BALANCED BILINEAR FORMS ON MATRIX AND MATRIX-LIKE COALGEBRAS

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ABSTRACT. In this short note, we determine all balanced bilinear forms for a matrix coalgebra and for one type of matrix-like coalgebra.

1. Introduction and preliminaries

In this note we determine all balanced bilinear forms for a matrix coalgebra over a field k and for a class of matrix-like coalgebras. In the first case, we show that the balanced bilinear forms for an n^2-dimensional matrix coalgebra with fixed comatrix basis β are in one-to-one correspondence with the set of $n \times n$ matrices over k. This description can be used to describe coalgebra automorphisms of a matrix coalgebra. In the second case, we see that our matrix-like coalgebras have only degenerate balanced bilinear forms which are determined by a scalar a.

Let C be a coalgebra over a field k and $B : C \otimes C \to k$ a bilinear map. Recall that B is called left (right) nondegenerate if $B(C, x) = 0$ ($B(x, C) = 0$) implies that $x = 0$ and B is nondegenerate if it is left and right nondegenerate. Also recall that B is called C^*-balanced if $B(x \cdot c^*, y) = B(x, c^* \cdot y)$ for all $x, y \in C$, $c^* \in C^*$, or equivalently, for all $x, y \in C$,

$$x_1 B(x_2, y) = B(x, y_1) y_2. \tag{1}$$

There is a bijective correspondence between the set of C^*-balanced bilinear forms on C and the set of left (right) C^*-module maps from C to C^* defined as follows [4, Section 3.3]. If B is a balanced bilinear form, let $B_l : C \to C^*$ be defined by $< B_l(c), d >= B(d \otimes c)$ and it is easy to check that B_l is a left C^*-module map. The map B_l is injective if and only if B is left nondegenerate. Conversely given φ, a left C^*-module map from C to C^*, then the bilinear form B_φ defined by $B_\varphi(c \otimes d) = \varphi(d)(c)$ is C^* balanced. Statements for the right hand case are similar.

Recall that C is called (left, right) co-Frobenius if there is a (left, right) nondegenerate balanced bilinear form B on C.

For example, let H be a Hopf algebra which is co-Frobenius as a coalgebra, and so has a nondegenerate balanced bilinear form B. Then $B(-, 1)$

Key words and phrases. balanced bilinear form, matrix coalgebra.

The first author's research is supported by NSERC.

The second author was funded by an NSERC Undergraduate Student Research Award, summer 2006.

$(B(1,-))$ is a nonzero left (right) integral for H in H^*. Conversely if λ is a nonzero left integral for H in H^*, define the nondegenerate balanced bilinear form B by $B(h,l) = \lambda(hS(l))$.

Symmetric coalgebras are defined in [3] to be those coalgebras for which there exists an injective morphism $\alpha : C \to C^*$ of (C^*, C^*)-bimodules. Equivalently a coalgebra C is symmetric if there exists a balanced bilinear form $B : C \otimes C \to k$ which is non-degenerate and symmetric, meaning that $B(b,c) = B(c,b)$ for all $b, c \in C$.

Recall [7], (see also [2, p. 318]), that a coalgebra C is coseparable if C has a coseparability idempotent, that is, a balanced bilinear form B such that $B \circ \Delta = \epsilon$.

Throughout, we work over a field k and use the Heyneman-Sweedler notation for comultiplication in a coalgebra with the summation sign omitted. Maps are assumed to be k-linear, $\otimes$ means $\otimes_k$, etc. We denote by E_{ij} the square matrix with a 1 in the $ijth$ position and zeroes elsewhere. The basics of the theory of coalgebras can be found in [11] or [4].

2. Matrix Coalgebras

Let $C = M^c(n,k)$, the n^2-dimensional matrix coalgebra over k. Then C has a comatrix basis $\beta = \{e_{11}, e_{12}, \ldots, e_{1n}, e_{21}, \ldots e_{nn}\}$ such that

$$\epsilon(e_{ij}) = \delta_{ij} \text{ and } \Delta(e_{ij}) = \sum_{k=1}^{n} e_{ik} \otimes e_{kj}. \tag{2}$$

We call the above ordering of a comatrix basis the standard ordering, and unless otherwise specified, this will be assumed to be the ordering used. We begin by finding all balanced bilinear forms for C.

