 1. *Let $\hat{c}_1(x_1, x_2)$ and $\hat{c}_2(x_1, x_2)$ be the solution functions for (P). Then*

$$\hat{c}_1(x_1, x_2) = \tilde{c}_1(x); \quad \hat{c}_2(x_1, x_2) = \tilde{c}_2(x). \tag{3}$$

Condition (3) states that for a given total family after–tax income, the allocation of consumption within the family does not depend on the identity of the family member who receives the income. This is know as the income pooling condition on family behavior. It holds when families maximize any Bergson–Samuelson social welfare function, not just for the additive form specified here. Its major implication for this study is the reduction of the number of planner's choice variables per family from four to three. Because the choice variables are consumption levels, the formulation of (P) may lead one to believe that the family is not making any labor supply decisions. This suspicion is untrue. The problem (P) describes how after–tax income is allocated between family members. Labor supply choices are modeled as the choice from the tax menu offered by the planner. Families take the division of consumption and its effects on family welfare into account when choosing how much to work.

Using (3), it is possible to define the function

$$V(x) := U(\tilde{c}_1(x)) + \gamma U(\tilde{c}_2(x)), \tag{4}$$

the component of family welfare owing to after–tax income. With this notation, I can now state some further properties of solutions to the problem (P).

Proposition 2. *Let $\tilde{c}_1(\cdot)$ and $\tilde{c}_2(\cdot)$ be as defined in Proposition 1. Then $V(\cdot)$ is increasing and strictly concave.*

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[5] Throughout this analysis it is assumed that non–negativity conditions (which are not stated explicitly) are satisfied.

The function $V(\cdot)$ is the sum of concave transformations of the optimal consumption choices. While the choice functions need not be concave, they are not so convex as to lead to a violation of strict concavity of $V(\cdot)$.

In what follows I need to consider the value function for (P), which depends on the before–tax incomes of the family members and on joint family after–tax income. Denote the value function for a family of type i by W^i. Then

$$W^i(x^i, y_1^i, y_2^i) = V(x^i) - h\Big(\frac{y_1^i}{w_1^i}\Big) - \gamma h\Big(\frac{y_2^i}{w_2^i}\Big). \tag{5}$$

I take the consumption good as numeraire and assume that the producer price of effective labor is one. Let π^i be the proportion of families of type i in the population. Then the materials balance constraint for the economy can be written as

$$\sum_i \pi^i x^i \leq \sum_i \pi^i y_1^i + \sum_i \pi^i y_2^i. \tag{F}$$

The lack of complete information prohibits the use of optimal lump–sum taxation. Instead, the planner must design an allocation of goods that satisfies the self–selection conditions

$$W^i(x^i, y_1^i, y_2^i) \geq W^i(x^j, y_1^j, y_2^j),\ \forall i \neq j. \tag{SS}$$

That is, each family must (weakly) prefer the bundle of goods designed for it to the bundle intended for any other family. The relations (SS) represent the natural incentive–compatibility constraints in this environment, as decisions are made by families.

It follows from the taxation principle (Guesnerie (1981)) that the problem of designing a tax schedule for these families is equivalent to offering a menu of alternatives satisfying (SS). Because of income pooling, there is no loss of generality in considering tax functions that specify the total tax liability of a family for a given ordered pair of before–tax incomes. For fixed before–tax incomes it is merely a transformation of variables to consider total family after–tax income rather than total tax liability as the decision variable of the taxation authority. Such a tax schedule must be anonymous in that each family faces the same budget set. It need not be anonymous at the individual level, because the tax paid by one member of a family may depend on the choices of her partner. Clearly, when families choose labor–consumption bundles to maximize their welfare from an anonymous tax schedule the outcomes satisfy (SS).

Furthermore, it has been shown by Guesnerie (1981) that for any allocation that satisfies (SS) and (F) a tax schedule can be constructed that induces the families to choose that allocation. The tax schedule constructed by Guesnerie requires the possibility of offering an infinitely negative amount of after–tax income to a family with before–tax incomes other than those that arise from a truth–telling game. When negative after–tax incomes are infeasible (as they are assumed to be here), these punishment allocations cannot be used. In the current model, it is possible to support any allocation which satisfies (SS) and (F) with a tax schedule that does not require negative consumption.

To see this, take an allocation that satisfies (SS) and (F). Let $\mathcal{S}$ denote the set of family bundles in that allocation. Define the sets

$$\mathcal{L} := \{(y_1, y_2) | (y_1, y_2, x) \in \mathcal{S}\}; \quad \chi(y_1, y_2) := \{x | (y_1, y_2, x) \in \mathcal{S}\}. \tag{6}$$

For any ordered pair (y_1, y_2), $\chi(y_1, y_2)$ is a singleton, so with a slight misuse of notation, I consider $\chi(\cdot)$ to be a function from $\mathcal{L}$ into $\mathbb{R}_+$. Now, construct a tax function $T\colon \mathbb{R}^2_+ \to \mathbb{R}$ by

$$T(y_1, y_2) := \begin{cases} y_1 + y_2 - \chi(y_1, y_2), & \text{if } (y_1, y_2) \in \mathcal{L}; \\ y_1 + y_2, & \text{if } (y_1, y_2) \notin \mathcal{L}. \end{cases} \tag{7}$$

Given the tax function $T(\cdot)$, the budget set faced by each family is

$$\mathcal{B} := \mathcal{S} \cup \{(y_1, y_2, 0) | (y_1, y_2) \notin \mathcal{L}\}. \tag{8}$$

Given that $U'(c)$ tends to positive infinity as c tends to zero, family indifference surfaces do not cross the (y_1, y_2)–plane. Thus, families choose only the bundles from $\mathcal{B}$ that are contained in $\mathcal{S}$. Moreover, because the bundles in $\mathcal{S}$ satisfy (SS), each family chooses the bundle intended for it in the allocation (SS). Hence, the resulting choices from the tax schedule also satisfy (F).

The planner seeks to maximize a social objective. There are two possible sets of arguments for a welfare–based objective: individual utilities and family welfare levels. The former is more closely related to the standard value judgments of individualism and welfarism. However, the arguments of the planner's objective function then fail to coincide with the criterion functions of the agents (here, families) in the economy. Hence, the formal analysis of this case would be closely related to the work on income taxation with non–welfarist objectives (Seade (1980); Kanbur et al. (1994)). Using family welfare levels as the arguments of a planner's objective arises inevitably from treating a family as a homogeneous unit, or can be interpreted as the planner respecting the ethics embodied in the family social welfare function. The resulting analysis is formally related to work on optimal nonlinear policies in environments of two–dimensional uncertainty. (Dana (1993); Rochet (1995); Armstrong (1996); Armstrong and Rochet (1999)).

In order to place this paper in the more familiar context of multi–dimensional screening, I choose to take the family as the basic unit of welfare analysis. Specifically, I analyze the solutions to the problem

$$\max_{\mathbf{y_1}, \mathbf{y_2}, \mathbf{x}} Z(W^{LL}, W^{HL}, W^{HH}, W^{LH}) \text{ subject to (SS) and (F),} \tag{PF}$$

where $\mathbf{x} := (x^{LL}, x^{HL}, x^{HH}, x^{LH})$, and $\mathbf{y_i} := (y_i^{LL}, y_i^{HL}, y_i^{HH}, y_i^{LH})$, $i = 1, 2$. The function $Z(\cdot)$ is assumed to be increasing, continuously differentiable and concave.[6]

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[6] The problem (PF) may also be interpreted as a the problem faced by a planner who wishes to design a tax scheme for individuals with multiple characteristics and preferences represented by (5).

3. Self–Selection

The structure of family objectives, summarized by (5), influences the nature of the self–selection constraints. I now turn to an elucidation of the important features of family objectives and the implications these features have for allocations that satisfy the constraints (SS).

Consider, first, the marginal rates of substitution between before–tax incomes and after–tax income, given by

$$MRS^i_{y_1,x} := \frac{h'(\frac{y^i_1}{w^i_1})}{w^i_1 V'(x^i)}, \quad MRS^i_{y_2,x} = \frac{\gamma h'(\frac{y^i_2}{w^i_2})}{w^i_2 V'(x^i)}. \tag{9}$$

It is clear from (9) that any differences among the marginal rates of substitution of different families arise from the structure of the function $h(\cdot)$. Two important properties of $h(\cdot)$ follow from its convexity. First, holding y_1 (resp. y_2) constant, preferences in (y_2, x)–space (resp. (y_1, x)–space) satisfy the single–crossing property. Projections of indifference surfaces onto (y_2, x)–space, say, are flatter for families with a person of higher ability in position 2. It is natural to expect such a property to hold, as individuals of low type must give up more leisure time than their high–type counterparts to gain an equal amount of before–tax income. Hence, families with low–type individuals would require more additional consumption to compensate them for increases in before–tax income. Moreover, viewed as a function of y and w, $h(\cdot)$ has decreasing–differences in y. It is important to note that no such property holds for $W(\cdot)$.[7]

A standard feature of optimal income tax models is that more able individuals receive both a higher before–tax and a higher after–tax income (Weymark (1986b)). One would not expect this result to obtain in the present context given that the notion of a more able family is not immediate. At least, the immediate concept of a more able family does not completely order the families. It is, however, possible to define a partial ordering on the set of families in a natural way.

Definition 1. *The relation $\geq_F$ on $\mathcal{F}$, the set of families, is defined by: $i \geq_F j$ if and only if $w^i_1 \geq w^j_1$ and $w^i_2 \geq w^j_2$. The relation $>_F$ is defined to be the asymmetric component of $\geq_F$.*

The relation $\geq_F$ orders all families but the pair (HL, LH).

The self–selection constraints place some structure on the pattern of before–tax incomes, especially for those families that can be compared using the partial order $\geq_F$. This is the content of the next two propositions.

Proposition 3. *Any allocation that satisfies (SS) also satisfies:*

(i) $y^{HH}_1 \geq y^{LH}_1$. *Moreover, if* $y^{HH}_2 > y^{LH}_2$, *then* $x^{HH} > x^{LH}$ *and if* $y^{HH}_2 = y^{LH}_2$, *then* $x^{HH} \geq x^{LH}$ *(with equality if and only if* $y^{HH}_1 = y^{LH}_1$*).*

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[7] These properties are stated formally as Lemmas A.1 and A.2 in the Appendix.

