Finiteness spaces and generalized power series

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Overview

- Ribenboim constructed rings of generalized power series for studies in number theory.
- While his construction gives a rich class of rings, it also seems ad hoc and non-functorial.
- We show that the conditions he imposes in fact can be used to construct internal monoids in a category of Ehrhard's *finiteness spaces* and the process is functorial.
- Furthermore any internal monoid of finiteness spaces induces a ring by Ehrhard's *linearization* process. So we get lots of new examples of generalized power series.

Ribenboim's generalized power series

We'll need the following technical condition:
Let $(M, +, \leq)$ be a partially ordered (commutative) monoid. M is *strictly ordered* if

$$s < s' \Rightarrow s + t < s' + t \; \forall s, s', t \in M \; .$$

We will henceforth assume that all the monoids we work with are strictly ordered.

Definition

An ordered monoid is *artinian* if all strictly descending chains are finite; that is, if any list $(m_1 > m_2 > \cdots)$ must be finite. It is *narrow* if all discrete subsets are finite; that is, if all subsets of elements mutually unrelated by $\leq$ must be finite.

Ribenboim's generalized power series II

Definition

Let V be a vector space, and recall that the *support* of a function $f\colon M \to V$ is defined by $supp(f) = \{m \in M | f(m) \neq 0\}$. Define the *space of Ribenboim power series from M with coefficients in V*, $G(M, V)$ to be the set of functions $f\colon M \to V$ whose support is artinian and narrow.

If A is also a commutative $\mathbb{K}$-algebra, then $G(M, A)$ is a commutative $\mathbb{K}$-algebra with

$$(f \cdot g)(m) = \sum_{(u,v)\in X_m(f,g)} f(u) \cdot g(v)$$

where

$$X_m(f, g) := \{(u, v) \in M \times M | u + v = m \text{ and } f(u) \neq 0, g(v) \neq 0\}$$

Ribenboim's generalized power series III

This requires the following observation. It is where the strictness property gets used:

Proposition

The set $X_m(f, g)$ is finite for $f, g \in G(M, V)$.

There are lots of examples.

- Let $M = \mathbb{N}$. The result is the usual ring of power series with coefficients in A.
- Let $M = \mathbb{Z}$. The result is the ring of Laurent series with coefficients in A.

Ribenboim's generalized power series IV: More examples

- Let $M = \mathbb{N}^n$, with pointwise order. The result is the usual ring of power series in n-variables with coefficients in A.

  This example is due to Ribenboim and was his motivation:

- Let $M = \mathbb{N} \backslash \{0\}$ with the operation of multiplication, equipped with the usual ordering. Then $G(M, \mathbb{R})$ is the ring of arithmetic functions (i.e. functions from the positive integers to the complex numbers), and multiplication is Dirichlet's convolution:

$$(f \star g)(n) = \sum_{d|n} f(d) g(\frac{n}{d})$$

Ehrhard's finiteness spaces I

- Let X be a set and let $\mathcal{U}$ be a set of subsets of X, i.e., $\mathcal{U} \subseteq \mathcal{P}(X)$. Define $\mathcal{U}^{\perp}$ by:

$$\mathcal{U}^{\perp} = \{u' \subseteq X \mid \text{the set } u' \cap u \text{ is finite for all } u \in \mathcal{U}\}$$

Lemma

We have $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$ and $\mathcal{U}^{\perp\perp\perp} = \mathcal{U}^{\perp}$.

- A *finiteness space* is a pair $\mathbb{X} = (X, \mathcal{U})$ with X a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ such that $\mathcal{U}^{\perp\perp} = \mathcal{U}$. We will sometimes denote X by $|\mathbb{X}|$ and $\mathcal{U}$ by $\mathcal{F}(\mathbb{X})$.

