The coradical of the dual of a lifting of a quantum plane

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Abstract

In this short note we compute the coradical of the dual A^* of a lifting A of a quantum linear space in the case that A^* has only trivial grouplike elements.

1 Introduction

The classification of finite dimensional Hopf algebras is known to be a difficult problem, for which only a few general techniques are available. One technique for studying Hopf algebras whose coradical is a sub Hopf algebra is the "lifting method" of Andruskiewitsch and Schneider. The simplest application of the lifting method is the description of all pointed Hopf algebras with finite abelian group of grouplikes Γ and with space V of $(1, g)$ primitives, $V \in_{K[\Gamma]}^{K[\Gamma]} \mathcal{YD}$, a quantum linear space.

*This work was partially supported by CONICET, Agencia Córdoba Ciencia, ANPCyT and Secyt (UNC).

†Research supported by NSERC.

1991 *Mathematics Subject Classification.* Primary: 17B37. Secondary: 16W30

A lifting of a quantum linear space is a pointed Hopf algebra A whose associated graded Hopf algebra is isomorphic to the Radford biproduct $\mathcal{B}(V)\#K[\Gamma]$ where $\mathcal{B}(V)$ is the Nichols algebra of the quantum linear space V, and is a Hopf algebra in ${}^{K[\Gamma]}_{K[\Gamma]}\mathcal{YD}$. Liftings of quantum linear spaces were constructed and completely described in [2], and also independently, using repeated Ore extensions, in [5].

Liftings of quantum linear spaces, although fairly easy to describe, have enough interesting structure to prove useful in various situations. They provided counterexamples to Kaplansky's Tenth Conjecture [2], [5], [9]. Suppose K is an algebraically closed field of characteristic 0. It is shown in [2] or [6] that every pointed nonsemisimple Hopf algebra over K of dimension p^3, p prime, is a lifting of a quantum linear space. In [10] it is shown that every pointed Hopf algebra of index p^2, p a prime, is the lifting of a quantum line or a quantum linear space; this result generalizes [1, Theorem 1.7] and [8, Theorem 1.2]. In [7], it is shown that over K, every pointed nonsemisimple Hopf algebra of dimension 16 is a lifting of a quantum linear space.

However, although the structure of liftings of quantum linear spaces is well understood, the coradicals of the duals (and thus the irreducible representations) of these Hopf algebras have yet to be completely described. In [12], and more recently in [3], for V a quantum linear space of dimension 1 or 2, the coradicals of the duals of some liftings of $\mathcal{B}(V)\#K[\Gamma]$ are described. In particular, if V has dimension 1, then the coradical of the dual A^* of a lifting A of $\mathcal{B}(V)\#K[\Gamma]$ is completely described. If V has dimension 2, then the coradical of the dual A^* is described in some cases where A^* has nontrivial grouplikes.

In this short note, we study some cases where V has dimension 2 and A^* has only the trivial grouplike element and we explicitly construct the coradical of the dual of a Hopf algebra in the family of pointed Hopf algebras of dimension $81 = 3^4$ which provided one of the early counterexamples to Kaplansky's Tenth Conjecture.

2 Preliminaries

Throughout, K will denote an algebraically closed field of characteristic 0, Γ a finite abelian group and ${}^{\Gamma}_{\Gamma}\mathcal{YD} = {}^{K[\Gamma]}_{K[\Gamma]}\mathcal{YD}$ the category of Yetter-Drinfel'd modules over the group algebra $K[\Gamma]$. For $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$, $g \in \Gamma, \chi \in \hat{\Gamma}$, we write V_g^χ for the set of $v \in V$ with the action of Γ on v given by $h \rightarrow v = \chi(h)v$ and the coaction by $\delta(v) = g \otimes v$. Since Γ is an abelian group, it is well-known that $V = \oplus_{\substack{g\in\Gamma \\ \chi\in\hat{\Gamma}}} V_g^{\chi}$.

Definition 2.1 *$V = \oplus_{i=1}^{t} Kv_i \in_{\Gamma}^{\Gamma} \mathcal{YD}$ with $0 \neq v_i \in V_{g_i}^{\chi_i}$ is called a quantum linear space if $\chi_i(g_j)\chi_j(g_i) = 1$ for $i \neq j$. As well, $\chi_i(g_i)$ is a primitive r_ith root of unity with $1 < r_i < \infty$.*

Recall that for V a quantum linear space as above, $\mathcal{B}(V)$ is the Nichols algebra for V, and the Radford biproduct, $H = \mathcal{B}(V)\#K[\Gamma]$, is the coradically graded Hopf algebra generated by the $(1, g_i)$-primitives $v_i\#1$ (usually written just v_i) and the grouplike elements $h = 1\#h$, with $h \in \Gamma$. Multiplication is given by $hv_i = \chi_i(h)v_ih$ and $v_iv_j = \chi_j(g_i)v_jv_i$. Also if $\chi_i(g_i)$ is a primitive r_ith root of unity, $v_i^{r_i} = 0$, and then dim $\mathcal{B}(V) = \Pi_{i=1}^{t} r_i$.

Proposition 2.2 *(see [2] or [5]) For V a quantum linear space with $\chi_i(g_i)$ a primitive r_ith root of unity, all liftings A of $\mathcal{B}(V)\#K[\Gamma]$ are Hopf algebras generated by the grouplikes and by $(1, g_i)$-primitives $x_i, 1 \leq i \leq t$ where*

$$\begin{aligned} hx_i &= \chi_i(h)x_ih; \\ x_i^{r_i} &= \alpha_{ii}(g_i^{r_i} - 1); \\ x_ix_j &= \chi_j(g_i)x_jx_i + \alpha_{ij}(g_ig_j - 1). \end{aligned}$$

We may assume $\alpha_{ii} \in \{0, 1\}$ and then we have that

$$\begin{aligned} \alpha_{ii} &= 0 \text{ if } g_i^{r_i} = 1 \text{ or } \chi_i^{r_i} \neq \epsilon; \\ \alpha_{ij} &= 0 \text{ if } g_ig_j = 1 \text{ or } \chi_i\chi_j \neq \epsilon. \end{aligned}$$

Note that $\alpha_{ji} = -\chi_j(g_i)^{-1}\alpha_{ij} = -\chi_i(g_j)\alpha_{ij}$. Thus the lifting A is described by a matrix $\mathcal{A} = (\alpha_{ij})$ with 0's or 1's on the diagonal and with $\alpha_{ji} = -\chi_i(g_j)\alpha_{ij}$ for $i \neq j$. ∎

Throughout, we use the usual Sweedler notation for Hopf algebra computations. Also, we use the notation $\mathcal{M}^c(r, K)$ for an $r \times r$ matrix coalgebra over K and we call a basis $e(i, j), 1 \leq i, j \leq r$, a matrix coalgebra basis if $\triangle(e(i, j)) = \sum_{k=1}^{r} e(i, k) \otimes e(k, j)$ and $\epsilon(e(i, j)) = \delta_{i,j}$.