(ii) $y_1^{HL} \geq y_1^{LL}$. *Moreover, if* $y_2^{HL} > y_2^{LL}$, *then* $x^{HL} > x^{LL}$ *and if* $y_2^{HL} = y_2^{LL}$, *then* $x^{HL} \geq x^{LL}$ *(with equality if and only if* $y_1^{HL} = y_1^{LL}$*).*

(iii) $y_2^{HH} \geq y_2^{HL}$. *Moreover, if* $y_1^{HH} > y_1^{HL}$, *then* $x^{HH} > x^{HL}$ *and if* $y_1^{HH} = y_1^{HL}$, *then* $x^{HH} \geq x^{HL}$ *(with equality if and only if* $y_2^{HH} = y_2^{HL}$*).*

(iv) $y_2^{LH} \geq y_2^{LL}$. *Moreover, if* $y_2^{LH} > y_2^{LL}$, *then* $x^{LH} > x^{LL}$ *and if* $y_2^{LH} = y_2^{LL}$, *then* $x^{LH} \geq x^{LL}$ *(with equality if and only if* $y_2^{LH} = y_2^{LL}$*).*

Proposition 3 states that given an equally productive partner, a person of high productivity will earn at least as much before–tax income as a low–productivity person. Abstracting from differences among individuals 2 (by, say, considering an economy populated entirely by families of types HH and LH) produces a three–good economy with one–dimensional uncertainty. Proposition 3 states that self–selection imposes a monotonicity property on the allocation of y_1, the good over which agents differ in a way that is unknown to the planner. Such worlds have been studied in an environmental regulation context by van Egteren (1996). He reports that self–selection requires a monotonicity property on pollution control standards, the cost of which is the only source of private information in his model.

When families differ in more than one dimension, the implications of self–selection are more complex.

Proposition 4. *Let (SS) hold. Then*

(i) $y_1^{HH} < y_1^{LL}$ *implies* $y_2^{HH} > y_2^{LL}$. *Moreover,* $y_1^{HH} = y_1^{LL}$ *implies either: a)* $y_2^{HH} > y_2^{LL}$ *and* $x^{HH} > x^{LL}$, *or b)* $y_2^{HH} = y_2^{LL}$ *and* $x^{HH} = x^{LL}$. *Furthermore,* $y_1^{HH} > y_1^{LL}$ *and* $y_2^{HH} \geq y_2^{LL}$ *implies* $x^{HH} > x^{LL}$.

(ii) $y_2^{HH} < y_2^{LL}$ *implies* $y_1^{HH} > y_1^{LL}$. *Moreover,* $y_2^{HH} = y_2^{LL}$ *implies either: a)* $y_1^{HH} > y_1^{LL}$ *and* $x^{HH} > x^{LL}$, *or b)* $y_1^{HH} = y_1^{LL}$ *and* $x^{HH} = x^{LL}$. *Furthermore,* $y_2^{HH} > y_2^{LL}$ *and* $y_1^{HH} \geq y_1^{LL}$ *implies* $x^{HH} > x^{LL}$.

(iii) $y_1^{HL} < y_1^{LH}$ *implies* $y_2^{LH} > y_2^{HL}$ *and* $x^{LH} > x^{HL}$. *Moreover,* $y_1^{HL} = y_1^{LH}$ *implies either: a)* $y_2^{LH} > y_2^{HL}$ *and* $x^{LH} > x^{HL}$, *or b)* $y_2^{LH} = y_2^{HL}$ *and* $x^{LH} = x^{HL}$.

(iv) $y_2^{LH} < y_2^{HL}$ *implies* $y_1^{HL} > y_1^{LH}$ *and* $x^{HL} > x^{LH}$. *Moreover,* $y_2^{LH} = y_2^{HL}$ *implies either: a)* $y_1^{HL} > y_1^{LH}$ *and* $x^{HL} > x^{LH}$, *or b)* $y_1^{HL} = y_1^{LH}$ *and* $x^{HL} = x^{LH}$.

Parts (i) and (ii) of Proposition 4 are consistent with a general notion of more productive individuals receiving a higher before–tax income, as at least one member of family HH must earn more before–tax income than the corresponding member of family LL. The only exception to this tendency occurs when families HH and LL receive exactly the same bundle.

Some high–productivity individuals may earn less than corresponding low–productivity individuals at an allocation consistent with self–selection. For example, if $\bar{y}_2^{LL} < \bar{y}_2^{HH}$, then self–selection requires that family HH enjoy more utility from the combination after–tax income and labor supply for its person 1 than it would under the allocation designed form family LL. This additional utility must be due to extra after-tax income or less work for person 1.

Parts (iii) and (iv) of Proposition 4 indicate that there are cases for which the self–selection constraints place restrictions on the ordering of the after–tax incomes of families LH and HL. There is, however, a caveat to this apparently strong result about after–tax incomes. The conditions under which this ordering is available appear to be rather strong. For instance, when y_2^{LH} is less than y_2^{HL}, it is the case that an individual of lower ability is earning more than a high–ability individual.

The indeterminacy of the ordering of after–tax incomes at this stage of the analysis is perhaps the most important obstacle to overcome in characterizing the optimal solution. It is also one of the most important distinctions between multi–dimensional optimal tax mechanisms and their single–dimensional counterparts. This indeterminacy is not the direct result of the two–dimensional uncertainty faced by the planner. Rather, as Proposition 3 illustrates, it results from the multiplicity of instruments. Nevertheless, when hidden information can be summarized by a one–dimensional characteristic, the use of multiple instruments leads to solutions that bear striking resemblance to the standard unidimensional problem with two goods as long as sufficient structure is imposed to ensure that agents of higher type receive more of all goods.[8]

4. Properties of Optimal Tax Schedules

The results of the previous section followed directly from the self–selection constraints. In particular, none of the analysis relied on the assumption that the planner seeks to maximize a social objective function. In this section I incorporate optimizing behavior on the part of the taxation authority into the analysis to derive some additional properties of solutions to the problem (PF). The first of these is standard in optimal income tax theory.

Proposition 5. *At any solution to (PF), the constraint (F) binds with positive multiplier.*

Proposition 5 can be interpreted as stating that any surplus output can always be distributed among the families in such a way that self–selection constraints are not violated.

Although simple, Proposition 5 is an important building block for further analysis of the problem. In particular, it allows us to conclude that if, starting from an initial allocation, a change can be made that slackens the resource constraint, does not violate

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[8] This property is known as attribute ordering (Matthews and Moore (1987)). Besley and Coate (1995) provide a model in which a monotonicity property different from, but in the same spirit as, attribute ordering is sufficient to render their income maintenance design problem well–behaved.

the self–selection constraints and makes no family worse off, then the initial allocation is not a solution to (PF). This is exactly the type of reasoning behind the next proposition.

Proposition 6. *Suppose that at a solution to (PF),* $W^i(x^i, y_1^i, y_2^i) = W^i(x^j, y_1^j, y_2^j)$ *for a pair of families* i, j. *Then*

$$y_1^i + y_2^i - x^i \geq y_1^j + y_2^j - x^j. \tag{10}$$

Proposition 6 is the restatement of a result of Brito et al. (1990, Proposition 1, p.66) in the current context. One would expect this result to hold here, for it states simply that a family never wishes to mimic a family that has a higher total tax bill than itself. In particular, the statement of Proposition 6 contains no reference to the labeling of families adopted in this analysis. It is well–known that, in the two–good model with unidimensional differences among agents, the pattern of binding self–selection constraints determines the qualitative features of optimal marginal tax rates (Röell (1985)). Proposition 6 tells us that the pattern of binding self–selection constraints can also be used to make statements concerning total tax liabilities.

When discussing the relationship between the self–selection constraints and tax functions, I considered a wide class of tax functions. In particular, the tax functions were allowed to be non–differentiable. At kinks in the tax schedule, it is impossible to define marginal tax rates as (partial) derivatives of the tax function. It is possible, however, to define implicit marginal tax rates at any allocation by:

$$t_1^i := 1 - MRS_{y_1,x}^i = 1 - \frac{h'(\frac{y_1^i}{w_1^i})}{w_1^i V'(x^i)}, \quad t_2^i := 1 - MRS_{y_2,x}^i = 1 - \frac{\gamma h'(\frac{y_2^i}{w_2^i})}{w_2^i V'(x^i)}. \tag{11}$$

It is clear from their definitions that implicit marginal tax rates are positive when the marginal rate of substitution between labor and consumption is less than one. Marginal wage subsidies (negative marginal tax rates) correspond to marginal rates of substitution in excess of one.

Marginal tax rates serve as an important summary statistic of the distortions arising from the planner's lack of information. In the current context, the following proposition demonstrates how the sign of marginal tax rates depends on the pattern of binding self–selection constraints.

Proposition 7.

(i) *For any family* i *and any family member* k, *if* $w_k^i = w_L$, *then* k *faces a non–negative marginal tax rate* t_k^i. *Furthermore,* $t_k^i > 0$ *if and only if there is a family* j *with* $w_k^j = w_H$ *and the* j*–*i *self–selection constraint binds with a positive multiplier.*

(ii) *For any family* i *and any family member* k, *if* $w_k^i = w_H$, *then* k *faces a non–positive marginal tax rate* t_k^i. *Furthermore,* $t_k^i < 0$ *if and only if there is a family* j *with* $w_k^j = w_L$ *and the* j*–*i *self–selection constraint binds with a positive multiplier.*

It follows directly from Proposition 7 that if, for all j, the j–i self–selection constraint does not bind at a solution to (PF), then both members of family i face a zero marginal tax rate. This is a direct analogue of a result due to Guesnerie and Seade (1982, Proposition 2, p.164), which states that if there an individual with a bundle that all other individuals view as strictly inferior to their own, then that individual faces a zero marginal tax rate. However, Proposition 7 is more than a restatement of the result of Guesnerie and Seade in the current context. It says: no distortions are introduced on the labor supply behavior of individual 1, say, when a self–selection constraint binds between families with identical persons 1. Because two such families view trade–offs between y_1 and x identically, any such trade–offs can be made without violating the self–selection constraints involving these two families. Were it not for the presence of other families, the planner could vary the allocations of y_1 and x for these two families until an undistorted bundle is achieved.

Proposition 7 speaks to the debate over the choice of tax base. In particular, it suggests that a tax based solely on total family income will fail to be optimal in many circumstances. Notice that the implied marginal tax rates for the individuals in family HL are, in general, different. Indeed, these rates can coincide only if both are zero. The person of lower productivity faces a higher marginal tax rate at the optimum. An analogous result holds for family LH. It seems reasonable that the planner might want to apply different tax rates within a family, given that total production in the economy is determined by how much individuals decide to work. Like one–dimensional taxation models, the choice of marginal tax rates for higher ability individuals is dominated by efficiency considerations. The planner wishes to extract effort from these individuals, since they are the ones whose work effort provides the most output. By using different marginal tax rates within the household, the planner is able to identify which of the two individuals in a mixed household is the one with higher productivity, enabling it to provide sufficient incentives for that individual to provide work effort. If the planner is forced to use a flat tax, then its power to identify high–productivity individuals is limited.

Proposition 7 has an important corollary.

Proposition 8. *No two families of different type receive the same allocation at a solution to (PF).*

Phrased in the language of screening models, there is no bunching at the optimum. Proposition 8 is an analogue to the no pooling result of Stiglitz (1982) for economies with two types of consumers. It shows that it is not the existence of only two types of consumers *per se* that is important in deriving the no–pooling result. As long as there are only two families along each dimension of uncertainty there must be separation at the optimum.[9]

Further progress in understanding the solutions to (PF) requires that something be said about the pattern of binding self–selection constraints at the optimum. One can

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[9] The argument establishing Proposition **??** relies on the implicit assumption that all individuals receive a positive before–tax income at the solution to (PF).

expect this pattern to be influenced by the form of the social objective function $Z(\cdot)$. The question arises: Can some patterns may be ruled out at any solution? I now turn to the task of answering this question.