Proposition 2.1. *For $C = M^c(n,k)$, let $B : C \otimes C \to k$ be a bilinear form on C. Then B is C^*-balanced if and only if $B(e_{ik}, e_{lm}) = 0$ for $i \neq m$, and $B(e_{ik}, e_{li}) = B(e_{jk}, e_{lj})$ for all $1 \leq i,j,k,l \leq n$. Thus, a balanced bilinear form B on C with basis β is completely determined by the n^2 scalars $a_{ij} = B(e_{mi}, e_{jm})$ for all m, $1 \leq m \leq n$.*

Proof. By (1), B is balanced if and only if, for all i,j,k,l, we have

$$\sum_{s=1}^{n} e_{is} B(e_{sj}, e_{kl}) = \sum_{t=1}^{n} B(e_{ij}, e_{kt}) e_{tl}. \tag{3}$$

But (3) holds if and only if $B(e_{sj}, e_{kl}) = B(e_{ij}, e_{kt}) = 0$ unless $t = i$ and $s = l$, and then $B(e_{lj}, e_{kl}) = B(e_{ij}, e_{ki})$. Since this holds for all i, l, the proof is complete. □

Let H be a Hopf algebra with antipode S and $\lambda \in H^*$ a nonzero left integral for H. The balanced bilinear form B defined by $B(h,l) = \lambda(hS(l))$ appears frequently in Hopf algebra theory, from the 1971 proof that the antipode of a cosemisimple Hopf algebra is bijective [6] to the recent discussion of Hopf algebras which are symmetric coalgebras in [3]. For example, the fact [6, Theorem 2.7], [9, 3.3] that if $C = D \oplus E$ as coalgebras where D, E are comatrix coalgebras, then for $d \in D, e \in E$, we have $\lambda(dS(e)) = 0$ follows directly from the definition of a balanced form and the comultiplication for a matrix coalgebra in (2). The following result of Larson follows immediately from Proposition 2.1.

Corollary 2.2. [6, Eqn (2.5)], [9, 3.2],[4, Proposition 7.3.8] *For H, λ as above with $C \simeq M^c(n,k)$ a subcoalgebra of H, then $\lambda(e_{ir}S(e_{sj})) = 0$ for $i \neq j$ and $\lambda(e_{jr}S(e_{sj})) = \lambda(e_{ir}S(e_{si}))$ for all i, j.*

For fixed β, given a balanced form B, let $A = A_B$ denote the matrix with $(i,j)th$ entry a_{ij}, the scalars from Proposition 2.1. Conversely given an $n \times n$ matrix A, define $B = B_A$ to be the bilinear form which maps $e_{ij} \otimes e_{km}$ to $\delta_{im}a_{jk}$. Some properties of the matrix A are independent of the comatrix basis β.

Proposition 2.3. *Given a balanced bilinear form B on $C = M^c(n,k)$, let A be the matrix associated to B with respect to some comatrix basis β and A' the matrix associated to B with respect to another basis γ. Then we have the following.*
(i) A has rank n if and only if B is left nondegenerate if and only if B is right nondegenerate. Thus A has rank n if and only if A' has rank n.
(ii) A has trace 1 if and only if B is a coseparability idempotent and thus A and A' have the same trace;
(iii) A and A' are scalar multiples of the $n \times n$ identity matrix if and only if B is symmetric.

Proof. (i) A has rank n if and only if for any $n \times n$ matrix Γ, if $A\Gamma = 0$ then $\Gamma = 0$. We show that $\text{rank}(A) < n$ if and only if B is left degenerate. Now, B is left degenerate if and only if there exists $0 \neq x \in C$ such that $B(C, x) = 0$, i.e., such that $B(e_{ij}, x) = 0$ for $1 \leq i, j \leq n$. Suppose $x = \sum_{1 \leq k,l \leq n} \gamma_{kl}e_{kl}$ and let Γ be the nonzero $n \times n$ matrix $\Gamma = (\gamma_{kl})$. Then it is easy to check that $B(C, x) = 0$ if and only if for all i, j, we have $\sum_{1 \leq k,l \leq n} \gamma_{kl}B(e_{ij}, e_{kl}) = 0$ if and only if $\sum_{k=1}^{n} a_{jk}\gamma_{ki} = 0$ if and only if $A\Gamma = 0$. The proof that $\text{rank}(A) < n$ if and only if B is right degenerate is the same. The last statement is clear.
(ii) A balanced form B is a coseparability idempotent for C if and only if $\sum_{k=1}^{n} B(e_{ik}, e_{kj}) = \epsilon(e_{ij}) = \delta_{ij}$ for any β and this occurs if and only if $\sum_{k=1}^{n} a_{kk} = 1$, i.e., the matrix A has trace 1.
Suppose that A has nonzero trace t, or equivalently $(1/t)A$ has trace 1. Then $(1/t)B$ is a coseparability idempotent for C. Thus $(1/t)A'$ has trace 1, so A' has trace t.
(iii) The balanced nondegenerate form B is symmetric if and only if we have $a_{ij} = B(e_{ki}, e_{jk}) = B(e_{jk}, e_{ki}) = \delta_{ji}a_{kk}$ for any k, i.e., if and only if A is a scalar times the identity. □

