Some Marginalist Intuition Concerning the Optimal Commodity Tax Problem

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Abstract:
This paper offers a simple intuition that can be exploited to derive and to help interpret some canonical results in the theory of optimal commodity taxation. Specifically, it develops and explores the principle that the marginal social welfare loss per last unit of tax revenue generated be equalized across tax instruments. A simple two-consumer, two-taxed-commodity economy is used to explore how this intuition can be used to derive the famous inverse elasticity rule and the modifications and extensions needed to account for the redistributive effects of commodity taxes.

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1. Introduction

One of the classical, and expositionally most difficult, problems of public finance is the task of choosing the "best" combination of commodity taxes. In the text-book formulation of this problem,[1] the taxation authority may levy per unit taxes on each good, allowing rates of taxation to vary among commodities. Accordingly, the possible justifications for non-uniform taxation need to be clearly set out. The two most celebrated reasons for deviations from uniform taxation are encapsulated in the following principles.

- *Goods with inelastic demand ought to be taxed more heavily.* This is a classic from introductory textbooks, in which triangles in supply–demand diagrams are used to illustrate the excess burden of taxation. But this is a partial equilibrium story. The goal of the optimal commodity tax literature is to examine how well this intuition carries over into a multi-good setting.

- *Luxuries (goods with high income elasticity of demand) should be taxed more heavily.* Judicious choice of differential rates can be used to adjust the burden of taxation so that it falls more on the rich than on the poor. The setting of this argument assumes that other forms of redistributive taxes (for example, income taxes) are not sufficient to produce the favored distribution of well-being.[2]

Because commodity taxes do not depend directly on the identity of the consumer, their redistributive effects are, of necessity, indirect. One justification of this anonymous taxation is that the government simply cannot observe how many chocolate bars, root vegetables and hours of legal advice every consumer enjoys. The best it can hope to do is to ask providers of goods and services to remit taxes on the basis of their aggregate sales. A tax per unit can be computed quite easily on this basis. A second justification for linear taxation is that consumers may be able to retrade certain goods, so that any differences in the ticket prices that different consumers may face do not translate into differences in prices actually paid, once reallocation through informal markets takes place. Thus, the government can do no better than to devise taxes in such a way that all consumers face the same consumer prices in the first place.

General principles of commodity taxation are difficult to come by and notoriously difficult to explain to students in a first course in public economics. This note offers a relatively accessible introduction to the inverse elasticity principle based entirely on a simple marginalist principle: the marginal loss in social welfare per last unit of tax revenue raised should be equal across tax sources. Straightforward arguments are provided that justify statements concerning the scope of taxation according to the inverse elasticity rule. These arguments are not too mathematically demanding, as they build on marginalist intuition using only simple algebra.

2. The Principle

Consider a world in which there are three goods and services, a numeraire good that remains untaxed, and two taxed commodities, xylophones (x for short) and yams (y for short).[3] Suppose xylophones and yams are produced with constant returns to scale and have constant producer prices p_x and p_y, respectively. Consumers pay $p_x + t_x$ and $p_y + t_y$, where t_i denotes the tax per unit on commodity i. The market demands for each taxed commodity depends, in general, on the consumer prices of every good. Typically, an increase in the tax on xylophones will decrease the quantity of xylophones demanded. If xylophones and yams are substitutes, an increase in t_x will also increase the demand for yams