In order to make some of the discussion easier, I adopt some terminology of Wilson (1993).

Definition 2. *Family i attracts family j if the j–i self–selection constraint is binding.*

Proposition 9. *At a solution to (PF), any pair of families i and j, except possibly the pair (HH, LL), are mutually attracted to each other only if they receive the same allocation.*

The one–dimensional analogue to Proposition 9 is an immediate consequence of the usual single–crossing property, and holds for all allocations. In the present context indifference surfaces intersect at more than one point. Hence, it is important to emphasize that the conclusion of Proposition 9 holds at the optimum. The exclusion of the pair (HH, LL) at this point is due to the fact that there is insufficient structure on the general form of (PF) to ensure that the solution is attribute ordered (Matthews and Moore (1987)). In particular, one cannot conclude that both members of family HH receive more before–tax income than the corresponding members of family LL.

Proposition 9 is closely related to a result of Brito et al. (1990, Proposition 2, p.67). They show that when there are cycles of self–selection constraints, the planner can do no worse by pooling the agents involved in the cycle. The additional structure of this model allows this conclusion to be strengthened for most pairs of families to say that the planner can always do better by pooling. However, in view of Proposition 8, the planner can do better still. This statement is formalized in the next proposition.

Proposition 10. *No pair of families is mutually attracted at a solution to (PF).*

For all pairs save (HH, LL) Proposition 10 follows directly from Propositions 8 and 9. For the pair (HH, LL) the proposition follows from Proposition 8 and the aforementioned result of Brito et al. Indeed, Proposition 10 is a special case of a more general result, itself a direct consequence of Proposition 8 and the work of Brito et al. In order to state this result, I need the following definition.

Definition 3. *A self selection cycle is said to exist when there is a set of families $\{i_1, \ldots, i_n\}$ such that family i_1 attracts family i_n and for all $k < n$, family i_{k+1} attracts family i_k.*

Proposition 11. *There are no self–selection cycles at a solution to (PF).*

Another standard feature of optimal nonlinear tax schedules is the existence of an agent who faces no distortions. In one–dimensional models with single–crossing and two goods, this is often the agent of highest ability, and, for some continuous models, the agent of lowest ability as well (Seade (1977)). A version of this result also holds in the present context.

Proposition 12. *For any family i, if*

$$y_1^i + y_2^i - x^i \geq y_1^j + y_2^j - x^j, \text{ for all } j \neq i, \tag{12}$$

then family i faces no marginal distortions at a solution to (PF).

It follows from Proposition 12 that there is a family whose allocation is not subject to marginal distortions. Moreover, it identifies undistorted families: those that pay the highest taxes. This is exactly the finding of Brito et al. (1990). Of course, a family that faces no marginal distortions is not pooled with any other family, just as was found by Guesnerie and Seade (1982).

At this point, it is useful to define two classes of constraints.

Definition 4. *The HH–LL and LL–HH self–selection constraints are called diagonal. The HL–LH and LH–HL self–selection constraints are called transverse.*

Proposition 13. *At a solution to (PF), it cannot be the case that a diagonal constraint and a transverse constraint bind simultaneously.*

Proposition 13 greatly simplifies the search for solutions to (PF). Moreover, it shows that it is not by coincidence that no solutions discovered by Dana (1993), Rochet (1995), and Armstrong and Rochet (1999) feature a binding diagonal constraint and a binding transverse constraint. Note that quasi–linearity (which is assumed in these earlier studies) is not needed to generate this result.

5. Redistributive Taxation

The requirement of Pareto–efficiency has yielded quite a bit of information about the properties of optimal tax schedules. Further progress in characterizing the solutions to the problem (PF) requires that some structure be placed on the objectives of the planner. There are two ways to provide this structure. One is to posit a form for the objective function of the planner, say a weighted sum of family objectives. The other method involves making a redistributive assumption. It is usual in the literature to assume that, when incentive effects are ignored, the planner wishes to transfer consumption from the more able to the less able (Guesnerie and Seade (1982); Chambers (1989)). In the spirit of this literature I employ the following redistributive assumption.

Assumption 1 (Redistribution). *Suppose that $i >_F j$ for a pair of families (i, j). Then there exists some sufficiently small $\epsilon > 0$ such that, at the optimum, it is socially desirable to transfer ϵ units of after–tax income from family i to family j if the constraints (SS) are ignored.*

Assumption 1 is not equivalent, in general, to a desire to increase the welfare of families with less able individuals (Dixit and Seade (1979)). It is also important to note that Assumption R places no restriction on how the planner views the consumption of family HL *vis–a–vis* the consumption of family LH. Despite this, Assumption 1 has implications for the structure of solutions to (PF). It is not sufficient, however, to ensure that all marginal tax rates are non–negative.

It is also necessary to say something about the distribution of families in order to classify more succinctly the optimal taxation mechanisms. Employing Assumption 1 is much more straightforward when all of the redistributions it declares to be desirable are production–feasible. This condition is guaranteed by the following assumption about the distribution of families.

Assumption 2 (Uniform Distribution). $\pi^{LL} = \pi^{HL} = \pi^{HH} = \pi^{LH}$.

Unlike the situation that obtains in the models of Dana (1993) and Armstrong and Rochet (1999), Assumption 2 does not rule out asymmetric solutions in this model. Asymmetries may arise due to the form of the planner's objectives or due to the asymmetry of the family decision process.

The remainder of this section is devoted to cataloging the qualitative features of possible optimal configurations. In order to compare the current results with others in the literature, it is necessary at times to impose additional structure on family objectives. The following assumption will sometimes be cited.

Assumption 3 (Quasi–linearity). $V(x) = x$.

Assumption 3 is consistent with the model of family decision making only if $U(x) = x$.[10] This specification can lead to one member of the family consuming everything. Assumption 3 may be more palatable in other screening contexts. The main reason for its use in this study is to compare the more general results obtained without it to the optimal mechanisms that arise when it is satisfied. In this way, the full force of this common assumption can be ascertained.

One more piece of notation is required.

Definition 5. *The set $\mathcal{C}$ is defined by saying that an ordered pair of indices (i, j) is an element of $\mathcal{C}$ if and only if the i–j self–selection constraint binds at the solution to (PF).*

5.1. The usual cases

As emphasized by Dana (1993), Rochet (1995), and Armstrong and Rochet (1999), much of the analytical difficulty in multi–dimensional screening problems arises from the possibility of binding diagonal or transverse constraints. In this subsection, patterns of

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[10] Strictly speaking, Assumption 3 is inconsistent with the boundary condition required on preferences required to guarantee that, for any allocation satisfying (SS), the planner can construct an equivalent tax schedule. However, the boundary condition considered earlier is not required for this equivalence to hold. It suffices that, at the optimum, no family strictly prefer the origin to the bundle designed for it.

binding constraints that can occur when diagonal constraints are assumed not to bind are presented. There is a surprisingly large number of configurations that are compatible with Assumption 1. The main qualitative implications of Assumption 1 are given by the next proposition.

Proposition 14.

(i) *Let Assumptions 1 and 2 hold and let* $(HH, LL) \notin \mathcal{C}$. *Then* $\{(HH, HL), (LH, LL)\} \subseteq \mathcal{C}$ *or* $\{(HH, LH), (HL, LL)\} \subseteq \mathcal{C}$.

(ii) *Let Assumptions 1, 2 and 3 hold and let* $(HH, LL) \notin \mathcal{C}$. *Then* $\{(HH, HL), (HL, LL), (LH, LL)\} \subseteq \mathcal{C}$ *or* $\{(HH, LH), (LH, LL), (HL, LL)\} \subseteq \mathcal{C}$. *Moreover,* $(LL, HH) \notin \mathcal{C}$.

The implication of Proposition 14 for tax rates is that either all low–productivity individuals occupying position 1 in a family or all low–productivity individuals in position 2 in a family face a strictly positive marginal tax rate.[11] (See Proposition 7.) Assumption 1 implies that family HH is attracted to some other family, say family HL. Then individual 2 of family HL has its labor supply distorted downward. Also by Assumption 1, family LL must attract some other family. If family HL is attracted to family LL (while family HL attracts family HH), satisfying the HH–LL self–selection constraint requires that $y_2^{HL} \geq y_2^{LL}$.[12] Then, individual 2 of family LL must also have its labor supply distorted downward. Part (ii) of Proposition 14 shows the force of Assumption 3. When Assumption 3 is maintained, both members of family LL face a strictly positive marginal tax rate at the optimum. This result is consistent with the findings of Armstrong and Rochet (1999), in that each of the possible configurations of optimally binding constraints in their relaxed problem satisfies the set inclusions in part (ii) of Proposition 14.

Quasi–linearity rules out the possibility that, for example, the HH–LH and HL–LL constraints are simultaneously slack at an optimum. In light of the part (i) of Proposition 14, the maintained hypothesis of a slack HH–LL constraint implies HH–HL constraint is binding. Under quasi-linearity, a simultaneous transfer of a small and equal amounts of consumption from HH to LH and from HL to LL has no effect on the HH–HL constraint, and, by Assumption 1, is welfare–improving. It is shown in the Appendix that this transfer does not lead to a violation any other self–selection constraints. The quasi–linearity assumption is used again to show that the HL–LH constraint could not have been binding. This is crucial to the argument, for otherwise the posited redistribution would violate this very constraint. Thus, at least one of the HH-LH or HL-LL constraints must bind. Moreover, it cannot be the HH-LH constraint alone, for then it is possible to engineer a feasible transfer from HL to LL.

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[11] For the remainder of the discussion, it is assumed that all binding constraints bind with positive shadow values. A proof of this assertion requires a slight strengthening of Assumption 1 to state that the social gains from a downward transfer of after–tax income are of the first order. Guesnerie and Seade (1982, p.174) provide the argument.

[12] This assertion is proven in the Appendix.

Proposition 14 is silent about the pattern of binding constraints when family LL attracts family HH. However, Proposition 10 ensures that family LL cannot be attracted to family HH when the HH–LL self–selection constraint binds. Then, part (ii) of Proposition 14 permits the conclusion that the LL–HH self–selection constraint can never bind under Assumptions 1, 2, and 3. Notice, however, that the combination of Assumptions 1 and 3 is not strong enough to rule out a binding LL–HH constraint at the optimum.

Given Assumption 1, any redistribution of after–tax income to family LL from any other family is socially desirable. A binding HH–LL self–selection constraint is an impediment to this kind of redistribution. However, transfers to family LL can only slacken the LL–HH self–selection constraint. Although it is possible for the LL–HH self–selection constraint to bind at the optimum, the planner can, for a given pattern of other binding self–selection constraints, carry out the same downward redistributions regardless of whether family HH attracts family LL or not. Hence, there is no interesting analogue to Proposition 14 for the case of $(LL, HH) \notin \mathcal{C}$.