3 The Main Theorem

Let A be a lifting of $\mathcal{B}(V)\#K[\Gamma]$ with $V = \oplus_{i=1}^{t} Kv_i \in_{\Gamma}^{\Gamma} \mathcal{YD}$ a quantum linear space and with $\chi_i(g_i)$ a primitive r_ith root of unity. Let $x_i \in A$ be the lifting of $v_i\#1$ to A, i.e. in A, x_i is $(1,g_i)$-primitive and $hx_i = \chi_i(h)x_ih$ for all $h \in \Gamma$. Then A has a vector space basis hz with $h \in \Gamma$ and $z \in \mathcal{Z} = \{x_1^{m_1} \dots x_t^{m_t} | 0 \leq m_i \leq r_i - 1\}$.

For $\gamma \in \hat{\Gamma}$, let $w(\gamma, x_1^{m_1} \dots x_t^{m_t}) \in A^*$ be the map which takes $hx_1^{m_1} \dots x_t^{m_t}$ to $\gamma(h)$ and all other basis elements to 0, so that $\mathcal{W} = \{w(\gamma, z) | \gamma \in \hat{\Gamma}, z \in \mathcal{Z}\}$ is a vector space basis for A^*. The map $w(\gamma, 1)$ is usually just written γ. Multiplication in A^* is given by the following lemma.

Lemma 3.1 *[3, Lemma 3.1] Let A be a lifting of $\mathcal{B}(V)\#K[\Gamma]$ with V a quantum linear space as above. Let $\gamma, \lambda \in \hat{\Gamma}$, $0 \leq m_i \leq r_i - 1$ and $1 \leq i < j \leq t$.*

(i) Then

$$w(\gamma, x_i^{m_i}) * w(\lambda, x_j^{m_j}) = (\gamma\chi_i^{-m_i})(g_j^{m_j}) w(\gamma\lambda, x_i^{m_i} x_j^{m_j});$$

$$w(\lambda, x_j^{m_j}) * w(\gamma, x_i^{m_i}) = \lambda(g_i^{m_i}) w(\gamma\lambda, x_i^{m_i} x_j^{m_j}).$$

In particular, $w(\chi_i, x_i)w(\chi_j, x_j) = w(\chi_i\chi_j, x_ix_j) = \chi_i(g_j)w(\chi_j, x_j)w(\chi_i, x_i)$ for $i < j$.

(ii)

$$\gamma * w(\lambda, x_1^{m_1} \dots x_t^{m_t}) = \gamma(g_1^{m_1} \dots g_t^{m_t}) w(\lambda\gamma, x_1^{m_1} \dots x_t^{m_t});$$
$$w(\lambda, x_1^{m_1} \dots x_t^{m_t}) * \gamma = w(\gamma\lambda, x_1^{m_1} \dots x_t^{m_t}).$$

In particular $\gamma * w(\chi_i, x_i) = \gamma(g_i)w(\chi_i, x_i) * \gamma = \gamma(g_i)w(\chi_i\gamma, x_i)$.

(iii) For $0 \leq m_i \leq r_i - 1$ we have that

$$w(\lambda, x_i) * w(\gamma, x_i^{m_i}) = \lambda(g_i^{m_i})(q^{m_i} + q^{m_i - 1} + \dots 1) w(\lambda\gamma, x_i^{m_i+1})$$

where $q = \chi_i(g_i)^{-1}$, a primitive r_ith root of 1, so that $w(\chi_i, x_i)^{r_i} = 0$ in A^*.

The multiplication formulas above show that $\gamma \in \hat{\Gamma}$ and the elements $w(\chi_i, x_i)$ generate A^* as an algebra with $\gamma * w(\chi_i, x_i) = \gamma(g_i)w(\chi_i, x_i) * \gamma$, $w(\chi_i, x_i) * w(\chi_j, x_j) = \chi_i(g_j)w(\chi_j, x_j) * w(\chi_i, x_i)$ for $i \neq j$, and $w(\chi_i, x_i)^{r_i} = 0$.

From now on, let $V \in_{\Gamma}^{\Gamma} \mathcal{YD}$, $V = Kv_1 \oplus Kv_2$ be a quantum linear space of dimension 2, with $v_i \in V_{g_i}^{\chi_i}$ and with $g_1 = g, \chi_1 = \chi, \chi_2 = \chi^m$. Also, for $i = 1, 2$,

1. $\chi_i(g_i)$ is a primitive rth root of unity for some odd integer $r > 1$.
2. χ_i has order r in the group $\hat{\Gamma}$, so that $(m, r) = 1$.

3. $g_i^r \neq 1$.

Let q denote the primitive rth root of unity $\chi(g)$. Let $H = \mathcal{B}(V)\#K\Gamma$ and let A be the nontrivial lifting of H with matrix $\mathcal{A} = \begin{bmatrix} 1 & \nu \\ -q^{-m}\nu & 1 \end{bmatrix}$. In A, let x be the lifting of v_1 and y the lifting of v_2. Then A is the pointed Hopf algebra generated by its group of grouplikes Γ and the skew-primitives x and y where for $h \in \Gamma$,

$$\begin{aligned} hx &= \chi(h)xh, \text{ and } hy = \chi^m(h)yh, && (1) \\ x^r &= g^r - 1, \text{ and } y^r = g_2^r - 1, && (2) \\ xy &= \chi^m(g)yx + \nu(g_1 g_2 - 1). && (3) \end{aligned}$$

If $\nu \neq 0$, then $m = -1$, i.e., $\chi_2 = \chi^{-1}$.

As a K-space, A has basis $\{gz | g \in \Gamma, z \in \mathcal{Z}\}$ where $\mathcal{Z} = \{y^i x^j | 0 \leq i, j \leq r-1\}$. Then as a K-space, A^* has basis $\mathcal{W} = \{w(\gamma, z) | \gamma \in \hat{\Gamma}, z \in \mathcal{Z}\}$.

Definition 3.2 $C(1)$ *is defined to be the sub Hopf algebra of* A^* *of dimension* r^3 *with basis* $\{w(\chi^i, z) | 0 \leq i \leq r-1, z \in \mathcal{Z}\}$.

Note that Lemma 3.1 and the fact that $\triangle w(\gamma, y^j x^i)$ is a linear combination of terms of the form $w(\gamma, y^t x^s) \otimes w(\gamma\chi^{-mt-s}, y^n x^l)$ for some $0 \leq t, s, n, l \leq r-1$, guarantees that $C(1)$ is a sub Hopf algebra of A^*. Also we see that $\gamma C(1)$ is the subcoalgebra of A^* with basis $\mathcal{W}_\gamma = \{w(\gamma\chi^k, y^j x^l) | 0 \leq k, j, l \leq r-1\}$. From Lemma 3.1, we see that as a coalgebra, A^* is the direct sum of its subcoalgebras $\gamma C(1)$ where $\gamma \in \overline{\gamma} \in \hat{\Gamma}/ < \chi >$.