The government sets taxes in order to raise a fixed amount of revenue, R. That is, the government faces a revenue constraint of the form $t_x x + t_y y = R$. There are many combinations of the two tax rates that give rise to the same revenue; thus, it is meaningful to talk about trading off one source of revenue for another. To better understand this tradeoff, it is helpful to first analyze the change in revenue due to a change in a single tax, say t_x. If the demand for (the taxed) goods were insensitive to taxes, then the government would receive exactly $x\Delta t_x$ units of additional tax revenue if it were to increase t_x by a small amount Δt_x, where x is the (pre-existing) overall demand for xylophones. This is because each of the x units of the good is subject to the extra Δt_x in taxation. This direct effect of the tax increase is supplemented by indirect effects that result from tax-induced changes in behavior. The increase in the price of xylophones causes the quantity demanded to change by some negative amount, say Δx. Tax receipts fall by the tax rate multiplied by this change in quantity demand. If the change in t_x also induces a change in the demand for yams, then revenue from the tax on yams is also affected. The net change in revenue is $t_y \Delta y$, where Δy is the change in the demand for yams due to the change in the price of xylophones. Δy is positive if the two goods are substitutes, negative if the goods are complements. In summary, the total change in government revenue per small change, Δt_x, of the tax on xylophones (denoted MR_{t_x} for marginal revenue) can be expressed as follows:

$$MR_{t_x} = \frac{\Delta \text{Revenue}}{\Delta t_x} = x + t_x \frac{\Delta x}{\Delta t_x} + t_y \frac{\Delta y}{\Delta t_x}.$$

Similar logic allows one to conclude that the additional revenue from the tax on yams is:

$$MR_{t_y} = \frac{\Delta \text{Revenue}}{\Delta t_y} = y + t_x \frac{\Delta x}{\Delta t_y} + t_y \frac{\Delta y}{\Delta t_y}.$$

Despite the possibility that the indirect revenue effects may be larger than the direct effects, it is helpful to maintain the assumption that both MR_{t_x} and MR_{t_y} are positive. Indeed, should MR_{t_y}, say, be negative, a reduction in the tax on yams would actually increase tax revenue. This could not possibly be the best state of affairs, as the taxation authority would do better by reducing the levy on yams.

The taxation authority is also concerned about *how* it raises the required revenue. The goal of the optimal taxation problem is to understand how the government can secure its revenue in a way that is least burdensome to citizens. Equivalently, we can think of trying to find a way to assign taxes so as to maximize overall social well-being. Because commodity taxes lead to an increase in consumer prices, they reduce purchasing power and consumer well-being. The amount by which a small increase in a tax reduces overall well-being is called the marginal social loss (MSL for short) due to the tax. Typically, the marginal social loss differs between the two taxes, which motivates the following definitions.

$$MSL_{t_x} = -\frac{\Delta(\text{Social well-being})}{\Delta t_x}, \quad \text{and} \quad MSL_{t_y} = -\frac{\Delta(\text{Social well-being})}{\Delta t_y}.$$

At this stage, it is not important to quantify the marginal social losses due to commodity taxes, only to recognize that they exist and are positive quantities.

It is now possible to state and interpret a general principle of commodity tax design. Social well-being is maximized when the marginal social loss per last unit of tax revenue is the same for all taxes. In the notation developed here:

$$\frac{MSL_{t_x}}{MR_{t_x}} = \frac{MSL_{t_y}}{MR_{t_y}}.$$

The intuition for this condition can be discovered by showing that the taxation authority can rearrange taxes so as to improve social well-being whenever this condition is violated. Suppose, for the sake of argument that the tax on xylophones has a lower marginal social loss per unit of tax revenue than the tax on yams. Consider a small (one-unit) increase in the tax on xylophones. This increases revenue by MR_{t_x}, while reducing well-being by MSL_{t_x}. The increased revenue from the tax on xylophones allows the government to reduce the tax on yams. If it reduces the tax by $\frac{MR_{t_x}}{MR_{t_y}}$ units, then total revenue does not change. This reduction in the tax on yams brings about an increase in well-being. The amount of this increase is the change in the tax rate multiplied by the marginal social loss per unit of change in the tax rate, namely $\frac{MR_{t_x}}{MR_{t_y}} \times MSL_{t_y}$. The combined effect on well-being is given by

$$\Delta(\text{Well-being}) = \frac{MR_{t_x}}{MR_{t_y}} \times MSL_{t_y} - MSL_{t_x}.$$