5.2. The unusual cases

Up to this point, little has been said about families who are not ordered by $\geq_F$. It is the presence of these families that makes the current model substantively different from a one–dimensional model. The possibility of the LH–HL or HL–LH constraint binding implies the possibility of negative marginal tax rates for high–productivity individuals in families HL and LH. This is a major departure from the standard results of the literature on nonlinear redistributive taxation. In this subsection, I show that Assumption 1 does not provide sufficient structure to rule out such phenomena. It does, however, place some restrictions on the pattern of marginal tax rates when a transverse constraint binds.

The next proposition shows that a negative marginal rate to person 1 in family HL (resp. person 2 in family LH) arising from a binding transverse constraint must be accompanied by a negative marginal tax rate for an individual 1 (resp. 2) in family HH.

Proposition 15. *Let Assumptions 1 and 2 hold. Then*

(i) $(LH, HL) \in \mathcal{C}$ *implies* $\{(HH, HL), (LH, LL), (LH, HH)\} \subseteq \mathcal{C}$.

(ii) $(HL, LH) \in \mathcal{C}$ *implies* $\{(HH, LH), (HL, LL), (HL, HH)\} \subseteq \mathcal{C}$.

Proposition 15 implies that when one of the transverse self–selection constraints binds, the pattern of binding constraints for family HH is completely determined. For instance, when the LH–HL self–selection constraint binds, family HH is attracted to family HL and family LH is attracted to family HH. Because a transverse constraint binds, it follows from Proposition 13 that neither family HH nor family LL is attracted to the other. Notice, however, that the pattern remains undetermined for family LL. This asymmetry is a direct result of Assumption 1 and the concomitant tendency for high–ability families to be tempted to under–report their productivity at the optimum. This indeterminacy vanishes, however, when Assumption 3 is satisfied. This is the content of the next proposition.

Proposition 16. *Let Assumptions 1, 2, and 3 hold. Then*

(i) $(LH, HL) \in \mathcal{C}$ *implies* $\mathcal{C} = \{(HH, HL), (HL, LL), (LH, LL), (LH, HH), (LH, HL)\}$.

(ii) $(HL, LH) \in \mathcal{C}$ *implies* $\mathcal{C} = \{(HH, LH), (HL, LL), (LH, LL), (HL, HH), (HL, LH)\}$.

It is interesting to compare Propositions 15 and 16 with the “singular” case of (Rochet, 1995, Proposition 3.4) and the results on binding diagonal constraints in Armstrong and Rochet (1999, Proposition 3). The introduction of income effects leads to some indeterminacy not found in the existing literature. Armstrong and Rochet find that only the cases covered in Proposition 16 may arise. This is neither ruled out nor required in the absence of Assumption 3. Proposition 16 extends the result of Armstrong and Rochet beyond the case of additive objective functions, highlighting the importance of quasi–linearity in deriving their characterization of binding incentive constraints.

The only possibility not considered thus far is that the HH–LL self–selection constraint may bind. In previous work (Dana (1993), Armstrong and Rochet (1999, Case A)), this case has arisen only in the so–called “separable” context, in which family i attracts family j if and only if $j >_F i$. In the present model, separability means that, at the optimum, all low–productivity individuals receive the same before–tax income. Income effects once again destroy the simplicity of the solution. However, the separable case can occur only if the j–i self–selection constraint binds for all pairs of families with $j >_F i$.

5.3. A summary

At first glance, Assumption 1 appears to provide only limited structure on the nature of optimal tax mechanisms. While it is true that the situation is much more complex than is usually found in the literature, some basic properties are common to all solutions. First, as the next proposition shows, when high–ability individuals occupy the same position in a household, the individual with a low–productivity partner works at least as hard as the person with a high–productivity partner.

Proposition 17. *Let Assumptions 1 and 2 hold. Then, at any solution to (PF),* $y_1^{HL} \geq y_1^{HH}$ *and* $y_2^{LH} \geq y_2^{HH}$.

In view of Proposition 17, any marginal wage subsidies offered to individuals in family HH must not be large enough to induce them to work more than high–productivity individuals in other types of families. Phrased in the language of screening models, the solution fails to be attribute ordered. Moreover, it seems natural for this violation to occur. We would expect part of the gain to having a high–productivity partner to be consumed as leisure. A similar finding was reported by Schroyen (2003) for the case of preferences that are quasi–linear in labor supply. Because it is possible to interpret the current model as a taxation scheme with joint filing, it is clear that the individual filing assumption of Schroyen is not needed to generate this result.

Second, if a transverse constraint binds, Proposition 15 has implications for the optimal use of marginal wage subsidies. Whenever family LL is attracted to no other family, the use of wage subsidies for high–productivity individuals is concentrated on individuals with high–productivity partners. When no family attracts family LL, high–productivity individuals with low–productivity partners receive wage subsidies only if individuals in the same position in a family with high–productivity partners also receive marginal subsidies. Suppose, for the sake of concreteness, that family HL attracts family LH, so that individual 1 in family HL faces a negative marginal tax rate. In the quasi–linear context it is clear that this can happen only if marginal increases in the welfare of family HL are weighted more heavily than marginal increases in the welfare of family LH at the optimum. This desire to redistribute resources away from family LH conflicts with the need to prevent family LH from masquerading as family HH. The only way to prevent this occurring, it turns out, is to distort the labor supply of family HH upward. In this case, it is precisely the desire to make transverse redistributions that upsets the standard non–negative marginal tax rate result. Assumption 1 is silent about the desirability of such redistributions. In the nonlinear pricing model studied by Armstrong and Rochet (1999), negative correlation between demand types and suitable asymmetry of preferences are required for binding transverse constraints. This suggests that transverse redistributions might be desirable when family members have disproportionate influence in household decisions.

Third, status in the household can exert an independent influence on optimal tax policy. That is, two equally–productive individuals with equally–productive partners can face different marginal tax rates on the basis of observed demographics that have a known effect on household decisions. Once again, let family HL attract family LH. It follows from Proposition 15 that the low–productivity individual in family HL faces a positive marginal tax rate, while, because no family is attracted to family LH, the low–productivity individual in family LH faces a zero marginal tax rate.

6. Conclusion

This study has presented a model of family income taxation based on the notion of multi–dimensional screening. Viewing the problem in this way gives arguments for members of the same family to face different marginal income tax rates, casting doubt on total family income as appropriate income tax base. Both the productivity of the partner and the relative position of a person in the household have some bearing on the marginal tax rate faced by an individual.

It is interesting to reflect upon the possibility of negative marginal tax rates in this model. It was noted in the discussion following Proposition 17 that, when Assumptions 1 and 2 hold, this important difference from standard one–dimensional tax models arises when the planner has a reason to transfer consumption among families that differ in both characteristics. In the current model, the desirability of such redistributions depends on the social welfare function. Asymmetries in the distribution of the population or a desire to offset undesirable effects of family interactions may provide additional motivations for such redistributions. When there is no reason to use differences along both dimensions to identify families to whom it would be desirable to transfer consumption, there is no reason (at least not in the quasi–linear context) to use marginal wage subsidies.

The biggest obstacle to a more specific characterization of optimal solutions is the income effects embodied in family objectives. It seems unnatural to rule these out. However, the analysis of this paper could be made more sharp in environments where the quasi–linearity assumption is more palatable. For instance, the work of Besley and Coate (1995) on work requirements in income maintenance programs could be extended to cover the case where market work and required work are qualitatively different and individuals may be more or less productive at one or the other.

Appendix

In this appendix I present the proofs of the propositions and lemmas contained in this paper. First, I introduce some notation that will be used throughout. Relabel the families so that the symbols A, B, C and D correspond to families LL, HL, HH and LH, respectively. Also, let

$$\Psi^{ij} := W^i(x^i, y_1^i, y_2^i) - W^i(x^j, y_1^j, y_2^j). \tag{A.13}$$

Ψ^{ij} measures the additional well–being family i enjoys at the allocation intended for it over the well-being it would enjoy by pretending to be family j. Self–selection requires $\Psi^{ij} \geq 0$ for all i and j.

Proof of Proposition 1:

The statement follows immediately from the way that x_1 and x_2 enter the constraint of problem (P) and only the constraint. □

Proof of Proposition 2:

The first–order necessary conditions for the problem (P) are:

$$\begin{aligned} U'(c_1) - \kappa &= 0, \\ \gamma U'(c_2) - \kappa &= 0, \\ x - c_1 - c_2 &= 0, \end{aligned} \tag{A.14}$$

where κ is the multiplier associated with the budget constraint. Differentiating the system (A.14) yields

$$\begin{aligned} U''(c_1)dc_1 - d\kappa &= 0, \\ \gamma U''(c_2)dc_2 - d\kappa &= 0, \\ dx - dc_1 - dc_2 &= 0. \end{aligned} \tag{A.15}$$

Apply Cramer's rule to the system (A.15) conclude

$$\begin{aligned} \frac{\partial c_1(x)}{\partial x} &= \frac{\gamma U''(c_2)}{U''(c_1)+\gamma U''(c_2)}; \\ \frac{\partial c_2(x)}{\partial x} &= \frac{U''(c_1)}{U''(c_1)+\gamma U''(c_2)}. \end{aligned} \tag{A.16}$$

Strict concavity of $U(\cdot)$ implies that in each line of (A.16) both numerator and denominator are negative. Thus, each expression in (A.16) is positive.

Increasingness of $V(\cdot)$ is established by the calculation:

$$V'(x) = U'(\tilde{c}_1(x))\tilde{c}_1'(x) + \gamma U'(\tilde{c}_2(x))\tilde{c}_2'(x) > 0. \tag{A.17}$$

It follows from (A.17) that

$$V''(x) = U''(\tilde{c}_1(x))[\tilde{c}_1'(x)]^2 + U'(\tilde{c}_1(x))\tilde{c}_1''(x) + \gamma U''(\tilde{c}_2(x))[\tilde{c}_2'(x)]^2 + \gamma U'(\tilde{c}_2(x))\tilde{c}_2''(x). \tag{A.18}$$

By (A.14), we may group the second and fourth terms of (A.18) to yield

$$V''(x) = U''(\tilde{c}_1(x))[\tilde{c}_1'(x)]^2 + \gamma U''(\tilde{c}_2(x))[\tilde{c}_2'(x)]^2 + U'(\tilde{c}_1(x))\Big[\tilde{c}_1''(x) + \tilde{c}_2''(x)\Big]. \tag{A.19}$$

By (A.15), the part of the last term of (A.19) in square brackets is zero, so that

$$V''(x) = U''(\tilde{c}_1(x))[\tilde{c}_1'(x)]^2 + \gamma U''(\tilde{c}_2(x))[\tilde{c}_2'(x)]^2 < 0, \tag{A.20}$$

establishing the strict concavity of $V(\cdot)$. □

Lemma A.1. *For all* y, $\frac{1}{w_L}h'(\frac{y}{w_L}) > \frac{1}{w_H}h'(\frac{y}{w_H})$.