If $n \neq 0$, the next corollary gives an alternate proof to a result of Nichols proved using a Skolem-Noether argument.

Corollary 2.4. [9, Corollary 0.3][4, 7.3.4] *Suppose that char k does not divide n, and let B be the balanced form on $C = M^c(n,k)$ defined by $B(e_{ij}, e_{kl}) = \delta_{jk}\delta_{il}$ for some comatrix basis β. Then if $\gamma = \{f_{ij} | 1 \leq i,j \leq n\}$ is another comatrix basis for C then $B(f_{ij}, f_{kl}) = \delta_{jk}\delta_{il}$ also.*

Proof. By Proposition 2.3 (ii) and (iii), since the matrix A for B associated with β is $I_{n\times n}$, so is the matrix A' for B associated with γ. □

Let $I_{n^2\times n^2}$ be the identity matrix with rows and columns indexed by β with the standard ordering and let $\mathcal{S}$ be the matrix obtained from $I_{n^2\times n^2}$ by switching the e_{ij} and e_{ji} columns for all $i \neq j$. Equivalently, $\mathcal{S}$ is obtained from the identity by switching the e_{ij} and e_{ji} rows for all $i \neq j$. In other words, $\mathcal{S}$ is the change of basis matrix that takes the standard ordered basis β to $\beta' = \{e_{11}, e_{21}, e_{31}, \ldots, e_{n1}, e_{12}, e_{22}, \ldots, e_{nn}\}$.
If B is a balanced bilinear form on C with $\mathcal{A}_\beta$ the $n^2 \times n^2$ matrix for B with respect to β, i.e., $B(\mathbf{x}, \mathbf{y}) = \mathbf{x^t}\mathcal{A}\mathbf{y}$ where $\mathbf{x}, \mathbf{y}$ are vectors in the k-space C written with respect to the ordered basis β, then $\mathcal{A} \circ \mathcal{S}$ is the matrix consisting of n copies of A along the diagonal and zeroes elsewhere.

Proposition 2.5. *Let $\gamma = \{f_{11}, f_{12}, \ldots, f_{nn}\}$ be another comatrix basis for C with the standard ordering and Q the change of basis matrix from γ to β. Then $Q^{-1} = \mathcal{S} \circ Q^t \circ \mathcal{S}$.*

Proof. Let $B(e_{ij}, e_{rs}) = \delta_{is}\delta_{jr}$ and let $\mathcal{A}$ be the matrix for B. Then $\mathcal{A} \circ \mathcal{S} = I_{n^2\times n^2}$ since the $n \times n$ matrix A for this form is the identity. Thus $\mathcal{A} = \mathcal{S}$. Also by Corollary 2.4, $\mathcal{A} = \mathcal{S}$ is the matrix for the form with respect to γ. But we also have that the matrix for the bilinear form with respect to γ is $Q^t \circ \mathcal{A} \circ Q = Q^t \circ \mathcal{S} \circ Q$. Thus $Q^t \circ \mathcal{S} \circ Q \circ \mathcal{S}$ is the $n^2 \times n^2$ identity. The statement is now immediate. □

The results of [10, 1.2] show that the change of basis matrix Q from one standard ordered comatrix basis to another has the form $\mathcal{S} \circ \mathcal{D}^t \circ \mathcal{S} \circ \mathcal{D}^{-1}$ where $\mathcal{D}$ is an $n^2 \times n^2$ matrix consisting of n copies of an invertible $n \times n$ matrix D along the diagonal and zeroes elsewhere.