Ehrhard's finiteness spaces II: Morphisms

- A *morphism* of finiteness spaces $R\colon \mathbb{X} \to \mathbb{Y}$ is a relation $R\colon |\mathbb{X}| \to |\mathbb{Y}|$ such that the following two conditions hold:
  (1) For all $u \in \mathcal{F}(\mathbb{X})$, we have $uR \in \mathcal{F}(\mathbb{Y})$, where $uR = \{y \in |\mathbb{Y}| \mid \exists x \in u, xRy\}$.
  (2) For all $v' \in \mathcal{F}(\mathbb{Y})^\perp$, we have $Rv' \in \mathcal{F}(\mathbb{X})^\perp$.

It is straightforward to verify that this is a category. We denote it FinRel.

Lemma

In the definition of morphism of finiteness spaces, condition (2) can be replaced with:

$$(2')\ \textit{For all } b \in |\mathbb{Y}|, \textit{ we have } R\{b\} \in \mathcal{F}(\mathbb{X})^\perp.$$

Ehrhard's finiteness spaces III: It's a model of linear logic

Theorem

FinRel *is a* $*$*-autonomous category. The tensor*

$$\mathbb{X} \otimes \mathbb{Y} = (|\mathbb{X} \otimes \mathbb{Y}|, \mathcal{F}(\mathbb{X} \otimes \mathbb{Y}))$$

is given by setting $|\mathbb{X} \otimes \mathbb{Y}| = |\mathbb{X}| \times |\mathbb{Y}|$ *and*

$$\begin{aligned} \mathcal{F}(\mathbb{X} \otimes \mathbb{Y}) &= \{u \times v \mid u \in \mathcal{F}(\mathbb{X}), v \in \mathcal{F}(\mathbb{Y})\}^{\perp\perp} \\ &= \{w \mid \exists u \in \mathcal{F}(\mathbb{X}), \exists v \in \mathcal{F}(\mathbb{Y}), w \subseteq u \times v\}. \end{aligned}$$

We note that it also has sufficient structure to model the rest of the connectives of linear logic.

Ehrhard's finiteness spaces IV: Another choice of morphism

Ehrhard was motivated by linear logic to construct a $*$-autonomous category and hence chose relations as morphisms. But the choice has issues. Much like the usual category of relations, FinRel is lacking most limits and colimits. Another choice is possible:

Definition

We define the category FinPf*. Objects are finiteness spaces and a morphism* $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ *is a partial function satisfying the same conditions as above.*

Proposition

The category FinPf *is a symmetric monoidal closed, complete and cocomplete category.*

Posets as finiteness spaces I

Ribenboim's use of artinian and narrow subsets may seem unmotivated, but it in fact is precisely what we need to embed posets into finiteness spaces:

Theorem

Let $(P, \leq)$ be a poset. Let $\mathcal{U}$ be the set of artinian and narrow subsets. Then $(P, \mathcal{U})$ is a finiteness space.

Lemma

Under the above assumptions, $\mathcal{U}^{\perp}$ is the set of noetherian subsets of P.

Posets as finiteness spaces II: Functoriality

Unfortunately, if we consider the above construction from the usual category Pos of posets to any of the categories of finiteness spaces we have considered, it isn't functorial. Indeed, the inverse image under an order-preserving map of a noetherian subset may be not noetherian. However, the problem disappears if we consider *strict maps.*

Definition

If $(P, \leq)$ and $(Q, \leq)$ are two posets, a map $f\colon P \to Q$ is said to be *strict* if $p < p'$ implies $f(p) < f(p')$. In particular, it is a morphism of posets. We denote the category of posets and strict maps by StrPos.

Proposition

The above construction is a strict symmetric monoidal functor $E\colon \mathrm{StrPos} \to \mathrm{FinPf}$.

Posets as finiteness spaces III: Internal monoids

As such, it takes monoids to monoids:

Theorem

The functor E induces a functor $Mon(E)\colon Mon(\mathrm{StrPos}) \to Mon(\mathrm{FinPf})$ from the category of strict pomonoids to the category of partial finiteness monoids.