Proof of Lemma A.1:

$w_H > w_L$ implies that for all y, $y/w_L > y/w_H$. It follows from strict convexity of $h(\cdot)$ that $h'\big(\frac{y}{w_L}\big) > h'\big(\frac{y}{w_H}\big)$. Lemma 1 follows immediately. □

Lemma A.2. *For any* $(\tilde{y}, \hat{y})$, $\tilde{y} \geq \hat{y}$ *if and only if*

$$h\Big(\frac{\tilde{y}}{w_L}\Big) - h\Big(\frac{\hat{y}}{w_L}\Big) \geq h\Big(\frac{\tilde{y}}{w_H}\Big) - h\Big(\frac{\hat{y}}{w_H}\Big). \tag{A.21}$$

Proof of Lemma A.2

By the first fundamental theorem of calculus, (A.21) holds if and only if

$$\int_{\hat{y}}^{\tilde{y}} \frac{1}{w_L} h'\Big(\frac{y}{w_L}\Big) dy \geq \int_{\hat{y}}^{\tilde{y}} \frac{1}{w_H} h'\Big(\frac{y}{w_H}\Big) dy. \tag{A.22}$$

$h(\cdot)$ strictly increasing implies that the integrands on each side of (A.22) are positive. Lemma A.1 implies that the integrand on the left–hand side of (A.22) is everywhere greater than the integrand on the right–hand side. Thus, (A.22) holds if and only if $\tilde{y} \geq \hat{y}$. $\square$

Proof of Proposition 3:

The proof of statement (i) is presented. Statements (ii)–(iv) are proven analogously to statement (i).

Let the C–D and D–C self–selection constraints be satisfied. Then

$$V(x^C) - h\Big(\frac{y_1^C}{w_H}\Big) - \gamma h\Big(\frac{y_2^C}{w_H}\Big) - V(x^D) + h\Big(\frac{y_1^D}{w_H}\Big) + \gamma h\Big(\frac{y_2^D}{w_H}\Big) \geq 0, \tag{A.23}$$

and

$$V(x^D) - h\Big(\frac{y_1^D}{w_L}\Big) - \gamma h\Big(\frac{y_2^D}{w_H}\Big) - V(x^C) + h\Big(\frac{y_1^C}{w_L}\Big) + \gamma h\Big(\frac{y_2^C}{w_H}\Big) \geq 0. \tag{A.24}$$

Adding (A.23) and (A.24) yields

$$h\Big(\frac{y_1^C}{w_L}\Big) - h\Big(\frac{y_1^D}{w_L}\Big) + h\Big(\frac{y_1^D}{w_H}\Big) - h\Big(\frac{y_1^C}{w_H}\Big) \geq 0. \tag{A.25}$$

Apply Lemma A.2 to conclude that $y_1^C \geq y_1^D$.

Now let $y_2^C > y_2^D$. Because $y_1^C \geq y_1^D$, (A.23) implies $V(x^C) > V(x^D)$. But $V(\cdot)$ is increasing, so $x^C > x^D$.

Suppose, instead, that $y_2^C = y_2^D$. Then $y_1^C > y_1^D$ and (A.23) imply $V(x^C) > V(x^D)$, and, hence, $x^C > x^D$. On the other hand, $y_1^C = y_1^D$, (A.23) and (A.24) imply $V(x^C) = V(x^D)$. Then $x^C = x^D$. $\square$

Proof of Proposition 4:

The proof of statements (i) and (iii) are presented. Statement (ii) can be proved in an analogous manner to statement (i); statement (iv), to that of statement (iii).

Let the C–A and A–C self-selection constraints be satisfied. Then

$$V(x^C) - h\Big(\frac{y_1^C}{w_H}\Big) - \gamma h\Big(\frac{y_2^C}{w_H}\Big) - V(x^A) + h\Big(\frac{y_1^A}{w_H}\Big) + \gamma h\Big(\frac{y_2^A}{w_H}\Big) \geq 0, \tag{A.26}$$

and

$$V(x^A) - h\Big(\frac{y_1^A}{w_L}\Big) - \gamma h\Big(\frac{y_2^A}{w_L}\Big) - V(x^C) + h\Big(\frac{y_1^C}{w_L}\Big) + \gamma h\Big(\frac{y_2^C}{w_L}\Big) \geq 0. \tag{A.27}$$

Adding (A.26) and (A.27) yields

$$\Big[h\Big(\frac{y_1^C}{w_L}\Big) - h\Big(\frac{y_1^A}{w_L}\Big) + h\Big(\frac{y_1^A}{w_H}\Big) - h\Big(\frac{y_1^C}{w_H}\Big)\Big] + \gamma\Big[h\Big(\frac{y_2^C}{w_L}\Big) - h\Big(\frac{y_2^A}{w_L}\Big) + h\Big(\frac{y_2^A}{w_H}\Big) - h\Big(\frac{y_2^C}{w_H}\Big)\Big] \geq 0. \tag{A.28}$$

Now let $y_1^C < y_1^A$. Apply Lemma A.2 to conclude that the first term in square brackets in (A.28) is negative. Hence, the second term is positive. Apply Lemma A.2 once more to conclude $y_2^C > y_2^A$.

Suppose $y_1^C = y_1^A$. Then, by (A.28) and Lemma A.2, $y_2^C \geq y_2^A$. If $y_2^C = y_2^A$ then (A.26) and (A.27) can both be satisfied only if $V(x^A) = V(x^C)$. Given increasingness of $V(\cdot)$, this implies $x^A = x^C$. If $y_2^C > y_2^A$ then (A.26) can be satisfied only if $V(x^C) > V(x^A)$; that is, when $x^C > x^A$. The final sentence of (i) follows directly from (A.26).

Let the B–D and D–B self-selection constraints be satisfied. Then

$$V(x^B) - h\Big(\frac{y_1^B}{w_H}\Big) - \gamma h\Big(\frac{y_2^B}{w_L}\Big) - V(x^D) + h\Big(\frac{y_1^D}{w_H}\Big) + \gamma h\Big(\frac{y_2^D}{w_L}\Big) \geq 0, \tag{A.29}$$

and

$$V(x^D) - h\Big(\frac{y_1^D}{w_L}\Big) - \gamma h\Big(\frac{y_2^D}{w_H}\Big) - V(x^B) + h\Big(\frac{y_1^B}{w_L}\Big) + \gamma h\Big(\frac{y_2^B}{w_H}\Big) \geq 0. \tag{A.30}$$

Adding (A.29) and (A.30) yields

$$\Big[h\Big(\frac{y_1^B}{w_L}\Big) - h\Big(\frac{y_1^D}{w_L}\Big) + h\Big(\frac{y_1^D}{w_H}\Big) - h\Big(\frac{y_1^B}{w_H}\Big)\Big] + \gamma\Big[h\Big(\frac{y_2^D}{w_L}\Big) - h\Big(\frac{y_2^B}{w_L}\Big) + h\Big(\frac{y_2^B}{w_H}\Big) - h\Big(\frac{y_2^D}{w_H}\Big)\Big] \geq 0. \tag{A.31}$$

Now let $y_1^B < y_1^D$. By Lemma A.2, the first term in square brackets in (A.31) is negative. Then, the second term must be positive. Apply Lemma A.2 to conclude that $y_2^D > y_2^B$. Furthermore, by (A.30), $V(x^D) > V(x^B)$. Hence, $x^D > x^B$.

Suppose $y_1^B = y_1^D$. Then, by Lemma A.2 and (A.31), $y_2^D \geq y_2^B$. If $y_2^D > y_2^B$, then (A.30) can be satisfied only if $x^D > x^B$. If $y_2^D = y_2^B$, then (A.29) and (A.30) imply $x^D = x^B$. □

Now, let μ_{ij} denote the Lagrange multiplier associated with the i–j self–selection constraint, and let λ denote the Lagrange multiplier associated with the constraint (F). The first order necessary conditions for a solution to (PF) can be written as:

$$\Big[Z_{W^i} + \sum_{j\neq i}(\mu_{ij} - \mu_{ji})\Big]V'(x^i) = \lambda\pi^i, \quad \forall i; \tag{A.32}$$

$$\Big[Z_{W^i} + \sum_{j\neq i}\mu_{ij}\Big]\frac{1}{w_1^i}h'(\frac{y_1^i}{w_1^i}) - \sum_{j\neq i}\mu_{ji}\frac{1}{w_1^j}h'(\frac{y_1^i}{w_1^j}) = \lambda\pi^i, \quad \forall i; \tag{A.33}$$

$$\Big[Z_{W^i} + \sum_{j\neq i}\mu_{ij}\Big]\frac{\gamma}{w_2^i}h'(\frac{y_2^i}{w_2^i}) - \sum_{j\neq i}\mu_{ji}\frac{\gamma}{w_2^j}h'(\frac{y_2^i}{w_2^j}) = \lambda\pi^i, \quad \forall i \tag{A.34}$$

Proof of Proposition 5:

Consider the first order necessary conditions for a solution to (PF). Suppose $\lambda = 0$. Then, the first–order necessary condition associated with y_2^A becomes

$$\Big[Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD} - \mu_{BA}\Big]\frac{\gamma}{w_L}h'\Big(\frac{y_2^A}{w_L}\Big) - \Big[\mu_{CA} + \mu_{DA}\Big]\frac{\gamma}{w_H}h'\Big(\frac{y_2^A}{w_H}\Big) = 0. \tag{A.35}$$

Suppose that $\mu_{CA} + \mu_{DA} > 0$. It follows from Lemma A.1 that

$$\Big[Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD} - \mu_{BA} - \mu_{CA} - \mu_{DA}\Big]\frac{\gamma}{w_L}h'\Big(\frac{y_2^A}{w_L}\Big) < 0. \tag{A.36}$$

Because $h(\cdot)$ is increasing,

$$Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD} - \mu_{BA} - \mu_{CA} - \mu_{DA} < 0. \tag{A.37}$$

However, the first–order condition associated with x^A is

$$\Big[Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD} - \mu_{BA} - \mu_{CA} - \mu_{DA}\Big]V'(x^A) = 0. \tag{A.38}$$

Hence, $V'(x^A) = 0$, a contradiction. Thus, $\mu_{CA} + \mu_{DA} = 0$.

A similar argument, using the first–order necessary condition associated with y_1^A, establishes that $\mu_{CA} + \mu_{BA} = 0$. Because all the multipliers are non–negative, $\mu_{CA} = \mu_{DA} = \mu_{BA} = 0$. Then (A.38) becomes

$$\Big[Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD}\Big]V'(x^A) = 0. \tag{A.39}$$

But $Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD} > 0$, since $Z(\cdot)$ is increasing. Thus, $V'(x^A) = 0$, a contradiction. □

Proof of Proposition 6:

The argument given here is essentially due to Brito et al. (1990), except that they analyze a "dual" problem to which any solution to (PF) must also be a solution. The proof below makes reference to (PF) alone.