Corollary 2.6. *If $Q = \mathcal{S} \circ \mathcal{D}^t \circ \mathcal{S} \circ \mathcal{D}^{-1}$ as above, then*

$$Q^{-1} = \mathcal{D} \circ \mathcal{S} \circ (\mathcal{D}^{-1})^t \circ \mathcal{S} = \mathcal{S} \circ (\mathcal{D}^{-1})^t \circ \mathcal{S} \circ \mathcal{D} \circ \mathcal{S} \circ \mathcal{S} = \mathcal{S} \circ (\mathcal{D}^{-1})^t \circ \mathcal{S} \circ \mathcal{D}.$$

3. Matrix-like Coalgebras

A coalgebra is called matrix-like if it is generated by elements e_{ij}, $1 \leq i,j \leq n$, not necessarily linearly independent, satisfying (2). For example, if C is a coalgebra and V a finite dimensional right C-comodule with basis $\{v_1, \ldots, v_n\}$, and $\rho : V \to V \otimes C$ the coaction, then $\rho(v_j) = \sum_{i=1}^n v_i \otimes c_{ij}$ for elements $c_{ij} \in C$. Then the c_{ij} form a coalgebra and satisfy (2) but need not be linearly independent.

Let $L = L(n,k)$ denote the matrix-like coalgebra over k of dimension $n(n+1)/2$, with basis $\{e_{ij} | 1 \leq i \leq j \leq n\}$ and with comultiplication and the counit map defined as in (2) with $e_{kl} = 0$ for $k > l$. In other words, $L(n,k)$ is the quotient coalgebra of $M^c(k,n)$ by the coideal J with basis e_{kl}, $k > l$.

For example, recall [1, Theorem 2.1] that if $n = 2$, then any 3 dimensional matrix-like coalgebra with the generators of $M^c(2,k)$ is isomorphic to $L(2,k)$.

Any Hopf algebra with a skew-primitive element contains subcoalgebras isomorphic to $L(2,k)$. For if $\Delta(x) = g\otimes x + x\otimes h$ for g,h different grouplikes, then $\{g,x,h\}$ is a subcoalgebra isomorphic to $L(2,k)$

In the next example, we identify copies of $L(n,k)$ in the Taft Hopf algebras.

Example 3.1. *Recall that T_{n^2}, the Taft Hopf algebra of dimension n^2, has a basis $\{g^i x^j | 0 \le i,j \le n-1\}$. The coalgebra structure is given by $\epsilon(g^i x^j) = \delta_{j,0}$ and by $\Delta(g^i x^j) = \sum_{k=0}^{j} \binom{j}{k}_q g^{i+k} x^{j-k} \otimes g^i x^k$ where q is a primitive nth root of unity such that $xg = qgx$. Here $\binom{j}{k}_q$ denotes the q-binomial coefficient as usual (see [5]). Now define*

$$e_{n-s,n-i} = \binom{s}{i}_q g^i x^{s-i}, \text{ or equivalently } e_{tm} = \binom{n-t}{n-m}_q g^{n-m} x^{m-t}$$

for $0 \le i \le s \le n-1$ and $1 \le t \le m \le n$. Let the linearly independent set $\{e_{n-s,n-i} | s \ge i\}$ of $n(n+1)/2$ elements be denoted by $E = E_0$.

The k-span of the set $E = E_0$ defined above is a subcoalgebra of T_{n^2} isomorphic to $L(n,k)$. First we note that $\epsilon(e_{n-s,n-i}) = \epsilon(\binom{s}{i}_q g^i x^{s-i}) = \delta_{si}$. It remains to show that $\Delta(e_{n-(i+j),n-i}) = \sum_{t=n-(i+j)}^{n-i} e_{n-(i+j),t} \otimes e_{t,n-i}$. Recall [5, IV.2] that $\binom{n}{k}_q = \binom{n}{n-k}_q$ and then it follows easily that $\binom{i+j}{i+s}_q \binom{i+s}{i}_q = \binom{i+j}{i}_q \binom{j}{s}_q$. Using this fact, for $0 \le i+j \le n-1$, we compute