Under the maintained assumption that MR_{t_x} is positive, the sign of the change in well-being is the same as the sign of

$$\frac{MSL_{t_y}}{MR_{t_y}} - \frac{MSL_{t_x}}{MR_{t_x}},$$

which is positive whenever the marginal social loss per unit of tax revenue is lower for the tax on xylophones than it is for the tax on yams. Thus, it is possible to reallocate the tax burden so that it increases well-being while collecting the same amount of revenue. Hence, the original tax system cannot be the best one. The desired reallocation makes intuitive sense. Increase the tax that is less burdensome (per unit of revenue) at the margin, while reducing the more burdensome tax. This reallocation process should stop only when the marginal social loss per unit of tax revenue is equalized across the two tax sources.

3. The Inverse Elasticity Rule and Its Extensions

In order to translate the abstract principle of the previous section into more accessible and familiar intuition about demand elasticities, one needs to look at the marginal impact of a tax increase on social welfare. The most straightforward case to consider is when there is only one consumer. In this case, social well-being can reasonably be identified with the utility of the single consumer. Because producer prices are fixed, a small increase in the tax on xylophones induces an equal increase in the consumer price of xylophones. This price increase has both income and substitution effects on the behavior of the consumer. Because the substitution effect is the (theoretically constructed) change in demand that would result if relative prices change while maintaining the same level of consumer satisfaction, it is not a measure of the loss of consumer well-being owing to the tax. The income effect measures the loss in well-being. An increase in the tax on xylophones in the amount Δt_x increases the income required to consume the before-increase quantity of xylophones by exactly $x\Delta t_x$ units. Thus, all else equal, the increase in the tax takes $x\Delta t_x$ units of income out of the consumer's pocket. The value of this lost expenditure in utility terms is just the income loss multiplied by the marginal utility of income (MUI for short). This is the content of Roy's Identity,[4] which states:

$$\frac{\Delta(\text{Individual utility})}{\Delta t_x} = -(\text{MUI}) \times (\text{Demand for } x).$$

A good way to illustrate that utility loss is proportional to demand is to think of the effects of a tax increase on a good that one does not consume. It is easy to agree that there is no direct impact on consumer utility in this case.

Recognizing that completely analogous arguments justify a similar expression for the utility loss due to a small increase in t_y, the general principle of optimal commodity taxation reduces to:

$$\frac{(\text{MUI}) \times (\text{demand for } x)}{x + t_x \frac{\Delta x}{\Delta t_x} + t_y \frac{\Delta y}{\Delta t_x}} = \frac{(\text{MUI}) \times (\text{demand for } y)}{y + t_x \frac{\Delta x}{\Delta t_y} + t_y \frac{\Delta y}{\Delta t_y}};$$

$$\frac{x}{x + t_x \frac{\Delta x}{\Delta t_x} + t_y \frac{\Delta y}{\Delta t_x}} = \frac{y}{y + t_x \frac{\Delta x}{\Delta t_y} + t_y \frac{\Delta y}{\Delta t_y}}.$$

This condition is expressed entirely in terms of (potentially) observable variables, namely the quantity of each good consumed and the change in ordinary demands due to changes in prices.

Some insight into the workings of the optimal tax rule can be gained by looking at the (admittedly unrealistic) special case of no demand responses to either tax. In this circumstance, the optimality conditions would reduce to

$$\frac{x}{x} = \frac{y}{y}.$$

Thus, the optimality condition holds for any combination of taxes. This indicates that all combinations of taxes that raise the required revenue are equally good. Neither tax has a deadweight loss associated with it, so there is no basis on which to base a preference for one revenue source over the other.