Suppose, contrary to the statement of the proposition, that at a solution to (PF) there exists families i and j such that $W^i(x^i, y_1^i, y_2^i) = W^i(x^j, y_1^j, y_2^j)$ and

$$y_1^i + y_2^i - x^i < y_1^j + y_2^j - x^j. \tag{A.40}$$

Now construct a new allocation identical to the original solution, except replace the allocation of family i by that of family j (leaving family j with its original allocation). Clearly, $\Psi^{ij} = \Psi^{ji} = 0$ at the new allocation. Furthermore, because $\Psi^{kj} \geq 0$ for all $k \neq i, j$ at the old allocation, $\Psi^{ki} \geq 0$ for all $k \neq i, j$ in the new allocation. In addition, $\Psi^{ik} \geq 0$ for all $k \neq i, j$ at the new allocation because in the original allocation family i was indifferent between its own allocation and that of family j and weakly preferred its own bundle to that of other families. Thus, the new allocation satisfies all self–selection constraints, makes no family worse off than in the original allocation and, by (A.40), slackens the feasibility constraint. Hence, the original allocation could not have been a solution to (PF). □

Proof of Proposition 7:

The proof of statement (i) is presented. The proof of statement (ii) is almost identical.

Pick a family i with $w_1^i = w_L$. Let J be the set of indices not equal to i. Partition J into $\hat{J} := \{j \in J : w_1^j = w_H\}$ and $\tilde{J} := \{j \in J : w_1^j = w_L\}$. The first–order necessary conditions for a solution to (PF) include

$$\Big[Z_{W^i} + \sum_{j\in J}\mu_{ij} - \sum_{j\in\tilde{J}}\mu_{ji}\Big]\frac{1}{w_L}h'\Big(\frac{y_1^i}{w_L}\Big) - \sum_{j\in\hat{J}}\mu_{ji}\frac{1}{w_H}h'\Big(\frac{y_1^i}{w_H}\Big) = \lambda\pi^i, \tag{A.41}$$

and

$$\Big[Z_{W^i} + \sum_{j\in J}(\mu_{ij} - \mu_{ji})\Big]V'(x^i) = \lambda\pi^i. \tag{A.42}$$

If we replace each term in the sum over $\hat{J}$ in (A.41) by $\mu_{ji}\Big(h'\big(\frac{y_1^i}{w_L}\big)/w_L\Big)$, then we may use Lemma A.1 to conclude

$$\Big[Z_{W^i} + \sum_{j\in J}(\mu_{ij} - \mu_{ji})\Big]\frac{1}{w_L}h'\Big(\frac{y_1^i}{w_L}\Big) \le \lambda\pi^i, \tag{A.43}$$

with (A.43) holding with equality if and only if $\mu_{ji} = 0$ for all $j \in \hat{J}$. By Proposition 5, λ is positive. Hence we may divide (A.43) by (A.42) to yield

$$\frac{h'(\frac{y_1^i}{w_L})}{w_L V'(x^i)} \le 1, \tag{A.44}$$

with (A.44) holding with equality if and only if $\mu_{ji} = 0$ for all $j \in \hat{J}$. □

Proof of Proposition 8:

Suppose, by way of contradiction, that there exist two families i and j of different type that receive the same allocation, $(\bar{x}, \bar{y}_1, \bar{y}_2)$, at a solution to (PF). There exist $k \in \{1, 2\}$ such that $w_k^i = w_H$ and $w_k^j = w_L$. Then $MRS^j_{y_k,x}(\bar{x}, \bar{y}_1, \bar{y}_2) > MRS^i_{y_k,x}(\bar{x}, \bar{y}_1, \bar{y}_2)$. That is, person k in family j faces a lower marginal tax rate than person k in family i. But, by Proposition 7, the former faces a non–negative marginal tax rate while the latter faces a non–positive marginal tax rate. A contradiction ensues. □

Proof of Proposition 9

The result is proven for the pair of families (C,D) and for the pair of families (B,D). The proofs for the pairs (A,B), (A,D) and (C,B) are analogous to the demonstration for the pair (C,D).

Let the C–D and D–C constraints bind simultaneously. Then (A.23), (A.24), and (A.25) all hold with equality. By Lemma A.2, $y_1^C = y_1^D$. Thus, (A.24) reduces to

$$V(x^D) - \gamma h\Big(\frac{y_2^D}{w_H}\Big) - V(x^C) + \gamma h\Big(\frac{y_2^C}{w_H}\Big) = 0. \tag{A.45}$$

By definition,

$$\begin{aligned}\Psi^{AD} &= V(x^A) - h\Big(\frac{y_1^A}{w_L}\Big) - \gamma h\Big(\frac{y_2^A}{w_L}\Big) - V(x^D) + h\Big(\frac{y_1^D}{w_L}\Big) + \gamma h\Big(\frac{y_2^D}{w_L}\Big)\\ &= V(x^A) - h\Big(\frac{y_1^A}{w_L}\Big) - \gamma h\Big(\frac{y_2^A}{w_L}\Big) - V(x^D) + h\Big(\frac{y_1^C}{w_L}\Big) + \gamma h\Big(\frac{y_2^D}{w_L}\Big).\end{aligned} \tag{A.46}$$

Adding (A.45) and (A.46) yields

$$\begin{aligned}\Psi^{AD} &= V(x^A) - h\Big(\frac{y_1^A}{w_L}\Big) - \gamma h\Big(\frac{y_2^A}{w_L}\Big) - V(x^C) + h\Big(\frac{y_1^C}{w_L}\Big) + \gamma h\Big(\frac{y_2^C}{w_L}\Big) \\ &\quad + \gamma\Big[h\Big(\frac{y_2^D}{w_L}\Big) - h\Big(\frac{y_2^C}{w_L}\Big) + h\Big(\frac{y_2^C}{w_H}\Big) - h\Big(\frac{y_2^D}{w_H}\Big)\Big] \\ &= \Psi^{AC} + \gamma\Big[h\Big(\frac{y_2^D}{w_L}\Big) - h\Big(\frac{y_2^C}{w_L}\Big) + h\Big(\frac{y_2^C}{w_H}\Big) - h\Big(\frac{y_2^D}{w_H}\Big)\Big].\end{aligned} \tag{A.47}$$

Analogous calculations can be used to show

$$\Psi^{BD} = \Psi^{BC} + \gamma\Big[h\Big(\frac{y_2^D}{w_L}\Big) - h\Big(\frac{y_2^C}{w_L}\Big) + h\Big(\frac{y_2^C}{w_H}\Big) - h\Big(\frac{y_2^D}{w_H}\Big)\Big]. \tag{A.48}$$

Suppose that $y_2^D > y_2^C$. Then Lemma A.2, (A.47), and (A.48) imply that both the A–D and B–D constraints are slack. Applying Proposition 7 yields that $MRS^D_{y_2,x} = 1$ and $MRS^C_{y_2,x} \geq 1$. However, (A.45) implies $x^D > x^C$, so that $MRS^D_{y_2,x} > MRS^C_{y_2,x}$, a contradiction. The case of $y_2^C > y_2^D$ is ruled out by a symmetric argument. Thus, $y_2^C = y_2^D$. In order for (A.45) to hold, it must be the case that families C and D receive the same bundle.

Suppose that the D–B and B–D constraints both bind. Then

$$V(x^D) - h\Big(\frac{y_1^D}{w_L}\Big) - \gamma h\Big(\frac{y_2^D}{w_H}\Big) - V(x^B) + h\Big(\frac{y_1^B}{w_L}\Big) + \gamma h\Big(\frac{y_2^B}{w_H}\Big) = 0, \tag{A.49}$$

and

$$V(x^B) - h\Big(\frac{y_1^B}{w_H}\Big) - \gamma h\Big(\frac{y_2^B}{w_L}\Big) - V(x^D) + h\Big(\frac{y_1^D}{w_H}\Big) + \gamma h\Big(\frac{y_2^D}{w_L}\Big) = 0. \tag{A.50}$$

Adding (A.49) and (A.50) gives

$$h\Big(\frac{y_1^B}{w_L}\Big) - h\Big(\frac{y_1^D}{w_L}\Big) + h\Big(\frac{y_1^D}{w_H}\Big) - h\Big(\frac{y_1^B}{w_H}\Big) = \gamma\Big[h\Big(\frac{y_2^B}{w_L}\Big) - h\Big(\frac{y_2^D}{w_L}\Big) + h\Big(\frac{y_2^D}{w_H}\Big) - h\Big(\frac{y_2^B}{w_H}\Big)\Big]. \tag{A.51}$$

It follows from Lemma A.2 that $y_1^B \geq y_1^D$ if and only if $y_2^B \geq y_2^D$.

Assume that $y_1^B > y_1^D$. Then $y_2^B > y_2^D$, so self–selection requires $x^B > x^D$. Furthermore, $y_2^B/w_L > y_2^D/w_H$. Thus, $MRS^B_{y_2,x} > MRS^D_{y_2,x}$. However, Proposition 7 implies that at a solution to (PF), $MRS^B_{y_2,x} \leq 1$ and $MRS^D_{y_2,x} \geq 1$. A contradiction ensues.

The case of $y_1^D > y_1^B$ can be ruled out by similar arguments, involving $MRS^B_{y_1,x}$ and $MRS^D_{y_1,x}$. Hence, $y_1^B = y_1^D$ and $y_2^B = y_2^D$. Self–selection requires $x^B = x^D$, so that families B and D receive the same bundle. □

Proof of Proposition 10:

For all pairs of families except the pair (C, A) the proposition follows directly from Propositions 8 and 9. Suppose that families C and A are mutually attracted at a solution to (PF). Then, by Proposition 6,

$$y_1^A + y_2^A - x^A = y_1^C + y_2^C - x^C. \tag{A.52}$$

Now construct a new allocation by giving family A the allocation of family C, leaving the bundles received by all other families unchanged. By the argument outlined in the proof of Proposition 6, this new allocation satisfies all of the self–selection constraints. (A.52) ensures that the new allocation is production–feasible. Moreover, all families are indifferent between the new allocation and the original. Because the original allocation solves (PF), the new one does as well. That is, there is a solution to (PF) at which families A and C receive the same allocation, contradicting Proposition 10. □

Proof of Proposition 11:

Suppose, by way of contradiction, that there is a self–selection cycle at a solution to (PF). In view of Proposition 6, all families in the cycle have the same total tax liability. Now select a pair of families, (i, j), from the cycle for which family j attracts family i. Then, we may replace the bundle of family i by the bundle of family j without making any family worse off and without violating the materials balance constraint. This results in a solution to (PF) in which families i and j are pooled, contradicting Proposition 8. □

Proof of Proposition 12:

Suppose that relation (12) of the text holds for family i. If no families are attracted to family i, then the result follows immediately from Proposition 7. By Proposition 6, any families attracted to family i must pay the same total tax bill as family i. Let family j be such a family. Replicating the argument of Proposition 10 allows one to conclude that replacing the allocation of family i by that of family j results in another solution to (PF) that features pooling. This contradicts Proposition 8. □

The remaining proofs use the following Lemmas, which outline monotonicity properties of allocations in which self–selection constraints relating households that differ in both dimensions bind.