Also we note that the only elements w in the basis $\mathcal{W}_\gamma$ for $\gamma C(1)$ such that a scalar multiple of $w \otimes w$ is a summand of $\triangle v$ for some $v \in \mathcal{W}_\gamma$ are those of the form $w = w(\gamma\chi^k, y^j x^l)$, where $y^j x^l \in C_A(\Gamma) \cap \mathcal{Z}$, i.e., $y^j x^l \in \mathcal{Z}$ and commutes with all elements of Γ.

Now let $e(1,1)$ be a matrix coalgebra basis element for a matrix coalgebra in $\gamma C(1)$. Since $e(1,1) \otimes e(1,1)$ is a summand of $\triangle e(1,1)$, then $e(1,1)$ is a linear combination of basis elements $w(\gamma\chi^k, z)$, where $z \in C_A(\Gamma) \cap \mathcal{Z}$. Since terms of the form $\gamma\chi^k \otimes \gamma\chi^l, k \neq l$, do not occur in $\triangle(w)$ for any $w \in \mathcal{W}_\gamma$, then exactly one element of $\hat{\Gamma}$ is a summand of $e(1,1)$, say $\gamma\chi^k$. Then

$$e(1,1) = \gamma\chi^k + \sum_{1 \neq z \in C_A(\Gamma) \cap \mathcal{Z}} a_z w(\gamma\chi^{k_z}, z),$$

where the $a_z \in K$. Also since $\triangle a_z w(\gamma\chi^{k_z}, z)$ contains the summand $\gamma\chi^{k_z} \otimes a_z w(\gamma\chi^{k_z}, z)$ and does not contain the summand $\gamma\chi^k \otimes a_z w(\gamma\chi^{k_z}, z)$ for $k \neq k_z$, we must have $k_z = k$ for all z.

Theorem 3.3 *For A with lifting matrix $\mathcal{A} = \begin{bmatrix} 1 & \nu \\ -q^{-m}\nu & 1 \end{bmatrix}$ as above, let $\gamma \in \hat{\Gamma}$ with $\gamma(g_i^r - 1) \neq 0$ for $i = 1, 2$. Then $\gamma C(1) \cong \oplus_{j=1}^r \mathcal{M}^c(r, K)$, the sum of r matrix coalgebras of dimension r^2, except possibly when ν belongs to a finite set of nonzero scalars.*

Proof. First we show that the matrix coalgebras in $\gamma C(1)$ have dimension at least r^2. This is essentially the argument from the proof of Theorem 3.6 in [3] and depends on the fact that $\gamma(g^r - 1) \neq 0$.

Suppose $e(1,1)$ as above is a matrix coalgebra basis element. Since $x^r = g^r - 1$ and $\gamma\chi^k(g^r - 1) = \gamma(g^r - 1) \neq 0$, then $\triangle(\gamma\chi^k)$ contains a summand $\xi w(\gamma\chi^k, x^j) \otimes w(\gamma\chi^k\chi^{-j}, x^{r-j})$ for some $\xi \in K^*$, all $1 \leq j \leq r-1$. Thus $\xi' w(\gamma\chi^k\chi^{-j}, x^{r-j})$ is a summand of $e(l, 1)$ for some l, some $\xi' \in K^*$. But then $\gamma\chi^{k-j}$ is a summand of $e(l,l)$. Thus the matrix coalgebra contains at least r matrix coalgebra basis elements $e(l,l)$ and so has dimension at least r^2.

Now we find a family of right A-modules arising from matrix coalgebras in $\gamma C(1)$.

Let $E = \oplus_{i=1}^r K e_i$ and, with the understanding that subscripts on the e_i are taken modulo r, we define a right A-action on E by

$$\begin{aligned} e_i \leftharpoonup h &= (\gamma\chi^{i-1})(h) e_i; && (4)\\ e_i \leftharpoonup y &= e_{i-m} \text{ for } 1 \leq i \leq r-1 \text{ and } e_r \leftharpoonup y = \gamma(g_2^r - 1) e_{r-m}; && (5)\\ e_i \leftharpoonup x &= \eta_{i-1} e_{i-1} \text{ for } 1 \leq i \leq r, \eta_i \in K^*. && (6) \end{aligned}$$

Clearly $e_i \leftharpoonup y^r = e_i \leftharpoonup (g_2^r - 1)$, and also we have that

$$\begin{aligned} e_i \leftharpoonup (hx - \chi(h)xh) &= ((\gamma\chi^{i-1})(h) - \chi(h)(\gamma\chi^{i-2})(h)) e_i \leftharpoonup x = 0;\\ e_i \leftharpoonup (hy - \chi^m(h)yh) &= ((\gamma\chi^{i-1})(h) - \chi^m(h)(\gamma\chi^{i-m-1})(h)) e_i \leftharpoonup y = 0. \end{aligned}$$

Furthermore, we require that

$$\prod_{i=1}^r \eta_i = \gamma(g^r - 1) \neq 0, \tag{7}$$

and this condition ensures that $e_i \leftharpoonup x^r = e_i \leftharpoonup (g^r - 1)$. Thus the action respects relations (1) and (2). As well, in order for the action to respect (3), we must have

$$\begin{aligned} \eta_r \gamma(g_2^r - 1) - q^m \eta_{r-m} &= \nu\gamma(g_1 g_2 - 1), && (8)\\ \eta_{i-1} - q^m \eta_{i-1-m} &= \nu\gamma\chi^{i-1}(g_1 g_2 - 1) \text{ for } 2 \leq i \leq r-1, && (9)\\ \eta_{r-1} - q^m \gamma(g_2^r - 1)\eta_{r-m-1} &= \nu\gamma\chi^{r-1}(g_1 g_2 - 1). && (10) \end{aligned}$$

If $\nu = 0$, so that the right hand sides of equations (8),(9) and (10) are zero, then clearly this system of equations is consistent, each η_i, $2 \leq i \leq r$, is a nonzero multiple of η_1 and (7) becomes $\eta_1^r = \zeta$ for $\zeta \in K^*$, an equation with r distinct solutions in K (since K is algebraically closed).

If $\nu \neq 0$, so that $m = -1$, from equations (8) - (10), we obtain

$$\begin{aligned}\eta_r &= (q^{-1}\eta_1 + \nu\gamma(g_1g_2 - 1))/\gamma(g_2^r - 1)\\ \eta_{r-1} &= q^{-1}(q^{-1}\eta_1 + \nu\gamma(g_1g_2 - 1)) + \nu\gamma\chi^{-1}(g_1g_2 - 1)\\ &= q^{-2}\eta_1 + \nu\gamma(q^{-1} + \chi^{-1})(g_1g_2 - 1),\end{aligned}$$

and, in general, for $1 \leq t \leq r - 1$,

$$\eta_{r-t} = q^{-(t+1)}\eta_1 + \nu\gamma(q^{-t} + q^{-t+1}\chi^{-1} + \ldots + \chi^{-t})(g_1g_2 - 1).$$

Then equation (7) is again an equation of degree r in $\eta = \eta_1$ and thus, since K is algebraically closed, has a solution.