More definite statements about the optimal tax rates can be derived when the goods in question are neither complements nor substitutes. In this circumstance, the following chain of reasoning holds:

$$\frac{x + t_x \frac{\Delta x}{\Delta t_x}}{x} = \frac{y + t_y \frac{\Delta y}{\Delta t_y}}{y};$$

$$1 + \frac{t_x}{x} \frac{\Delta x}{\Delta t_x} = 1 + \frac{t_y}{y} \frac{\Delta y}{\Delta t_y};$$

$$\frac{t_x}{t_y} = \frac{x}{\frac{\Delta x}{\Delta t_x}} \frac{\frac{\Delta y}{\Delta t_y}}{y};$$

$$\frac{t_x}{t_y} = \frac{\epsilon_y}{\epsilon_x},$$

where ϵ_i is the tax (price) elasticity of demand for good i. The final line of this chain contains the canonical result that relative taxes vary in inverse proportion to the relative elasticity of demand. The more price elastic the demand for yams, the lower the extra revenue that is collected when the tax on yams is increased (because consumers shift consumption to other goods) and the lower the benefit to the government in terms of additional revenue from increases in the tax on yams. In this circumstance, it makes sense to tax xylophones more heavily.

3.1. Redistribution

The inverse elasticity rule may require modification in multiple-consumer settings, even when the goods are neither complements nor substitutes. Suppose that there are two consumers, A and B, each of whom consume xylophones and yams. Denote individual demands for the goods by x^A, x^B, y^A, and y^B, and reserve the symbols x and y for aggregate demands. Suppose that social well-being is a weighted sum of the utilities of these two consumers. Then the marginal social loss of the tax on xylophones is the weighted sum of the losses felt by the two consumers. The weights, typically denoted by β^A and β^B, are known as the social marginal valuations of income. For each consumer, the weight is made up of up of two components, the individual's own valuation of extra income and the social valuation of that individual's utility. Recalling that, for each consumer, the loss in purchasing power per unit of tax increase is equal to the quantity demanded, the marginal social loss due to the tax on xylophones is calculated to be:

$$MSL_{t_x} = \beta^A x^A + \beta^B x^B.$$

When the goods are neither substitutes nor complements, marginal welfare loss per last unit of tax revenue is equalized across the two taxes when:

$$\frac{\beta^A x^A + \beta^B x^B}{x + t_x \frac{\Delta x}{\Delta t_x}} = \frac{\beta^A y^A + \beta^B y^B}{y + t_y \frac{\Delta y}{\Delta t_y}}.$$

Of special note in this expression is that the relevant changes in x and y are changes in aggregate demands. Taxes influence tax receipts according to their effects on aggregate demand.

If the optimal commodity tax regime deviates from the inverse elasticity rule, it must do so in order to redistribute the tax burden between the two consumers. For the sake of argument, assume that consumer A has a higher social marginal valuation of income. This might be because that person is deemed to be more needy or more deserving of extra income at the margin. Also assume that this person consumes relatively more yams than xylophones; that is, assume the following:

$$\beta^A > \beta^B, \quad \text{and} \quad \frac{y^A}{x^A} > \frac{y^B}{x^B}. \tag{$\clubsuit$}$$

Because both consumers face the same prices, this means that consumer A spends a larger share of income on yams that consumer B does. If the difference in marginal social valuation of income is reflective of A having a lower income than B, then we are considering a situation in which yams are more of a necessity than xylophones. As shown in the Appendix, a direct implication of these assumptions is that

$$\frac{\beta^A y^A + \beta^B y^B}{\beta^A x^A + \beta^B x^B} > \frac{y^A + y^B}{x^A + x^B} = \frac{y}{x}.$$

This says that the ratio of the loss in social well-being due to an increase in the tax on yams to the loss in social well-being due to an increase in the tax on xylophones is greater than the ratio of aggregate demands for the two commodities. Because yams are a relatively important part of the consumption of the consumer with the larger social marginal utility of income, the loss in purchasing power caused by the tax on yams is viewed as socially more damaging than the loss brought about by the tax on xylophones.