Lemma A.3. *Let (SS) hold.*

(i) *If, in addition, the HH–LL self–selection constraint holds with equality, then* $y_1^{LH} \leq y_1^{LL}$ *(with equality only if the HH–LH and LH–LL self–selection constraints also hold with equality) and* $y_2^{HL} \leq y_2^{LL}$ *(with equality only if the HH–HL and HL–LL self–selection constraints also hold with equality).*

(ii) *If, in addition, the LL–HH self–selection constraint holds with equality, then* $y_1^{HH} \leq y_1^{HL}$ *(with equality only if the LL–HL and HL–HH self–selection constraints also hold with equality) and* $y_2^{HH} \leq y_2^{LH}$ *(with equality only if the LL–LH and LH–HH self–selection constraints also hold with equality).*

Proof of Lemma A.3:

The proof of statement (i) is presented. (ii) can be proved in analogous fashion.

Notice that

$$\begin{aligned}\Psi^{CA}-\Psi^{CD}-\Psi^{DA}=&-V(x^A)+h\Big(\frac{y_1^A}{w_H}\Big)+\gamma h\Big(\frac{y_2^A}{w_H}\Big)+V(x^D)-h\Big(\frac{y_1^D}{w_H}\Big)-\gamma h\Big(\frac{y_2^D}{w_H}\Big)\\&-V(x^D)+h\Big(\frac{y_1^D}{w_L}\Big)+\gamma h\Big(\frac{y_2^D}{w_H}\Big)+V(x^A)-h\Big(\frac{y_1^A}{w_L}\Big)-\gamma h\Big(\frac{y_2^A}{w_H}\Big).\end{aligned}\tag{A.53}$$

From (A.53) it follows that

$$\Psi^{CA}-\Psi^{CD}-\Psi^{DA}=h\Big(\frac{y_1^D}{w_L}\Big)-h\Big(\frac{y_1^A}{w_L}\Big)+h\Big(\frac{y_1^A}{w_H}\Big)-h\Big(\frac{y_1^D}{w_H}\Big).\tag{A.54}$$

Let the C–A self–selection constraint bind. Then $\Psi^{CA}=0$, so that the left–hand side of (A.54) must be non–positive when (SS) holds. Apply Lemma A.2 to the right–hand side of (A.54) to conclude that $y_1^D \leq y_1^A$, with $y_1^D = y_1^A$ only if $\Psi^{CD}=\Psi^{DA}=0$.

The argument establishing $y_2^B \leq y_2^A$ is analogous. □

Lemma A.4. *Let (SS) hold.*

(i) *If, in addition, the LH–HL self–selection constraint holds with equality, then* $y_1^{HL} \leq y_1^{HH}$ *(with equality only if the LH–HH and HH–HL self–selection constraints also hold with equality) and* $y_2^{LL} \leq y_2^{HL}$ *(with equality only if the LH–LL and LL–HL self–selection constraints also hold with equality).*

(ii) *If, in addition, the HL–LH self–selection constraint holds with equality, then* $y_1^{LL} \leq y_1^{HL}$ *(with equality only if the HL–LL and LL–LH self–selection constraints also hold with equality) and* $y_2^{LH} \leq y_2^{HH}$ *(with equality only if the HL–HH and HH–LH self–selection constraints also hold with equality).*

Proof of Lemma A.4:

The proof of (i) is presented. Statement (ii) has a similar proof

It can be shown that

$$\Psi^{DB}-\Psi^{DC}-\Psi^{CB}=h\Big(\frac{y_1^B}{w_L}\Big)-h\Big(\frac{y_1^C}{w_L}\Big)+h\Big(\frac{y_1^C}{w_H}\Big)-h\Big(\frac{y_1^B}{w_H}\Big).\tag{A.55}$$

Now let the D–B self–selection constraint bind, so $\Psi^{DB}=0$. By (SS), $\Psi^{DC}\geq 0$ and $\Psi^{CB}\geq 0$. Thus, the left–hand side of (A.55) is non–positive. Apply Lemma A.2 to conclude that $y_1^C \geq y_1^B$, with equality only if $\Psi^{DC}=\Psi^{CB}=0$.

The argument establishing $y_2^B \geq y_2^A$ is analogous. □

Proof of Proposition 13:

I show that the C–A and D–B self–selection constraints cannot bind simultaneously. All other cases have identical proofs.

Suppose, by way of contradiction, that the C–A and D–B self–selection constraints both bind at a solution to (PF). By Lemma A.3, $y_2^B \leq y_2^A$. But, by Lemma A.4, $y_2^A \leq y_2^B$. Hence, $y_2^B = y_2^A$. Now apply Lemmas A.3 and A.4 once more to conclude that the B–A and A–B self–selection constraints must both bind. This contradicts Proposition 10. □

Proof of Proposition 14:

The proof employs the following claim:

Claim 1: If $(C, A) \notin \mathcal{C}$ and if $\{(C, B), (B, A)\} \subseteq \mathcal{C}$, then $(D, A) \in \mathcal{C}$.

Suppose, by way of contradiction, that the D–A constraint is slack. It can be shown that

$$\Psi^{CA} - \Psi^{CB} - \Psi^{BA} = \gamma\Big[h\Big(\frac{y_2^B}{w_L}\Big) - h\Big(\frac{y_2^A}{w_L}\Big) - h\Big(\frac{y_2^B}{w_H}\Big) + h\Big(\frac{y_2^A}{w_H}\Big)\Big]. \tag{A.56}$$

Now, by the hypothesis of the claim, $\Psi^{CB} = \Psi^{BA} = 0$. Thus, by Lemma A.2, the C–A self–selection constraint can be satisfied only if $y_2^B \geq y_2^A$. By Proposition 3, $y_1^B \geq y_1^A$. Then self–selection requires that $x^B \geq x^A$. Hence, $MRS^A_{y_2,x} \leq MRS^B_{y_2,x}$. Moreover, $x^B > x^A$ if either $y_1^B > y_1^A$ or $y_2^B > y_2^A$. When $x^B > x^A$, $MRS^A_{y_2,x} < MRS^B_{y_2,x}$ (because $y_2^B \geq y_2^A$). Hence, if $MRS^A_{y_2,x} = MRS^B_{y_2,x}$, then $y_1^B = y_1^A$ and $y_2^B = y_2^A$. But then self–selection implies $x^B = x^A$, violating the no–pooling condition. Consequently, $MRS^A_{y_2,x} < MRS^B_{y_2,x}$. Because the C–A constraint is slack, Proposition 7 implies $MRS^A_{y_2,x} = 1$. But, by Proposition 7, $MRS^B_{y_2,x} \leq 1$ at a solution to (PF), a contradiction. Hence, the D–A self–selection constraint must bind. This establishes the claim.

Suppose that Assumptions 1 and 2 hold. Consider a pair of families (i, j) with $i >_F j$. By Assumption 1, a sufficiently small transfer of x from family i to family j is socially desirable and does not lead to a violation of the materials balance constraint (by Assumption 2). Moreover, such a redistribution will not lead to a violation of the constraints (SS) as long as, at the before–transfer allocation, the "donor" family is not attracted to any other family and the "recipient" family attracts no other family.

Now, let the C–A constraint be slack. Suppose that the C–D constraint is slack as well. I will show that the C–B constraint must bind. Suppose otherwise. Then a small redistribution of x from family C to family B would not violate the materials balance constraint (by Assumption 2) and would be socially desirable (by Assumption 1). Because family C is not attracted to any family at the original candidate solution, the posited redistribution can be infeasible only if either the D–B constraint or the A–B constraint binds at the candidate solution.

Case 1: D–B binds and A–B binds.

The D–A self selection constraint is slack, for otherwise there would be a cycle, contradicting Proposition 11. By Proposition 10, the B–A self–selection constraint is slack. Then a small redistribution from family C to family A is both feasible and desirable. Thus, the original allocation is not a solution to (PF).

Case 2: D–B binds and A–B does not bind.

By Proposition 10, the B–D self–selection constraint is slack. Suppose the A–D constraint is slack as well. Because the C–D constraint is slack by assumption, a small redistribution of x from family C to family D is feasible and socially desirable, contradicting the optimality of the original allocation. Now suppose the A–D constraint binds. Then the B–A constraint must not bind, for otherwise there is a cycle ($\{(D, B), (B, A), (A, D)\}$). By Proposition 10 the D–A constraint must be slack. Consequently, a small redistribution from family C to family A is feasible and socially desirable. Hence, the candidate solution is not optimal.

Case 3: D–B does not bind and A–B binds.

The B–A constraint does not bind (by Proposition 10). If the D–A constraint is slack, a small redistribution of x from family C to family A is both feasible and socially desirable. If, instead, the D–A constraint is binding, then Proposition 10 ensures that the A–D constraint is slack. Now, a binding B–D constraint produces the cycle $\{(B, D), (D, A), (A, B)\}$. Hence, the B–D constraint is slack. Thus, a small redistribution from family C to family D is feasible and socially desirable, violating the optimality of the original allocation.

Thus, it has been shown that the C–B constraint must bind whenever the C–A and C–D constraints are both slack.

It is also the case that either the D–A constraint or the B–A constraint must bind (possibly both). Otherwise a small redistribution of x from family C to family A would be socially desirable (by Assumption 1), production–feasible (by Assumption 2) and will not lead to a violation of the self–selection constraints (because the C–A constraint is slack). If the D–A constraint binds, we have immediately that $\{(C, B), (D, A)\} \subseteq \mathcal{C}$. Now, suppose that the D–A constraint does not bind. Then the B–A constraint must bind. Employ the claim to conclude that the D–A constraint must bind as well, a contradiction.

Now suppose that the C–D constraint is binding. Again, we must have either the D–A or the B–A constraint (or both) binding. If the B–A constraint binds, we are done. If the D–A constraint binds, an argument symmetric to the one establishing Claim 1 applies, and the proof of statement (i) is complete.

Now suppose that Assumptions 1, 2 and 3 hold and that the C–A and C–D constraints do not bind. Then, by statement (i) of the proposition, the C–B and D–A constraints must bind. Suppose that the B–A constraint does not bind. Then, by Assumption 3, we can transfer sufficiently small and equal amounts of x from C to D and from B to A without affecting the C–B or D–A constraints. By Assumption 2, this redistribution does not affect the production constraint, and it is socially desirable (by Assumption 1). There are now two cases to consider.

Case 1: B–D does not bind.

By Proposition 10 and the D–A constraint binding, the A–D constraint is slack. Thus, no family is attracted to family D. Because family A attracts family D alone, the posited redistribution does not violate (SS).

Case 2: B–D binds.

When $\Psi^{BD} = 0$, it can be shown that

$$\Psi^{CD} = \Psi^{CB} + \gamma\Big[h\Big(\frac{y_2^B}{w_L}\Big) - h\Big(\frac{y_2^B}{w_H}\Big) + h\Big(\frac{y_2^D}{w_H}\Big) - h\Big(\frac{y_2^D}{w_L}\Big)\Big]. \tag{A.57}$$

Given that the C–B constraint binds, Lemma A.2 implies $\Psi^{CD} \geq 0$ if and only if $y_2^B \geq y_2^D$. Because the C–D constraint is slack, $y_2^B > y_2^D$. From Assumption 3 it follows that $MRS^B_{y_2,x} > MRS^D_{y_2,x}$, violating Proposition 7. Hence, this case cannot arise.