$$\begin{aligned}
\sum_{t=n-(i+j)}^{n-i} e_{n-(i+j),t} \otimes e_{t,n-i} &= \sum_{s=0}^{j} e_{n-(i+j),n-(i+s)} \otimes e_{n-(i+s),n-i} \\
&= \sum_{s=0}^{j} \binom{i+j}{i+s}_q g^{i+s} x^{j-s} \otimes \binom{i+s}{i}_q g^i x^s \\
&= \binom{i+j}{i}_q \sum_{s=0}^{j} \binom{j}{s}_q g^{i+s} x^{j-s} \otimes g^i x^s \\
&= \binom{i+j}{i}_q \triangle(g^i x^j) \\
&= \triangle(e_{n-(i+j),n-i}).
\end{aligned}$$

Since the g^m are grouplike, for $0 \le m \le n-1$, the k-span of the set $E_m = g^m E = \{g^m e_{n-s,n-i} | s \ge i\}$ is also a subcoalgebra of T_{n^2} isomorphic to $L(n,k)$. Finally, we note that for $m \ne l$, $E_m \ne E_l$ since $g^m x^{n-1} \in g^m E$

but does not lie in any other $g^l E$. *For if* $g^{l+i}x^{s-i} = g^m x^{n-1}$ *for some* $s \geq i$ *then we must have* $s = n-1, i = 0$, *and then* $l = m$.

Every basis element of T_{n^2} *lies in some* E_m *since* $x^t = e_{n-t,n} \in E$ *for* $0 \leq t \leq n-1$. *Note that* E *contains exactly* t *elements of the form* $g^i s^{n-t}$ *for some* i, *and thus* $g^i x^{n-t}$ *lies in exactly* t *of the subcoalgebras* E_m.

In the next proposition, we describe all balanced bilinear forms for $L(n,k)$.

Proposition 3.2. *Let* $B : L \otimes L \to k$ *be a bilinear form on* L. *Then* B *is* L^**-balanced if and only if* $B(e_{ij}, e_{kl}) = \delta_{il}\delta_{jn}\delta_{k1}a$ *for some scalar* a.

Proof. The bilinear form B is balanced if and only if for all $i \leq j$, $k \leq l$, we have

$$\sum_{s=i}^{j} e_{is}B(e_{sj}, e_{kl}) = \sum_{t=k}^{l} B(e_{ij}, e_{kt})e_{tl}. \tag{4}$$

Thus $B(e_{sj}, e_{kl}) = 0$ for $s \neq l$ for all $i \leq s \leq j$ and $B(e_{ij}, e_{kt}) = 0$ for $t \neq i$ for all $k \leq t \leq l$. If $s = l$ and $t = i$, then we must have $k \leq i \leq l \leq j$ and then $e_{il}B(e_{lj}, e_{kl}) = B(e_{ij}, e_{ki})e_{il}$. As in Proposition 2.1, we write a_{jk} for $B(e_{lj}, e_{kl}) = B(e_{ij}, e_{ki})$ and form the lower triangular matrix $A = (a_{jk})$.

Now let $1 < i \leq r$, and apply (1) to $e_{1r} \otimes e_{ii}$ to obtain that

$$\sum_{k=1}^{r} e_{1k}B(e_{kr}, e_{ii}) = e_{1i}B(e_{ir}, e_{ii}) = B(e_{1r}, e_{ii})e_{ii} = 0.$$

Thus $B(e_{ir}, e_{ii}) = a_{ri} = 0$ unless $i = 1$ so that the matrix A has nonzero entries only in the first column.

Finally apply (1) to $e_{ii} \otimes e_{rn}$ with $r \leq i < n$ to see that $a_{ir} = B(e_{ii}, e_{ri}) = 0$, so that the matrix A has nonzero entries only in the *nth* row. This completes the proof. □

Thus we see that for $n > 1$, $L(n,k)$ has no nondegenerate balanced bilinear forms so cannot be co-Frobenius, and also is not coseparable since the elements e_{ii} are grouplike and for B a coseparability idempotent, $B(g,g) = 1$ for any grouplike g.

Corollary 3.3. *No coalgebra with a subcoalgebra isomorphic to* $L(n,k)$ *for* $n > 1$ *is coseparable.*

For example, this means that no coalgebra containing a skew-primitive element is coseparable.