Recalling the assumption that aggregate demands display neither complementarity nor substitutability, we can conclude that the optimal tax regime features

$$\frac{y + t_y \frac{\Delta y}{\Delta t_y}}{x + t_x \frac{\Delta x}{\Delta t_x}} = \frac{\beta^A y^A + \beta^B y^B}{\beta^A x^A + \beta^B x^B} > \frac{y}{x}.$$

Hence, the following chain of reasoning holds:

$$\frac{y + t_y \frac{\Delta y}{\Delta t_y}}{y} > \frac{x + t_x \frac{\Delta x}{\Delta t_x}}{x};$$

$$1 + t_y \epsilon_y > 1 + t_x \epsilon x;$$

$$t_y \epsilon_y > t_x \epsilon_x.$$

If it is the case that the tax rate on each good is positive, we may use the fact that the demand elasticities are negative quantities to conclude that

$$\frac{t_y}{t_x} < \frac{\epsilon_x}{\epsilon_y}.$$

The foregoing analysis illustrates an important feature of the optimal tax scheme: The ratio of the tax on yams to the tax on xylophones is smaller than is called for by the aggregate inverse elasticity rule. This is because yams are consumed relatively more by individual A, who is deemed to be more deserving at the margin. It is interesting to note that if *either* of the inequalities in (♣) is replaced with an equality, then equality holds throughout the all of the preceding expressions. In this case, the argument confirms that the inverse elasticity rule continues to hold when xylophones and yams are neither substitutes nor complements. If $\frac{y^A}{x^A} = \frac{y^B}{x^B}$ then the relative burden of the two commodity taxes is the same for both consumers. Hence, there is no scope for using commodity taxes for redistributive purposes. If $\beta^A = \beta^B$, the taxation authority has no desire to redistribute income at the margin, so no desire to deviate from the inverse elasticity rule in order to bring about a different distribution of well-being.

The question of whether deviations from the inverse elasticity rule are quantitatively significant depend on the extent to which the β^A differs from β^B and the extent to which $\frac{y^A}{x^A}$ differs from $\frac{y^B}{x^B}$. Deviations from the inverse elasticity rule are significant only if consumption behavior differs markedly among individuals *and* the taxation authority has a strong motivation for redistribution at the margin (sizeably different βs). Deviations from the inverse elasticity rule, therefore, require both positive and a normative justification.

4. Conclusion

This note has presented a non-standard exposition of the optimal commodity tax problem, aimed at illustrating the inverse elasticity principle and deviations from this principle on redistributive grounds. The exposition has the advantage of making explicit the revenue constraint faced by the taxation authority in a way that supply-demand diagrams cannot. Yet, the concepts used are accessible to students familiar with introductory economics and basic algebra. The general principle of equal welfare loss per last unit of tax revenue is a natural analog of the condition that describes optimal consumer decisions, namely equal marginal utility per last dollar spent on each commodity. This principle can be applied to taxes other than commodity taxes, and to the problem of the mix between commodity taxes and other revenue instruments.

The exposition given here makes no use of more advanced concepts like compensated demands (except for a passing reference to substitution effects). This is not to minimize the importance of tax-induced distortions to consumption behavior. Behavioral responses are front-and-center in any discussion of the inverse elasticity rule. In this version of the story, however, changes in behavior have the effect of reducing tax collections per unit of loss in social welfare. This is just the flip-side of the more canonical view of behavioral responses leading to increased burden per unit of tax collections. The current exposition has the advantage of emphasizing the observed behavioral effects and their real effect on tax collections. It does so without introducing hypothetical lump-sum compensation for taxes paid, the very instrument whose infeasibilty renders optimal taxation problematic in the first place.

Notes

[1]Stiglitz (2000) offers two accounts of the famous inverse elasticity formula, which are related to, but slightly different from, the current discussion. For a graduate level textbook account, see Myles (1995).

[2]J.S. Mill (1894, p.476) argues that charging the same duty on all commodities constitutes "a flagrant injustice to the poorer class of contributors, unless compensated by the existence of other taxes from which, as the present income-tax, they are altogether exempt." Exemption is a type of differential rating.

[3]All goods are assumed to be produced under constant returns to scale, so that the price of the untaxed commodity remains fixed.

[4]Deaton and Muellbauer (1980) provide a standard derivation of this result.