Thus E is a right A-module. Also, from the action of Γ on A, we see that any simple submodule of E must be isomorphic to a right A-module $\oplus_j Ke(j,k)$ where $e(i,l)$ are a basis for a matrix coalgebra in $\gamma C(1)$. But we have already shown that such a matrix coalgebra has dimension at least r^2 and so E is a simple right A-module.

Now we show that any two distinct solutions to equation (7) yield nonisomorphic modules. Let η and η' be two different solutions, and E, E' the corresponding modules.

Since Ke_i is the eigenspace for $\gamma\chi^{i-1}(g) = q^{i-1}\gamma(g)$ under the action of g, any isomorphism ϕ from E to E' must map e_i to $\alpha_i e_i'$ for some $\alpha_i \in K^*$. But then for $j < r$, $\phi(e_j \leftharpoonup y) = \phi(e_{j-m}) = \alpha_{j-m}e'_{j-m}$ and $\phi(e_j) \leftharpoonup y = \alpha_j e_j' \leftharpoonup y = \alpha_j e'_{j-m}$. Also $\phi(e_r \leftharpoonup y) = \phi(\gamma(g_2^r - 1)e_{r-m}) = \alpha_{r-m}\gamma(g_2^r - 1)e'_{r-m}$ while $\phi(e_r) \leftharpoonup y = \alpha_r e_r' \leftharpoonup y = \alpha_r\gamma(g_2^r - 1)e'_{r-m}$, so for all i, j, $\alpha_j = \alpha_i = \alpha$. Also $\phi(e_2 \leftharpoonup x) = \eta_1\alpha e_1'$ and $\phi(e_2) \leftharpoonup x = \alpha\eta_1' e_1'$, which is impossible unless $\eta_1 = \eta_1'$.

Finally we prove that for ν different from a finite set of nonzero scalars, equation (7) has r distinct solutions, and so $\gamma C(1)$ is the direct sum of r matrix coalgebras, each of dimension r^2.

Now suppose $\nu \neq 0$. Let $s_t = \gamma(q^{-t} + q^{-t+1}\chi^{-1} + \ldots + \chi^{-t})(g_1g_2 - 1)$, for $0 \leq t \leq r - 1$; note that $s_{r-1} = 0$.

Then writing (7) as a polynomial equation in the variable X, we have

$$\prod_{i=o}^{r-1}(q^iX + \nu s_{r-i-1}) - \gamma(g^r - 1)\gamma(g_2^r - 1) = 0.$$

Then, with $X = \nu Y$, (7) can be written as

$$g(Y) = Y^r + a_{r-1}Y^{r-1} + \ldots + a_1 Y + a_0 = 0,$$

where $a_0 = -\gamma(g^r - 1)\gamma(g_2^r - 1)/\nu^r$ and the scalar ν does not appear in any other coefficient a_i. If $g(Y)$ has a repeated root z then z is also a zero of $g'(Y) = 0$. Note that $g'(Y)$ is a polynomial whose coefficients do not depend upon ν. Let $\mathcal{Z}$ be the finite set of roots of $g'(Y)$. If $z \in \mathcal{Z}$ is also a zero of $g(Y)$ then $\nu^r = -\gamma(g^r - 1)\gamma(g_2^r - 1)/(z^r + a_{r-1}z^{r-1} + \ldots + a_1 z)$. If we choose ν so that this equation does not hold for any $z \in \mathcal{Z}$, then (7) has r distinct zeros. ∎

Proposition 3.4 *For V as above, suppose A is a lifting of $\mathcal{B}(V)\#K[\Gamma]$ with lifting matrix $\mathcal{A} = \begin{bmatrix} 1 & 1 \\ -q & 0 \end{bmatrix}$. Let $\gamma \in \hat{\Gamma}$ with $\gamma(g^r - 1) \neq 0$. Then $\gamma C(1) \cong \oplus_{j=1}^r \mathcal{M}^c(r, K)$, the sum of r matrix coalgebras of dimension r^2.*

Proof. Note that here $m = -1$. Also since $y^r = 0$, we may assume that if ν is nonzero then $\nu = 1$. The proof that any matrix coalgebra in $\gamma C(1)$ has dimension at least r^2 still holds and an A-module E may still be constructed using conditions (4) to (10) with $\gamma(g_2^r - 1)$ replaced by 0. However, here the equations (8) to (10) determine $\eta_1, \ldots, \eta_{r-1}$ and equation (7) is a linear equation in η_r, so that we obtain one simple A-module of dimension r in this way replacing γ in (8) - (10) by each of the r elements of $\gamma < \chi >$. Suppose two of these modules are isomorphic, say $E = \oplus_{i=1}^r Ke_i$, constructed using γ and $F = \oplus_{i=1}^r Kf_i$, constructed using $\gamma\chi^s$ with $s \neq 0$. Under the right action of g, Ke_i and Kf_{i-s} are the eigenspaces for $\gamma\chi^{i-1}(g)$. Thus any isomorphism from E to F must map e_i to $\alpha_i f_{i-s}$ for some nonzero scalar α_i. But $e_1 \leftharpoonup y^{r-1} = e_r$ while $f_{1-s} \leftharpoonup y^{r-1} = 0$. ∎

To describe the coradical of $C(1)$, note that the sub Hopf algebra $C(1)$ of A^* can be described in the following way. Let J be the Hopf ideal in A generated by $g_i^r - 1$ for $i = 1, 2$ and let $A' = A/J$ be the quotient Hopf algebra. In A', $x'^r = 0 = y'^r$ where x', y' are the images of x and y in A'. Then the dual of A' is isomorphic to a sub Hopf algebra of A^* and $C(1)$ is a sub Hopf algebra of A'^*. The structure of the sub Hopf algebra $C(1)$ of A'^* is different depending on whether ν is zero or nonzero.

If $\nu = 0$, then $C(1)$ is a Radford biproduct. Let $\overline{\Gamma}$ be the quotient of Γ by the subgroup generated by g_1^r, g_2^r and it was noted in [3] that $\hat{\overline{\Gamma}} \cong G(A^*) = \{\gamma \in \hat{\Gamma} | \gamma(g_1^r) = 1 = \gamma(g_2^r)\}$. The quotient Hopf algebra A' is isomorphic to the bicrossed product $\mathcal{B}(V)\#K[\overline{\Gamma}]$, so that A'^* is isomorphic to the bicrossed product $\mathcal{B}(W)\#K[G(A^*)]$ where $W = Kw_1 \oplus Kw_2 \in^{\hat{\Gamma}}_{\hat{\Gamma}} \mathcal{YD}$ with $w_i \in W^{g_i}_{\chi_i}$. Since $\chi \in G(A^*)$, then $\mathcal{B}(W)\#K[< \chi >]$ is a sub Hopf algebra of A'^* and this is $C(1)$. Thus $C(1)$ is pointed with coradical $K[< \chi >]$. More details can be found in [3].