Example 3.4. *Sweedler's example* [8, p.361], [4], *of a coalgebra* C *which is left but not right co-Frobenius consists of a countable number of copies of coalgebras* $L_i \cong L(2,k)$ *such that* $L_i \cap L_{i+1} = \{g_{i+1}\}$ *and* $L_i \cap L_j = \emptyset$ *otherwise. Take* x_i, g_i, $i \geq 1$ *to be a basis for* C *where the* g_i *are grouplike and the* x_i *are skew primitive with* $\triangle(x_i) = g_i \otimes x_i + x_i \otimes g_{i+1}$. *Define* B_i *to be the balanced bilinear form on* L_i *such that*

$$B_i(x_i, g_i) = B_i(g_{i+1}, x_i) = a_i$$

and B_i is 0 otherwise. It is easy to check that if B is a balanced bilinear form on C, then $B(z,w) = B_i(z,w)$ if $z, w \in L_i$ and B is zero otherwise. Since $B(g_1, C) = 0$, B is always right degenerate but if the a_i are all nonzero, then B is left nondegenerate.

We end this note by discussing bilinear forms on the matrix-like subcoalgebras of T_{n^2}. We have shown that T_{n^2} is the union of subcoalgebras $E_m \cong L(n,k)$. Any balanced bilinear form on T_{n^2} induces a form on each E_m which is determined by a scalar a_m. Conversely, given an n-tuple $(a_0, \dots, a_{n-1})$ with a_m determining a form on E_m, we can paste these together to obtain a form on T_{n^2}.

Proposition 3.5. *Let B_m be the balanced bilinear form on E_m defined as in Proposition 3.2 with $B_m(g^m e_{ij}, g^m e_{kl}) = \delta_{il}\delta_{jn}\delta_{k1}a_m$ for some scalar a_m. Define a bilinear map B from $T_{n^2} \otimes T_{n^2}$ to k by*

$$\text{(5)} \quad \binom{n-1}{t}_q B(g^m x^t, g^l x^k) = \delta_{k+t,n-1}\delta^{[n]}_{l,m+t} B_m(g^m e_{n-t,n}, g^m e_{1,n-t}) = \delta_{k+t,n-1}\delta^{[n]}_{l,m+t} a_m$$

for $0 \le m, t, l, s, k \le n-1$ and where $\delta^{[n]}$ means that the subscripts must be equivalent mod n. Then B is a balanced bilinear form.

Proof. First we show that B is well defined, i.e. that $e_{in} \otimes e_{1i} \in E_m \otimes E_m$ implies $m = 0$, or equivalently $g^m e_{in} \otimes g^m e_{1i} \in E_t \otimes E_t$ implies that $t = m$. Suppose that $e_{1i} = g^m e_{rs}$ so that $\binom{n-1}{n-i}_q g^{n-i}x^{i-1} = \binom{n-r}{n-s}_q g^{m-s}x^{s-r}$ and also $e_{in} = g^m e_{tp}$, so that $x^{n-i} = \binom{n-t}{n-p}_q g^{m-p}x^{p-t}$. Then we have
(i) $i - 1 = s - r$;
(ii) $n - i = p - t$;
(iii) $m \equiv s - i \equiv p \bmod n$;
(iv) $\binom{n-t}{n-p}_q = 1$.
This last means that either $p = t$ or $p = n$. If $p = t$, then $n = i$ by (ii) and by (i), $s = n + r - 1$. Since $s \le n$, we must have $s = n$ and $r = 1$. Then $g^m = 1$ by (iii). If $p = n$, then by (iii) directly, we have $g^m = 1$.

Now, to show that B is balanced, we must show that for all $g^i x^j, g^l x^r \in T_{n^2}$ we have

$$\text{(6)} \qquad (g^i x^j)_1 B((g^i x^j)_2, g^l x^r) = B(g^i x^j, (g^l x^r)_1)(g^l x^r)_2.$$

From the comultiplication on $_{n^2}$, this is equivalent to showing that

$$\sum_{p=0}^{j} \binom{j}{p}_q g^{i+p} x^{j-p} B(g^i x^p, g^l x^r) = \sum_{w=0}^{r} \binom{r}{w}_q B(g^i x^j, g^{l+w} x^{r-w}) g^l x^w.$$