References

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Appendix

This appendix demonstrates and elaborates on the following claim:

Claim 1 For any positive numbers β^A, x^A, y^A, β^B, x^B, and y^B: $\beta^A > \beta^B$ and $\frac{y^A}{x^A} > \frac{y^B}{x^B}$ implies

$$\frac{\beta^A y^A + \beta^B y^B}{\beta^A x^A + \beta^B x^B} > \frac{y^A + y^B}{x^A + x^B}.$$

An algebraic demonstration is contained in the following two-column proof:

Statement	Reason
1. $\dfrac{y^A}{x^A} > \dfrac{y^B}{x^B}$	1. Assumption.
2. $x^B y^A > x^A y^B$	2. Algebra.
3. $(\beta^A - \beta^B) x^B y^A > (\beta^A - \beta^B) x^A y^B$	3. By assumption, $\beta^A > \beta^B$.
4. $\beta^A y^A x^B + \beta^B y^B x^A > \beta^A x^A y^B + \beta^B x^B y^A$	4. Algebra.
5. $\beta^A y^A x^B + \beta^B y^B x^A + \beta^A y^A x^A + \beta^B y^B x^B >$ $\beta^A x^A y^B + \beta^B x^B y^A + \beta^A y^A x^A + \beta^B y^B x^B$	5. Algebra: add $\beta^A y^A x^A + \beta^B y^B x^B$ to both sides.
6. $(\beta^A y^A + \beta^B y^B)(x^A + x^B) > (\beta^A x^A + \beta^B x^B)(y^A + y^B)$	6. Algebra: factoring polynomials.
7. $\dfrac{\beta^A y^A + \beta^B y^B}{\beta^A x^A + \beta^B x^B} > \dfrac{y^A + y^B}{x^A + x^B}$	7. Algebra.

A geometric interpretation is available if we plot the points (x^A, y^A) and (x^B, y^B) in the xy-plane. The condition$\frac{y^A}{x^A} > \frac{y^B}{x^B}$ implies that the line segment connecting (x^A, y^A) to the origin is steeper than the line segment connecting (x^B, y^B) to the origin. These two line segments determine a parallelogram, the diagonal of which is the line segment connecting the aggregate demand bundle (x, y) to the origin. The solid lines in Figure 1 are descriptive of this arrangement. The dashed lines of Figure 1 demonstrate the effect of scaling up the commodity bundle of individual A to $(\bar{x}^A, \bar{y}^A) = (\beta^A x^A, \beta^A y^A)$, without rescaling the commodity bundle to person B. This is a way to represent the added weight that person A's commodity bundle (hence the welfare loss due to taxation) receives in social evaluation. The line connecting the new (weighted) aggregate demand bundle, $(\bar{x}, \bar{y})$ to the origin is steeper than the line connecting (x, y) to the origin. Thus, the ratio of the welfare loss due to the tax on yams to the welfare loss due to the tax on xylophones is greater than the ratio of the aggregate demands for the two commodities. This is the economic content of Claim 1.

Figure 1 About Here

If $\frac{y^A}{x^A} = \frac{y^B}{x^B}$, then all the line segments in Figure 1 lie along the same ray from the origin. Thus, the ratio of the welfare loss due to the tax on yams to the welfare loss due to the tax on xylophones coincides the ratio of the aggregate demands for the two commodities, and an aggregate form of the inverse elasticity rule can be recovered. If $\beta^A = \beta^B$, then both commodity bundles must be rescaled by the same amount, as demonstrated in Figure 2. The dashed lines represent a radial expansion of the solid lines, emphasising that the slope of the line segment connecting $(\bar{x}, \bar{y})$ to the origin is identical to the slope of the line segment connecting (x, y) to the origin. Again, an aggregate form of the inverse elasticity rule can be derived.

Figure 2 About Here

Figure 1. A comparison of relative welfare loss versus the ratio of aggregate demands.