If $\nu \neq 0$, then $C(1)$ is a Frobenius Lusztig kernel of dimension r^3. A' is isomorphic to the nontrivial lifting of $\mathcal{B}(V)\#K[\overline{\Gamma}]$ with lifting matrix $\mathcal{A} = \begin{bmatrix} 0 & 1 \\ -q & 0 \end{bmatrix}$. Again, $< \chi > \subseteq \hat{\overline{\Gamma}} = \{\gamma \in \hat{\Gamma} | \gamma(g_1^r) = 1 = \gamma(g_2^r)\}$ but here $G(C(1)) = \{\epsilon\}$ since for $i \neq 0$, $\chi^i(g_1g_2) = q^{2i} \neq 1$ since $(r, 2) = 1$.

The representation theory for the Frobenius Lusztig kernels is known; however we make the following remark since the construction in the proof of Theorem 3.3 still is valid here.

Remark 3.5 Suppose that A has lifting matrix $\mathcal{A} = \begin{bmatrix} 1 & \nu \\ -q^{-m}\nu & 1 \end{bmatrix}$ but γ is such that $\gamma(g_i^r - 1) = 0$, for example, we could have $\gamma = \chi$. Then the argument that the simple modules must have dimension greater than or equal to r no longer holds. However, we still may define a right A-module E as in the proof of Theorem 3.3, where equations (4)-(10) hold with $\gamma(g_i^r - 1)$ replaced by 0. Suppose $\gamma \in < \chi >$ and $\nu \neq 0$. The various choices of γ will yield simple submodules of E of dimensions $1, \ldots, r$.
For example, if $\gamma = \epsilon$, then by (8), $\eta_1 = 0$ and then by (10), $\eta_{r-1} = \nu\chi^{r-1}(g_1g_2 - 1) = \nu(q^{-2} - 1)$, then by (9), $\eta_{r-2} = q^{-1}\eta_{r-1} + \nu\chi^{r-2}(g^2 - 1) = \nu(q^{-3} - q^{-1} + q^{-4} - 1)$, and then $\eta_{r-3} = \nu(q^{-4} - q^{-2} + q^{-5} - q^{-1} + q^{-6} - 1)$, etc. Thus $\eta_2, \ldots, \eta_{r-1}$ are nonzero and the submodule $Ke_2 \oplus \ldots \oplus Ke_r$ of E is a simple right $(r-1)$-dimensional A-module.
If $\eta_2 = 0$, and $\eta_3, \ldots, \eta_{r-1}$ are nonzero, then $Ke_3 \oplus \ldots \oplus Ke_r$ is a simple right $(r-2)$-dimensional A-module, etc. An example of this construction where $C(1)$ has dimension 27 and has coradical $\mathcal{M}^c(K, 3) \oplus \mathcal{M}^c(K, 2) \oplus K\epsilon$ appears in the next section. ∎

4 Examples of dimension 81

In the remainder of this note, we let $\Gamma = C_9 =< c >, \hat{\Gamma} =< c^* >, V = Kv_1 \oplus Kv_2$ with $v_1 \in V_c^{c^{*3}}, v_2 \in V_c^{c^{*6}}$. Let λ denote c^{*3} and let $q = \lambda(c)$, a primitive cube root of unity. Let ν be a nonzero scalar. Let A be the lifting of $\mathcal{B}(V)\#K[\Gamma]$ with lifting matrix $\mathcal{A} = \begin{bmatrix} 1 & \nu \\ -q\nu & \alpha_{2,2} \end{bmatrix}$ and we explicitly determine the matrix coalgebras in each of the subcoalgebras $c^{*i}C(1)$, $i = 1, 2$. Noice that here $G(A^*) = \{\epsilon\}$. D. Stefan has remarked that A cannot satisfy the Chevalley property. Otherwise the coradical of A^* would be a semisimple Hopf algebra of dimension 3^i, which, then, by a theorem of Masuoka, would have a nontrivial grouplike element. The irreducible representations of the sub Hopf algebra $C(1)$ are known [11], but the structure of the coradical of $C(1)$ also follows easily from our computations, and so we describe it as well.

By Theorem 3.3, or Proposition 3.4 the coalgebra $c^{*i}C(1)$, $1 \leq i \leq 2$, is the sum of three 9-dimensional matrix coalgebras except possibly when ν lies in a finite set of nonzero scalars. We find these coalgebras explicitly in terms of the basis $\mathcal{W}$ of A^*, $\mathcal{W} = \{w(\gamma, z) | \gamma \in \hat{\Gamma}, z \in \mathcal{Z}\}$, $\mathcal{Z} = \{y^i x^j | 0 \leq i, j \leq 2\}$.

4.1 Lifting matrix $\mathcal{A}$ with two nonzero diagonal elements

In this section, we let A be the lifting of $\mathcal{B}(V)\#K[\Gamma]$ with lifting matrix $\mathcal{A} = \begin{bmatrix} 1 & \nu \\ -q\nu & 1 \end{bmatrix}$.

Let $\gamma \in \hat{\Gamma}$. From the discussion in the previous section, some matrix coalgebra basis element say $e(1,1)$ in $\gamma C(1)$ has the form $e(1,1) = \gamma + aw(\gamma, yx) + bw(\gamma, y^2x^2)$ and noting that $\gamma(c^3 - 1) = \gamma\lambda^i(c^3 - 1)$ since $\lambda(c) = q$, we compute

