SOLUTIONS TO CHAPTER 2 EXERCISES IN “HOPF ALGEBRAS” BY M.E. SWEEDLER

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Abstract. This is a solution set for all the exercises in the second chapter of “Hopf Algebras” by M.E. Sweedler, W.A. Benjamin (1969).

Preliminaries

The same conventions hold as in Chapter 1; k denotes a field of characteristic zero and all maps are k-linear. In addition, if V is a vector space, S a subset of V, then by $S^{\perp} \subseteq V^*$ we mean $\{v^* \in V^* \mid v^*(S) = 0\}$, and we note that $S^{\perp\perp} = S$.

We will appeal to the following propositions and theorems from the text in some of the exercises.

Proposition 1.4.5. Let C be a coalgebra.

(a) If $J \subseteq C$ is a right (left) coideal, then $J^{\perp}$ is a right (left) ideal in C^*.
(b) If I is a right (left) ideal in C^*, then $I^{\perp}$ is a right (left) coideal in C.

Proposition 1.4.6. Let C be a coalgebra.

(a) If $J \subseteq C$ is a coideal, then $J^{\perp}$ is a subalgebra of C^*.
(b) If I is a subalgebra of C^*, then $I^{\perp}$ is a coideal in C.

Theorem 2.1.3(b). If C is a coalgebra, M a rational C^*-module, then every cyclic submodule of M is finite dimensional.

Theorem 2.2.1. (Fundamental Theorem of Coalgebras). Let C be a coalgebra, $c \in C$. The subcoalgebra generated by c is finite dimensional.

Proposition A.5. Suppose U is a vector space with subspaces $\{V_\alpha\}$, V, W. Let $\{X_\alpha\}$ be subspaces of U^*. Then

(a) $\bigcap_\alpha V_\alpha^{\perp} = (\sum_\alpha V_\alpha)^{\perp}$ in U^*.
(b) $V^{\perp} + W^{\perp} = (V \cap W)^{\perp}$ in U^*.
(c) $\bigcap_\alpha X_\alpha^{\perp} = (\sum_\alpha X_\alpha)^{\perp}$ in U.

This was part of two summer projects at Mount Allison University in 2008 and 2010, supervised by M. Beattie. Funding for the first author was provided through M. Beattie’s NSERC Discovery Grant, while the second author was funded through an NSERC USRA.

§2.0 Comodules.

Exercise. (p.33)

Let M be a right C-comodule with comodule structure map $\omega\colon M \to M \otimes C$, $m \mapsto \sum_m m_0 \otimes m_1$. Show $\sum_m m_0 \otimes m_1 \otimes \ldots \otimes \Delta(m_i) \otimes \ldots \otimes m_{n-1} = \sum_m m_0 \otimes m_1 \otimes \ldots \otimes m_n$ for any $i = 1, \ldots, n-1$.

Proof: We inductively define $\omega_{n+1} = \big(\omega \otimes I^n\big)\omega_n$, so that we have

$$\begin{aligned}\sum_m m_0 \otimes m_1 \otimes \ldots \otimes m_n &= \left(\sum_m \omega(m_0) \otimes m_1 \otimes \ldots \otimes m_{n-1}\right)\\ &= \left(\sum_m m_0 \otimes \Delta(m_1) \otimes \ldots \otimes m_{n-1}\right) \qquad \big(\text{since } (\omega \otimes I)\omega = (I \otimes \Delta)\omega\big)\\ &= \sum_m m_0 \otimes m_1 \otimes \ldots \otimes \Delta(m_i) \otimes \ldots \otimes m_{n-1}. \quad (\text{coassociativity of } \Delta)\end{aligned}$$

□

§2.1 Rational modules.

Corollary 2.1.4. (p.43)

Every finitely generated rational module is finite dimensional.

Proof: Let C be a coalgebra, and let M be a finitely generated rational C^*– module. Since M is finitely generated, there exists a finite set $A = \{a_1, a_2, \ldots, a_n\} \subseteq M$ such that $M = C^*A$, i.e. $M = C^*a_1 + C^*a_2 + \ldots + C^*a_n = C^*A$, where $a_i \in A$. In particular, M is a finite sum of cyclic submodules C^*a_i, and by Theorem 2.1.3(b) every cyclic submodule of a rational module is finite dimensional, hence M is also finite dimensional. □

Let C be a coalgebra, and let A be a dense subalgebra of C^*, i.e. A is a subalgebra of C^* for which $0 = A^\perp \subset C$, where $A^\perp = \{c \in C \mid f(c) = 0 \ \forall f \in A\}$, or equivalently, if for some $c \in C$, $f(c) = 0$ for all $f \in A$, then $c = 0$. Then there is an injection $\varphi\colon C \to A^*$, $c \mapsto \langle\ , c\rangle$ where $\langle\ , c\rangle(f) = \langle f, c\rangle = f(c)$. Let M be an A- module and let $\rho\colon M \to \operatorname{Hom}(A, M)$ be defined by $\rho(m)(a) = a \cdot m$. There are injections

$$M \otimes C \overset{M\otimes\varphi}{\hookrightarrow} M \otimes A^* \overset{\psi}{\hookrightarrow} \operatorname{Hom}(A, M) \text{ where } \psi(m \otimes a^*)(a) = a^*(a)m \text{ as usual,}$$

and we say that M is a *rational A- module* if $\rho(M) \subset M \otimes C$. Thus M is a rational A- module if there exist $m_i \in M$, $c_i \in C$, $i = 1, \ldots, n$ such that $a \cdot m = \sum_i c_i(a)m_i$ for all $a \in A$.