Now by (5), we must have $p = n - 1 - r$, $w = j + r - (n-1)$ and if $j + r < n - 1$ then both sides are 0. Now we must show that

$$\begin{aligned}&\binom{j}{n-1-r}_q g^{i-r-1}x^{j+r-(n-1)}B(g^ix^{(n-1)-r},g^lx^r)\\ &=\binom{r}{j+r-(n-1)}_q g^lx^{j+r-(n-1)}B(g^ix^j,g^{l+j+r+1}x^{(n-1)-j}).\end{aligned}$$

By (5), both sides are zero unless $l+r+1\equiv i \bmod n$. Then, assuming this, it remains to show that $\binom{j}{n-1-r}_q B(g^ix^{(n-1)-r},g^lx^r)$ is the same as $\binom{r}{j+r-(n-1)}_q B(g^ix^j,g^{i+j}x^{(n-1)-j})$.

This holds if and only if the equation holds when both sides are multiplied by $\binom{n-1}{n-1-r}_q\binom{n-1}{j}_q$, i.e., when

$$\binom{j}{n-1-r}_q\binom{n-1}{j}_q a_i=\binom{n-1}{n-1-r}_q\binom{r}{j+r-(n-1)}_q a_i,$$

and this last equality is easily checked. □

Corollary 3.6. *Every ordered n-tuple of (nonzero) scalars $(a_0,\dots,a_{n-1})$ determines a (nondegenerate) balanced bilinear form $B(a_0,\dots,a_{n-1})$ on T_{n^2}.*

Proof. It remains only to show that B as constructed in Proposition 3.5 is left and right nondegenerate when the a_i are all nonzero. But this follows from the observation that $x^j=e_{n-j,n}$ and $g^te_{1t}=x^{t-1}$. □

Corollary 3.7. *For nonzero scalars a_0,a_1, the maps $\lambda,\gamma\in T^*_{n^2}$ defined by $\lambda(gx^{n-1})=a_1$, $\lambda(g^mx^t)=0$ otherwise, and $\gamma(x^{n-1})=a_0$, $\gamma(g^mx^t)=0$ otherwise, are nonzero left and right integrals for T_{n^2}.*

Proof. Let B be a balanced bilinear form, as constructed in Proposition 3.5 with $a_1\neq 0$. We know that $\lambda=B(-,1)$ is a nonzero left integral for T_{n^2} in $T^*_{n^2}$. But if $1=g^me_{1i}=\binom{n-1}{n-i}_q g^mg^{n-i}x^{i-1}$ then $i=1$ and $m=1$, and thus $\binom{n-i}{0}_q g^mx^{n-i}=gx^{n-1}$. Thus $\lambda(gx^{n-1})=a_1\neq 0$ and λ is 0 elsewhere. The statement for γ follows by a similar argument but starting with a form B with $a_0\neq 0$. □

Remark 3.8. *Note that if we start with the degenerate form B as defined in Proposition 3.5 with $a_i=\delta_{i,1}$, then $\lambda=B(-,1)$ is the nonzero left integral in Corollary 3.7. If we then construct the nondegenerate bilinear form B' given by $B'(y,z)=\lambda(yS(z))$ we obtain the form $B(q,1,q^{-1},\dots)$ from Proposition 3.5 with $a_m=q^{(n-1)(m-1)}$, which is nondegenerate.*

In the Introduction, we noted that there is a one to one correspondence between balanced bilinear forms on a coalgebra C and the left C^*-module maps from C to C^*. For C finite dimensional and co-Frobenius, C and C^* are isomorphic as left C^*-modules, so this last is C^*. Thus, for $C=T_{n^2}$, the space of all balanced bilinear forms has dimension n^2, while the space of

n-tuples has dimension n. Since the element $g^i x^{n-1}$ occurs only in E_i, then the image of $g^i x^{n-1} \otimes g^j x^{n-1}$ is not defined by the form acting on the E_m.

For example, for T_4, the Sweedler 4-dimensional Hopf algebra with basis $\{1, g, x, gx\}$, then a general balanced bilinear form B is given by:

$$B(x,g) = B(1,x) = a_0; \quad B(gx,1) = B(g,gx) = a_1;$$
$$B(x,gx) = b; \qquad B(gx,x) = c,$$

for scalars a_0, a_1, b, c and B is 0 elsewhere.

Acknowledgement

Thanks to the referee for his/her helpful remarks.

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