$$\begin{aligned}
& \triangle(\gamma + aw(\gamma, yx) + bw(\gamma, y^2x^2)) \qquad (11) \\
= \; & [\gamma \otimes \gamma + \nu\gamma(c^2 - 1)w(\gamma, x) \otimes w(\gamma\lambda^2, y) \\
+ \; & \gamma(c^3 - 1)w(\gamma, x) \otimes w(\gamma\lambda^2, x^2) \\
+ \; & \gamma(c^3 - 1)w(\gamma, y) \otimes w(\gamma\lambda, y^2) + \gamma(c^3 - 1)w(\gamma, x^2) \otimes w(\gamma\lambda, x) \\
+ \; & \nu^2\gamma(q - c^2)\gamma(c^2 - 1)w(\gamma, x^2) \otimes w(\gamma\lambda, y^2) \\
+ \; & \nu\gamma(q - c^2)\gamma(c^3 - 1)w(\gamma, x^2) \otimes w(\gamma\lambda, yx^2) \\
+ \; & q\gamma(c^3 - 1)^2 w(\gamma, yx) \otimes w(\gamma, y^2x^2) + \gamma(c^3 - 1)w(\gamma, y^2) \otimes w(\gamma\lambda^2, y) \\
+ \; & q^2\gamma(c^3 - 1)^2 w(\gamma, yx^2) \otimes w(\gamma\lambda^2, y^2x) \\
+ \; & \nu\gamma(c^3 - 1)\gamma(q - c^2)w(\gamma, y^2x) \otimes w(\gamma\lambda, y^2) \\
+ \; & q^2\gamma(c^3 - 1)^2 w(\gamma, y^2x) \otimes w(\gamma\lambda, yx^2) \\
+ \; & q\gamma(c^3 - 1)^2 w(\gamma, y^2x^2) \otimes w(\gamma, yx) \\
+ \; & q\nu\gamma(c^2q - 1)\gamma(c^3 - 1)^2 w(\gamma, y^2x^2) \otimes w(\gamma, y^2x^2)] \\
\\
+ \; & a[\gamma \otimes w(\gamma, yx) + w(\gamma, yx) \otimes \gamma + q^2 w(\gamma, x) \otimes w(\gamma\lambda^2, y)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad q\nu\gamma(1-c^2q)w(\gamma,x)\otimes w(\gamma\lambda^2,y^2x)+w(\gamma,y)\otimes w(\gamma\lambda,x)\\
&+ \quad q\nu\gamma(c^2-1)w(\gamma,x^2)\otimes w(\gamma\lambda,y^2)+q\gamma(c^3-1)w(\gamma,x^2)\otimes w(\gamma\lambda,yx^2)\\
&+ \quad \nu\gamma(c^2q^2-1)w(\gamma,yx)\otimes w(\gamma,yx)+\gamma(c^3-1)w(\gamma,y^2)\otimes w(\gamma\lambda^2,y^2x)\\
&+ \quad q\nu\gamma(1-c^2q)w(\gamma,yx^2)\otimes w(\gamma\lambda^2,y)+\gamma(c^3-1)w(\gamma,yx^2)\otimes w(\gamma\lambda^2,x^2)\\
&+ \quad \nu^2\gamma(1-c^2q)\gamma(c^2-q)w(\gamma,yx^2)\otimes w(\gamma\lambda^2,y^2x)\\
&+ \quad q\gamma(c^3-1)w(\gamma,y^2x)\otimes w(\gamma\lambda,y^2)\\
&+ \quad q^2\gamma(c^3-1)^2w(\gamma,y^2x^2)\otimes w(\gamma,y^2x^2)]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad b[\gamma\otimes w(\gamma,y^2x^2)+w(\gamma,y^2x^2)\otimes\gamma+qw(\gamma,x)\otimes w(\gamma\lambda^2,y^2x)\\
&+ \quad w(\gamma,y)\otimes w(\gamma\lambda,yx^2)+q^2w(\gamma,x^2)\otimes w(\gamma\lambda,y^2)\\
&+ \quad q^2w(\gamma,yx)\otimes w(\gamma,yx)\\
&+ \quad q\nu\gamma(1-c^2)w(\gamma,yx)\otimes w(\gamma,y^2x^2)+w(\gamma,y^2)\otimes w(\gamma\lambda^2,x^2)\\
&+ \quad qw(\gamma,yx^2)\otimes w(\gamma\lambda^2,y)+\nu\gamma(c^2-q)w(\gamma,yx^2)\otimes w(\gamma\lambda^2,y^2x)\\
&+ \quad w(\gamma,y^2x)\otimes w(\gamma\lambda,x)+\nu\gamma(c^2q-1)w(\gamma,y^2x)\otimes w(\gamma\lambda,yx^2)\\
&+ \quad q\nu\gamma(1-c^2)w(\gamma,y^2x^2)\otimes w(\gamma,yx)\\
&+ \quad q\nu^2\gamma(1-c^2)\gamma(c^2q-1)w(\gamma,y^2x^2)\otimes w(\gamma,y^2x^2)].
\end{aligned}
$$

From the first two paragraphs of the proof of Theorem 3.3, we recall that γ is a summand of the matrix coalgebra basis element $e(j,j)$ only when $j=1$. But then since for $i=1,2$, we have $\gamma\otimes aw(\gamma,y^iz^i)$ a summand of $\triangle aw(\gamma,y^iz^i)$, then $aw(\gamma,y^iz^i)$ cannot be a summand of $e(j,1), j\neq 1$. Then no scalar multiple of $w(\gamma,y^ix^i)\otimes w(\gamma,y^ix^i)$ can occur in $e(1,j)\otimes e(j,1)$ unless $j=1$. Thus:

$$
\begin{aligned}
a^2 &= a\nu\gamma(c^2q^2-1)+bq^2; && (12)\\
b^2 &= (q\nu\gamma(c^2q-1)+aq^2)\gamma(c^3-1)^2+bq\nu^2\gamma(1-c^2)\gamma(c^2q-1), && (13)\\
ab &= q\gamma(c^3-1)^2+bq\nu\gamma(1-c^2). && (14)
\end{aligned}
$$

Then from equation (12) and equation (14) we have

$$
\begin{aligned}
b &= (a^2-a\nu\gamma(c^2q^2-1))q && (15)\\
b(a-q\nu\gamma(1-c^2)) &= q\gamma(c^3-1)^2. && (16)
\end{aligned}
$$

so that

$$a(a-\nu\gamma(c^2q^2-1))(a-q\nu\gamma(1-c^2))=\gamma(c^3-1)^2. \qquad (17)$$

Equation (17) is consistent with equation (13) so the system has solutions a and b. If $\gamma(c^3-1)\neq 0$ then both a and b are nonzero. (Note that equation (17) is just equation (7) with $r=3$, $a=\eta_1$, $\chi=\lambda$.)

4.1.1 Case: $\gamma = c^*$ or c^{*2}.

Now let $\gamma = c^{*i}$ with $i = 1$ or 2, let a be a solution to equation (17) and let b be defined by equation (15). Let $e(1,1) = \gamma + aw(\gamma, yx) + bw(\gamma, y^2x^2)$ as above.

Then define

$$e(2,1) = e(1,1) \leftharpoonup y = \gamma(c^3-1)w(\gamma\lambda, y^2) + aw(\gamma\lambda, x) + bw(\gamma\lambda, yx^2) \tag{18}$$

and

$$\begin{aligned} e(3,1) &= e(2,1) \leftharpoonup y = e(1,1) \leftharpoonup y^2 \\ &= \gamma(c^3-1)w(\gamma\lambda^2, y) + a\gamma(c^3-1)w(\gamma\lambda^2, y^2x) + bw(\gamma\lambda^2, x^2). \end{aligned} \tag{19}$$

Then

$$\begin{aligned} e(1,1) \leftharpoonup y^3 &= e(3,1) \leftharpoonup y \\ &= \gamma(c^3-1)\gamma + a\gamma(c^3-1)w(\gamma, yx) + b\gamma(c^3-1)w(\gamma, y^2x^2) \\ &= \gamma(c^3-1)e(1,1) = e(1,1) \leftharpoonup (c^3-1). \end{aligned}$$

Thus $e(i,1) \leftharpoonup y^3 = e(i,1) \leftharpoonup (c^3-1)$ for $1 \leq i \leq 3$.