Exercise. (p.44)

Let M be a rational A-module. Then ρ, as defined above, makes M a right C-comodule and the induced C^* action on M extends the A-action. If N is any rational C^* module, restricting scalars from C^* to A gives a rational A-module whose extension to C^* is just N.

Proof:

Since $\rho(M) \subset M \otimes C$, for any $m \in M$ there are $m_i \in M$ and $c_i \in C$ such that $\rho(m) = \sum_i m_i \otimes c_i$. Hence, for $a \in A \subseteq C^*$

$$a \cdot m = \rho(m)(a) = \left(\sum_i m_i \otimes c_i\right)(a) = \sum_i m_i \langle a, c_i\rangle.$$

We have that

$$(\rho \otimes I)\left(\sum_i m_i \otimes c_i\right) = \sum_i \left(\sum_j (m_i)_j \otimes c_j\right) \otimes c_i \in M \otimes C \otimes C.$$

We will show that $\sum_i \left(\sum_j (m_i)_j \otimes c_j\right) \otimes c_i = \sum_i m_i \otimes \Delta_C(c_i)$. If we identify $M \otimes C \otimes C$ with its image in $\mathrm{Hom}(A \otimes A, M)$, it suffices to show that they both act on $a \otimes b \in A \otimes A$ in the same way. On the left-hand side we have

$$\begin{aligned}\left(\sum_i \left(\sum_j (m_i)_j \otimes c_j\right) \otimes c_i\right)(a \otimes b) &= \sum_i \sum_j (m_i)_j \langle a, c_j\rangle \langle b, c_i\rangle = \sum_i (a \cdot m_i)\langle b, c_i\rangle = \sum_i a \cdot \big(\langle b, c_i\rangle m_i\big) \\ &= a \cdot (b \cdot m) = (a * b) \cdot m,\end{aligned}$$

where $*$ denotes the convolution product. Because A is a subalgebra of C^*, for $a, b \in A,\ c \in C$, $(a * b)(c) = \sum_c a(c_1)b(c_2)$, and so on the right-hand side we have

$$\begin{aligned}\left(\sum_i m_i \otimes \Delta_C(c_i)\right)(a \otimes b) = \left(\sum_{i,c_i} m_i \otimes c_{i1} \otimes c_{i2}\right)(a \otimes b) &= \sum_{i,c_i} m_i \langle a, c_{i1}\rangle \langle b, c_{i2}\rangle \\ &= \sum_i m_i \langle a * b, c_i\rangle = (a * b) \cdot m.\end{aligned}$$

Hence, $(\rho \otimes I)\rho = (I \otimes \Delta_C)\rho$.

Also, since $1_A = 1_{C^*} = \epsilon_C$, for any $m \in M$ we have that

$$\begin{aligned}(I \otimes \epsilon_C)\rho(m) = (I \otimes \epsilon_C)\left(\sum_i m_i \otimes c_i\right) = \sum_i m_i \epsilon_C(c_i) \otimes 1_k &= \sum_i m_i \langle \epsilon_C, c_i\rangle \otimes 1_k \\ &= \epsilon_C \cdot m \otimes 1_k = m \otimes 1_k.\end{aligned}$$

Hence, ρ makes M a right C-comodule.

Let N be any rational C^*-module so that $\rho(N) \subseteq N \otimes C$ where $\rho\colon N \to \mathrm{Hom}(C^*, N)$ by $\rho(n)(f) = f \cdot n$ for all $n \in N,\ f \in C^*$. Then N is also an A-module since $A \subseteq C^*$ and it is clearly rational. If N is a rational A-module (so that $\rho(N) \subseteq N \otimes C$), then N is a C^*-module by $c^* \cdot n = \sum_i n_i \langle c^*, c_i\rangle$. If the rational A-module structure on N was induced by a rational C^*-module structure, then these C^*-module structures are the same.

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§2.2 The Fundamental Theorem of Coalgebras.

Exercise. (p.45)

Show that the intersection of left coideals (right coideals) of a coalgebra C is again such.

Proof: Let $\{V_\alpha\}$ be a collection of left coideals. By 1.4.5(a) each $V_\alpha^\perp$ is a left ideal, so $I = \sum V_\alpha^\perp$ is a sum of left ideals, and so I must be a left ideal. By 1.4.5(b) $I^\perp$ is a left coideal, but by $A.5(a)$, $I^\perp = (\sum V_\alpha^\perp)^\perp = \bigcap V_\alpha^{\perp\perp} = \bigcap V_\alpha$, so the intersection of left coideals is a left coideal. Similarly, the intersection of right coideals is a right coideal. ☐

Corollary 2.2.2. (p.47)

The subcoalgebra of C generated by any finite set or finite dimensional subspace is again finite dimensional.

Proof: Let M be a subcoalgebra of C generated by a finite set A of C, i.e. M is the smallest subcoalgebra containing A. Then M is the sum of the subcoalgebras generated by each element of A (since the subcoalgebra generated by each element of A is the smallest subcoalgebra containing each element, so the sum of these subcoalgebras would be the smallest subcoalgebra containing A). By the Fundamental Theorem of Coalgebras, each subcoalgebra generated by an element in C is finite dimensional, hence the sum of subcoalgebras generated by single elements is finite dimensional, and so M is finite dimensional. If M is instead generated by a finite dimensional subspace, then since such a subspace must have a finite set as a basis, it follows from the above argument that M is finite dimensional. ☐