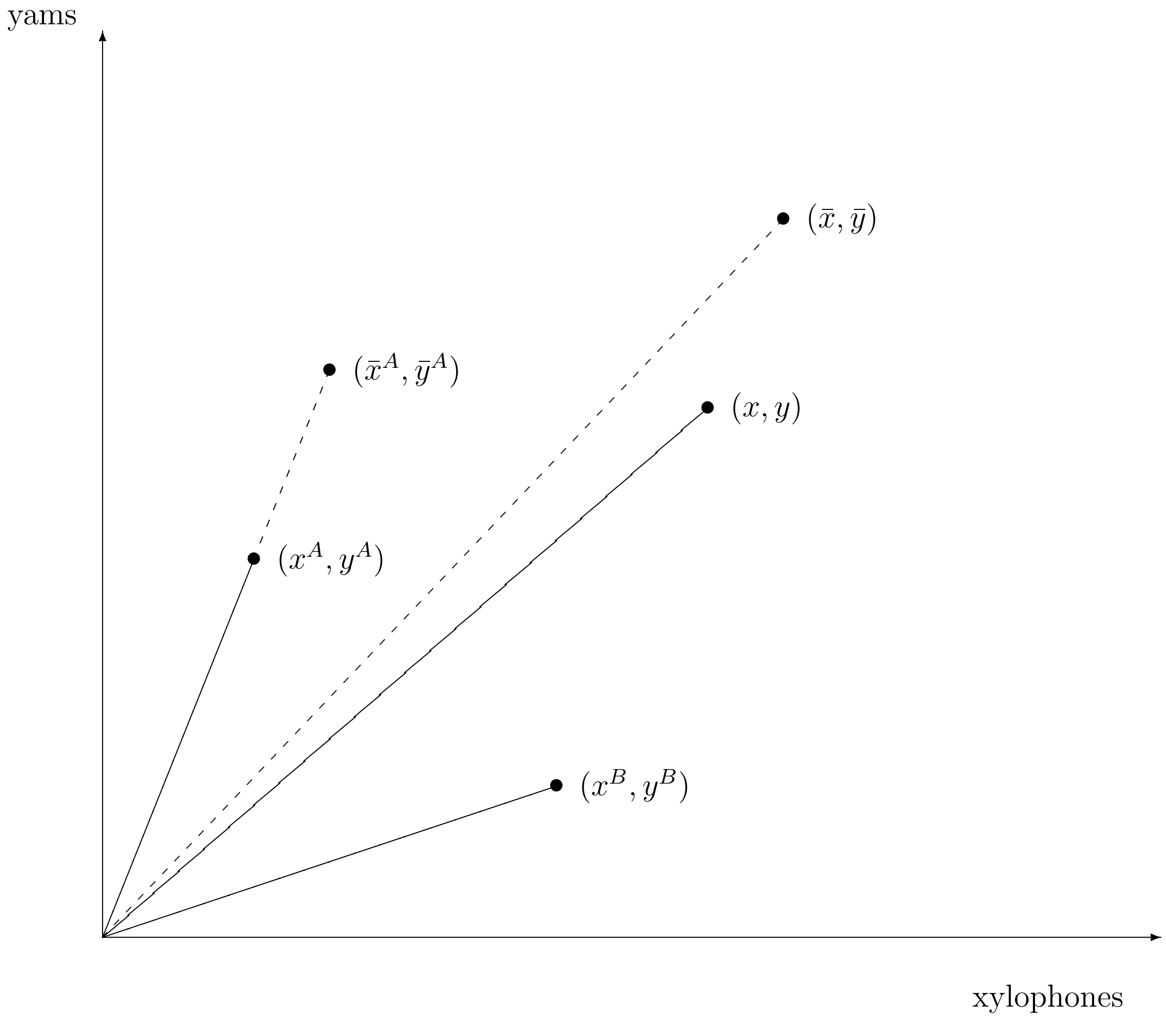

Figure 2. Relative welfare loss and the ratio of aggregate demands when marginal social valuations of income are equal.