Now from equation (11), we see that $\triangle e(1,1) = \sum_{i=1}^3 e(1,j) \otimes e(j,1)$ where

$$\begin{aligned} e(1,2) &= \alpha w(\gamma, x^2) + w(\gamma, y) + \frac{b}{a}w(\gamma, y^2x), \\ e(1,3) &= \beta w(\gamma, x) + w(\gamma, y^2) + a\beta w(\gamma, yx^2) \end{aligned}$$

with $\beta = \frac{\gamma(c^3-1)}{b}$, and $\alpha = \frac{\gamma(c^3-1)}{a}$. Now define

$$\begin{aligned} e(2,2) &= \gamma\lambda + \frac{b}{a}w(\gamma\lambda, yx) + \alpha\gamma(c^3-1)w(\gamma\lambda, y^2x^2), \\ e(2,3) &= w(\gamma\lambda, y) + a\beta w(\gamma\lambda, x^2) + \beta\gamma(c^3-1)w(\gamma\lambda, y^2x), \\ e(3,2) &= \frac{b}{a}w(\gamma\lambda^2, x) + \gamma(c^3-1)w(\gamma\lambda^2, y^2) + \alpha\gamma(c^3-1)w(\gamma\lambda^2, yx^2), \\ e(3,3) &= \gamma\lambda^2 + \beta\gamma(c^3-1)w(\gamma\lambda^2, yx) + a\beta\gamma(c^3-1)w(\gamma\lambda^2, y^2x^2). \end{aligned}$$

It is tedious but straightforward to check that the $e(i,j)$ are a matrix coalgebra basis. The 3-dimensional spaces $V_j = \oplus_{i=1}^3 Ke(i,j)$ are irreducible right A-modules. In general, the right action of $g \in \Gamma$ or y on V_j is given by

$$e(i,j) \leftharpoonup g = (\lambda^{i-1}\gamma)(g)e(i,j);$$

$$e(1,j) \leftharpoondown y = e(2,j),\ e(2,j) \leftharpoondown y = e(3,j),\ e(3,j) \leftharpoondown y = \gamma(c^3-1)e(1,j);$$

We compute the action of x on $e(1,1)$, $e(2,1)$ and $e(3,1)$.

From equation (11) we see that

$$\begin{aligned}
e(1,1) \leftharpoondown x &= (\nu\gamma(c^2-1)w(\gamma\lambda^2, y) + \gamma(c^3-1)w(\gamma\lambda^2, x^2)) \\
&+ a(q^2 w(\gamma\lambda^2, y) + q\nu\gamma(1-c^2q)w(\gamma\lambda^2, y^2x)) + bqw(\gamma\lambda^2, y^2x) \\
&= (\nu\gamma(c^2-1) + aq^2)w(\gamma\lambda^2, y) \\
&+ (aq\nu\gamma(1-c^2q) + bq)w(\gamma\lambda^2, y^2x) + \gamma(c^3-1)w(\gamma\lambda^2, x^2).
\end{aligned}$$

We see that $e(1,1) \leftharpoondown x = \beta e(3,1)$ if and only if

$$\nu\gamma(c^2-1) + aq^2 = \beta\gamma(c^3-1) = \frac{\gamma(c^3-1)^2}{b} \tag{20}$$

$$\text{and } aq\nu\gamma(1-c^2q) + bq = \beta a\gamma(c^3-1) = \frac{a\gamma(c^3-1)^2}{b}. \tag{21}$$

Now, (20) holds iff

$\nu\gamma(c^2-1)b + abq^2 = \gamma(c^3-1)^2$ iff
$\nu\gamma(c^2-1)b + \gamma(c^3-1)^2 + b\nu\gamma(1-c^2) = \gamma(c^3-1)^2$ by equation (14),

and this last equation clearly holds. Also, equation (21) holds if and only if

$$abq\nu\gamma(1-c^2q) + b^2q = a\gamma(c^3-1)^2$$

and using (13), we see that this equation reduces to equation (14). Thus $e(1,1) \leftharpoondown x = \beta e(3,1)$.

Similar computations now show that

$$e(1,1) \leftharpoondown x^2 = \beta e(3,1) \leftharpoondown x = \alpha e(2,1), \text{ and}$$

$$e(1,1) \leftharpoondown x^3 = \beta e(3,1) \leftharpoondown x^2 = \alpha e(2,1) \leftharpoondown x = \alpha a e(1,1) = \gamma(c^3-1)e(1,1),$$

and that, in general, the right action of x on the $e(i,j)$ is given by

$$e(1,j) \leftharpoondown x = \beta e(3,j),\ e(2,j) \leftharpoondown x = ae(1,j),\ e(3,j) \leftharpoondown x = \frac{b}{a}e(2,j).$$

This action preserves the relations (2) and (3) since,

$$\begin{aligned}
e(i,j) \leftharpoondown y^3 &= \gamma(c^3-1)e(i,j) = e(i,j) \leftharpoondown (c^3-1); \\
e(i,j) \leftharpoondown x^3 &= \left(\frac{\gamma(c^3-1)}{b}\right)(a)(\frac{b}{a})e(i,j) = e(i,j) \leftharpoondown (c^3-1),
\end{aligned}$$

and also denoting $xy - q^2yx$ by z and using (12),(13), (14), we have

$$\begin{aligned} e(1,j) \leftharpoondown z &= \left(\frac{\gamma(c^3-1)^2}{b} - q^2a\right) e(1,j) = \nu\gamma(c^2-1)e(1,j), \\ e(2,j) \leftharpoondown z &= \left(a - q^2\frac{b}{a}\right) e(2,j) = \nu(\gamma\lambda)(c^2-1)e(2,j), \\ e(3,j) \leftharpoondown z &= \left(\frac{b}{a} - q^2\frac{\gamma(c^3-1)^2}{b}\right) e(3,j) = \nu(\gamma\lambda^2)(c^2-1)e(3,j). \end{aligned}$$

We note that similarly, $U_1 = \oplus_{i=1}^3 Ke(1,i)$ is an irreducible left A-module. The left action of Γ is given by $h \rightharpoonup e(1,j) = \gamma\lambda^{j-1}(h)e(1,j)$. The left action of x is given by

$$x \rightharpoonup e(1,1) = ae(1,2);\ x \rightharpoonup e(1,2) = \frac{b}{a}e(1,3);\ x \rightharpoonup e(1,3) = \beta e(1,1),$$

and the left action of y is given by

$$y \rightharpoonup e(1,3) = e(1,2);\ y \rightharpoonup e(1,2) = e(1,1);\ y \rightharpoonup e(1,1) = \gamma(c^3-1)e(1,3).$$

Finally we consider the number of solutions to equation (17),

$$X(X - \nu\gamma(c^2q^2-1))(X - q\nu\gamma(1-c^2)) - \gamma(c^3-1)^2 = 0,$$

or equivalently,

$$Y^3 + \omega Y^2 + \theta Y = \gamma(c^3-1)^2/\nu^3$$

where $X = \nu Y$, $\omega = -\gamma(c^2q^2-1) - q\gamma(1-c^2)$,and $\theta = q\gamma(c^2q^2-1)\gamma(1-c^2)$. Then the solutions to $3Y^2 + 2\omega Y + \theta = 0$ are $\mathcal{Z} = \{(-\omega \pm \sqrt{\omega^2 - 3\theta})/3\}$, and in order to guarantee that $\gamma C(1)$ is $\oplus_{i=1}^3 \mathcal{M}^c(3,K)$, we should choose ν so that $\gamma(c^3-1)^2 \neq \nu^3(z^3 + \omega z^2 + \theta z)$ for either $z \in \mathcal{Z}$, i.e. 6 possible choices for ν are excluded.