Now suppose the C–D constraint binds. We may replicate the argument in the last paragraph of the proof of statement (i) to conclude that the B–A constraint must bind as well. If, in addition, the C–B constraint binds, it follows from Claim 1 that the D–A constraint is binding, establishing statement (ii). If the C–B constraint does not bind, then an argument symmetric to the one used when it was assumed that the C–D constraint is slack can be employed.

Now suppose that the A–C constraint binds. Then, if the C–D constraint binds, $\{(C,D),(D,A),(A,C)\}$ is a self–selection cycle. If the C–D constraint is slack, the C–B constraint must bind. Hence, $\{(C,B),(B,A),(A,C)\}$ is a self–selection cycle. Thus, in both cases, Proposition 11 is violated. □

Proof of Proposition 15:

Only the proof of statement (i) is presented. Proposition 13 implies that we may assume that the C–A and A–C self–selection constraints are slack throughout this proof.

Assume that the D–B constraint binds. Then $\Psi^{CB} = \Psi^{CB} - \Psi^{DB}$, or

$$\begin{aligned}\Psi^{CB} = {} & V(x^C) - h\Big(\frac{y_1^C}{w_H}\Big) - \gamma h\Big(\frac{y_2^C}{w_H}\Big) - V(x^B) + h\Big(\frac{y_1^B}{w_H}\Big) + \gamma h\Big(\frac{y_2^B}{w_H}\Big) \\ & + V(x^B) - h\Big(\frac{y_1^B}{w_L}\Big) - \gamma h\Big(\frac{y_2^B}{w_H}\Big) - V(x^D) + h\Big(\frac{y_1^D}{w_L}\Big) + \gamma h\Big(\frac{y_2^D}{w_H}\Big).\end{aligned} \tag{A.58}$$

Add and subtract $h\Big(\frac{y_1^D}{w_H}\Big)$ to the left–hand side of (A.58) to conclude that

$$\Psi^{CB} = \Psi^{CD} + h\Big(\frac{y_1^D}{w_L}\Big) - h\Big(\frac{y_1^D}{w_H}\Big) + h\Big(\frac{y_1^B}{w_H}\Big) - h\Big(\frac{y_1^B}{w_L}\Big). \tag{A.59}$$

Case 1: $y_1^D > y_1^B$.

Lemma A.2 and (A.59) imply that the C–B constraint must be slack, so, by Proposition 14, the C–D and B–A constraints bind. It follows from Proposition 10 that the D–C and A–B constraints are slack. Thus, by Proposition 7, $MRS^C_{y_1,x} = 1$ and $MRS^B_{y_1,x} \geq 1$. However, by Proposition 3, $y_1^C \geq y_1^D$, so that $y_1^C > y_1^B$. Apply the MRS conditions to conclude that $x^B > x^C$. But, by Proposition 3, $y_2^C \geq y_2^B$. Then the C–B constraint must be violated, a contradiction.

Case 2: $y_1^D = y_1^B$.

Then, by Lemma A.2, (A.59), and Proposition 14, both the C–D and C–B constraints bind. Thus, $MRS^C_{y_1,x} = 1$ and $MRS^B_{y_1,x} \geq 1$. From this, we may replicate the argument of Case 1 to deduce a contradiction.

Case 3: $y_1^B > y_1^D$.

Then, by Lemma A.2 and (A.59), the C–D constraint is slack. Thus, by Proposition 14, both the C–B and D–A constraints bind. Now, it must be the case that $\Psi^{DC} =$

$\Psi^{DC} - \Psi^{DB} + \Psi^{CB}$, or

$$\Psi^{DC} = h\Big(\frac{y_1^C}{w_L}\Big) - h\Big(\frac{y_1^B}{w_L}\Big) + h\Big(\frac{y_1^B}{w_H}\Big) - h\Big(\frac{y_1^C}{w_H}\Big). \tag{A.60}$$

Hence, the D–C constraint is satisfied only if $y_1^C \geq y_1^B$. Suppose $y_1^C > y_1^B$. By Proposition 3, $y_2^C \geq y_2^B$. Hence, the C–B constraint can be satisfied only if $x^C > x^B$. Thus, $MRS^C_{y_1,x} > MRS^B_{y_1,x} \geq 1$, where the second inequality follows from Proposition 7 and D–B binding. Because the A–C constraint is slack, we may apply Proposition 7 to conclude that the D–C constraint must bind. So that, by (A.60), $y_1^C = y_1^B$, contradicting $y_1^C = y_1^B$. Thus, we may conclude that $y_1^C = y_1^B$ and, in view of (A.60), that the D–C constraint binds.

The result follows from Case 3 being the only possibility that is not contradictory. □

Proof of Proposition 16:

In view of Proposition 13, this proposition follows immediately from Propositions 14 and 15. □

Proof of Proposition 17:

I give the proof of the first inequality. The second is proven in a symmetric fashion.

I first show that if $MRS^B_{y_1,x} \geq MRS^C_{y_1,x}$ at a solution to (PF), then $y_1^B \geq y_1^C$. Suppose, to the contrary, that $MRS^B_{y_1,x} \geq MRS^C_{y_1,x}$ and $y_1^B < y_1^C$. Then, $x^B > x^C$. But, by Proposition 3, $y_2^C \geq y_2^B$. Thus, the C–B constraint is violated.

Now suppose $MRS^B_{y_1,x} < MRS^C_{y_1,x}$. (Otherwise, we are done.) By Proposition 7, $MRS^B_{y_1,x} \geq 1$. Thus, $MRS^C_{y_1,x} > 1$. Hence, by Proposition 7, it must be the case that either the A–C constraint binds or the D–C constraint binds (or both). But, by Lemma A.3, when the A–C constraint binds, $y_1^B \geq y_1^C$.

It remains to consider the case of D–C binding. Two possibilities need to be considered.

Case 1: C–A does not bind.

By Proposition 10, the C–D constraint is slack. In view of Proposition 14, this implies that the C–B constraint binds. By the argument used to establish (A.60), we know that

$$\Psi^{DC} - \Psi^{DB} + \Psi^{CB} = h\Big(\frac{y_1^C}{w_L}\Big) - h\Big(\frac{y_1^B}{w_L}\Big) + h\Big(\frac{y_1^B}{w_H}\Big) - h\Big(\frac{y_1^C}{w_H}\Big). \tag{A.61}$$

Because the C–B and D–C constraints bind, (A.61) and Lemma A.2 imply that $\Psi^{DB} \geq 0$ if and only if $y_1^B \geq y_1^C$.

Case 2: C–A binds.

It can be shown that

$$\Psi^{CA} + \Psi^{DC} = \Psi^{DA} + h\Big(\frac{y_1^C}{w_L}\Big) - h\Big(\frac{y_1^A}{w_L}\Big) + h\Big(\frac{y_1^A}{w_H}\Big) - h\Big(\frac{y_1^C}{w_H}\Big). \tag{A.62}$$

Because both the C–A and D–C constraints are assumed to bind, the left–hand side of (A.62) is zero. Apply Lemma A.2 and (A.62) to conclude that $\Psi^{DA} \geq 0$ if and only if $y_1^A \geq y_1^C$. However, by Proposition 3, $y_1^B \geq y_1^A$. Thus, $y_1^B \geq y_1^C$. □

References

Armstrong, Mark (1996). "Multiproduct Nonlinear Pricing." *Econometrica* 64, 51–75.

Armstrong, Mark and Jean-Charles Rochet (1999). "Multi–dimensional Screening: A User's Guide." *European Economic Review* 43, 959–979.

Basov, Suren (2005). Multidimensional Screening. Berlin: Springer.

Besley, Timothy and Stephen Coate (1995). "The Design of Income Maintenance Programmes." *Review of Economic Studies* 62, 187–221.

Brito, Dagobert, Jonathan Hamilton, Steven Slutsky, and Joseph Stiglitz (1990). "Pareto Efficient Tax Structures." *Oxford Economic Papers* 42, 61–77.

Chambers, Robert (1989). "Concentrated Objective Functions for Nonlinear Taxation Models." *Journal of Public Economics* 39, 365–375.

Dana, James (1993). "The Organization and Scope of Agents: Regulating Multiproduct Industries." *Journal of Economic Theory* 59, 365–375.

Dixit, Avinash and Jesus Seade (1979). "Utilitarian Versus Egalitarian Redistributions." *Economics Letters* 4, 121–124.

Guesnerie, Roger (1981). "On Taxation and Incentives: Further Remarks on the Limits to Redistribution." Discussion paper 89, University of Bonn.

Guesnerie, Roger and Jesus Seade (1982). "Nonlinear Pricing in a Finite Economy." *Journal of Public Economics* 17, 157–179.

Kanbur, Ravi, Michael Keen, and Matti Tuomala (1994). "Optimal Nonlinear Income Taxation for the Alleviation of Income Poverty." *European Economic Review* 38, 1613–1632.

Matthews, Steven and John Moore (1987). "Monopoly Provision of Quality and Warranties: An Exploration in the Theory of Multi–dimensional Screening." *Econometrica* 55, 441–467.

Mirrlees, James A. (1971). "An Exploration in the Theory of Optimum Income Taxation." *Review of Economic Studies* 38, 175–208.

Mirrlees, James A. (1976). "Optimal Tax Theory: A Synthesis." *Journal of Public Economics* 6, 327–358.

Rochet, Jean-Charles (1995). "Ironing, Sweeping and Multidimensional Screening." Cahier de Recherche 95.11.374, GREMAQ, Unversité des Sciences Sociales, Toulouse.

Röell, Ailsa A. (1985). "A Note on the Marginal Tax Rate in a Finite Economy." *Journal of Public Economics* 28, 267–272.

Schroyen, Fred (2003). "Redistributive Taxation and the Household: The Case of Individual Filings." *Journal of Public Economics* 87, 2527–2547.

Seade, Jesus (1977). "On the Shape of Optimal Tax Schedules." *Journal of Public Economics* 7, 203–236.

Seade, Jesus (1979). "On the Optimal Taxation of Multidimensional Consumers." Working Paper 79–21, CEPREMAP.

Seade, Jesus (1980). "Optimal Nonlinear Policies for Non–Utilitarian Motives." In David Collard, Richard Lecomber, and Martin Slater (eds.), Income Distribution: The Limits to Redistribution, 53–68. Bristol: John Wright and Sons.

Stiglitz, Joseph E. (1982). "Self-Selection and Pareto Efficient Taxation." *Journal of Public Economics* 17, 213–240.

van Egteren, Henry (1996). "Regulating an Externality–Generating Public Utility: A Multidimensional Screening Approach." *European Economic Review* 40, 1773–1797.

Weymark, John A. (1986a). "Bunching Properties of Optimal Nonlinear Income Taxes." *Social Choice and Welfare* 3, 213–232.

Weymark, John A. (1986b). "A Reduced-Form Optimal Nonlinear Income Tax Problem." *Journal of Public Economics* 30, 199–217.

Weymark, John A. (1987). "Comparative Static Properties of Optimal Nonlinear Income Taxes." *Econometrica* 55, 1165–1185.

Wilson, Robert (1993). Nonlinear Pricing. Oxford: Oxford University Press.