Note that if $\gamma = c^*$, then $\omega^2 - 3\theta = -3\zeta^2q^2$ and if $\gamma = c^{*2}$, then $\omega^2 - 3\theta = -3\zeta$ where $\zeta = c^*(c)$ is a primitive 9th root of unity with $\zeta^3 = q$. Thus (17) always has at least 2 distinct solutions.

4.1.2 Case: $\gamma \in <\lambda>$.

Now consider the matrix coalgebras in $C(1)$, as described in Remark 3.5. For $\gamma \in <\lambda>$, $\gamma(c^3-1) = 0$ so that equations (13), (14) are different, and we have

$$\begin{aligned} a^2 &= a\nu\gamma(c^2q^2-1) + bq^2; \\ b^2 &= bq\nu^2\gamma(1-c^2)\gamma(c^2q-1); \\ ab &= b\nu q\gamma(1-c^2). \end{aligned}$$

If $\gamma = \epsilon$, then $b = 0$ and then either $a = 0$ or $a = \nu(q^2 - 1)$. If $\gamma = \lambda$, then $b = 0$ and then either $a = 0$ or $a = \nu(q - 1)$. If $\gamma = \lambda^2$, then $a^2 = bq^2$ and then either $a = b = 0$ or $a = \nu q(1 - q)$ and $b = \nu^2 q(1 - q)(q^2 - 1)$.

Then we obtain the 1 dimensional coalgebra $K \cdot \epsilon$, a 9-dimensional coalgebra E with basis $e(i, j)$, and a 4-dimensional F coalgebra with basis $f(i, j)$ where

$$\begin{aligned}
e(1,1) &= \lambda^2 + \nu q(1-q)w(\lambda^2, yx) + \nu^2(1-q)^2 w(\lambda^2, y^2x^2),\\
e(1,2) &= w(\lambda^2, y) + \nu(q^2-1)w(\lambda^2, y^2x);\\
e(1,3) &= w(\lambda^2, y^2);\\
e(2,1) &= \nu q(1-q)w(\epsilon, x) + \nu^2(1-q)^2 w(\epsilon, yx^2);\\
e(2,2) &= \epsilon + \nu(q^2-1)w(\epsilon, yx),\\
e(2,3) &= w(\epsilon, y);\\
e(3,1) &= \nu^2(1-q)^2 w(\lambda, x^2);\\
e(3,2) &= \nu(q^2-1)w(\lambda, x);\\
e(3,3) &= \lambda,
\end{aligned}$$

and

$$\begin{aligned}
f(1,1) &= \lambda^2,\\
f(1,2) &= \nu(q-1)w(\lambda^2, x);\\
f(2,1) &= w(\lambda, y);\\
f(2,2) &= \lambda + \nu(q-1)w(\lambda, yx).
\end{aligned}$$

4.2 Lifting matrix $\mathcal{A}$ with one nonzero diagonal element

Now let A have lifting matrix $\begin{bmatrix} 1 & \nu \\ -q\nu & 0 \end{bmatrix}$, i.e., $x^3 = c^3 - 1$, $y^3 = 0$ and we are in the situation of Proposition 3.4. Suppose $\gamma \notin <\lambda>$. Then equations (12),(13) and (14) become

$$\begin{aligned}
a^2 &= a\nu\gamma(c^2q^2 - 1) + bq^2; &(22)\\
b^2 &= bq\nu^2\gamma(1-c^2)\gamma(c^2q-1), &(23)\\
ab &= bq\nu\gamma(1-c^2). &(24)
\end{aligned}$$

The three solutions to this set of equations are $a = b = 0$, $b = 0$ and $a = \nu\gamma(c^2q - 1)$ or $b = q\nu^2\gamma(1 - c^2)\gamma(c^2q - 1)$ and $a = q\nu\gamma(1 - c^2)$. Let us choose the solution where both a and b are nonzero and then we can construct a matrix coalgebra exactly as in the previous subsection. As before, let $\beta = \frac{\gamma(c^3-1)}{b}$, and $\alpha = \frac{\gamma(c^3-1)}{a}$. Then $Ke(1, 1) \oplus Ke(2, 1) \oplus Ke(3, 1)$ is a simple right A-module and $e(i, j)$ with $1 \le i, j \le 3$ is a basis for a matrix coalgebra in $\gamma C(1)$ where

$$
\begin{aligned}
e(1,1) &= \gamma + aw(\gamma, yx) + bw(\gamma, y^2x^2), \\
e(1,2) &= \alpha w(\gamma, x^2) + w(\gamma, y) + \frac{b}{a}w(\gamma, y^2x), \\
e(1,3) &= \beta w(\gamma, x) + w(\gamma, y^2) + a\beta w(\gamma, yx^2), \\
e(2,1) &= aw(\gamma\lambda, x) + bw(\gamma\lambda, yx^2), \\
e(2,2) &= \gamma\lambda + \frac{b}{a}w(\gamma\lambda, yx), \\
e(2,3) &= w(\gamma\lambda, y) + a\beta w(\gamma\lambda, x^2), \\
e(3,1) &= bw(\gamma\lambda^2, x^2), \\
e(3,2) &= \frac{b}{a}w(\gamma\lambda^2, x), \\
e(3,3) &= \gamma\lambda^2.
\end{aligned}
$$

As in the previous section $e(1,j) \leftharpoonup y = e(2,j)$, $e(2,j) \leftharpoonup y = e(3,j)$ and $e(3,j) \leftharpoonup y = 0$. Had we not chosen a, b nonzero, then the action of y on $e(1,1)$ would not have yielded a 3-dimensional K-vector space.

To obtain the other two matrix coalgebras in $\gamma C(1)$, replace γ by $\gamma\lambda$ or $\gamma\lambda^2$ in equations (22)-(24) to obtain matrix coalgebras generated by elements

$$
\begin{aligned}
e'(1,1) &= \gamma\lambda + a'w(\gamma\lambda, yx) + b'w(\gamma\lambda, y^2x^2), \\
e''(1,1) &= \gamma\lambda^2 + a''w(\gamma\lambda^2, yx) + b''w(\gamma\lambda^2, y^2x^2),
\end{aligned}
$$

where $a' = q\nu\gamma(1 - c^2q^2)$, $b' = q\nu^2\gamma(1 - c^2q^2)\gamma(c^2 - 1)$, $a'' = q\nu\gamma(1 - c^2q)$, $b'' = q\nu^2\gamma(1 - c^2q)\gamma(c^2q^2 - 1)$.

Acknowledgement

Thanks to the referee for his/her careful reading of this paper and helpful comments.

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