Definition

A *partial finiteness monoid* is an internal monoid in FinPf.

Linearizing finiteness spaces and generalizing the Ribenboim construction

Let A be an abelian group and $\mathbb{X} = (X, \mathcal{U})$ a finiteness space. Ehrhard defined the abelian group $A\langle\mathbb{X}\rangle$ as the set

$$A\langle\mathbb{X}\rangle = \{f : X \to A \,|\, supp(f) \in \mathcal{U}\}$$

together with pointwise addition.

Lemma

In the case of a poset $(P, \leq)$ with its finiteness structure as determined as above, we recover $G(P, A)$.

Linearizing II

Theorem

If $(\mathbb{M}, \mu \colon \mathbb{M} \otimes \mathbb{M} \to \mathbb{M}, \eta \colon I \to \mathbb{M})$ *is a partial finiteness monoid and* R *a ring (not necessarily commutative, but with unit), then* $R\langle\mathbb{M}\rangle$ *canonically has the structure of a ring.*
The multiplication in $R\langle\mathbb{M}\rangle$ *is given by*

$$(f \cdot g)(m) = \sum_{(m_1, m_2) \in X_m(f,g)} f(m_1) \cdot g(m_2).$$

Note the obvious similarity to Ribenboim's definition. But here it is the second condition in the definition of morphism of finiteness spaces that ensures the finiteness of the sum.

Example I: Puiseaux series (Newton)

A *Puiseux series* with coefficients in the ring R is a series (with indeterminate T) which allow for negative and fractional exponents of the form

$$\sum_{i \geqslant a}^{+\infty} r_i T^{i/n}$$

for some integer $a \in \mathbb{Z}$, some positive integer $n \in \mathbb{N} \setminus \{0\}$ and where $r_i \in R$. With the usual sum and product law, they form the ring of Puiseux series with coefficients in R.

Our postdoc Pierre-Alain Jacqmin showed that these rings fit into the finiteness space framework. Details in our paper on the archive.

Example II: Formal power series

Let A be a set (called in this case the *alphabet*). Then, let M be the free monoid generated by A. The finiteness space $(M, \mathcal{P}(M))$ has a monoid structure in FinPf given by the classical monoid structure of M. The only non-trivial part here is to check that the multiplication

$$\cdot : (M, \mathcal{P}(M)) \otimes (M, \mathcal{P}(M)) \to (M, \mathcal{P}(M))$$

is a morphism.

But since M is freely generated by A, for each $m \in M$, there are only finitely many $(m_1, m_2) \in M^2$ such that $m_1 \cdot m_2 = m$.

Then the ring $R\langle(M, \mathcal{P}(M))\rangle$ is called the *ring of formal power series* with exponents in M and coefficients in R.

Example III: Polynomials of degree at most n

Let n be a natural number and $X = \{0, \ldots, n\}$. The finiteness space $(X, \mathcal{P}(X))$ has a monoid structure $((X, \mathcal{P}(X)), \mu, \eta)$ in FinPf:

$$\eta\colon (\{*\}, \mathcal{P}(\{*\})) \to (X, \mathcal{P}(X))$$

maps $*$ to 0 and

$$\mu\colon (X, \mathcal{P}(X)) \otimes (X, \mathcal{P}(X)) = (X \times X, \mathcal{P}(X \times X)) \to (X, \mathcal{P}(X))$$

is defined by

$$\mu(a, b) = \begin{cases} a + b & \text{if } a + b \leqslant n \\ \text{undefined} & \text{if } a + b > n. \end{cases}$$

The corresponding ring $R\langle(X, \mathcal{P}(X))\rangle$ is $R_{\leqslant n}[T]$, the ring of polynomials of degree at most n and coefficients in R.

Future work

- Etale groupoids yield C^*-algebras. The construction is very similar.
- Can generalized power series be differentiated?
- Do all rings that arise as above have a *Rota-Baxter operator*? One place RB-operators arise is in renormalization in quantum field theory. Rings of Laurent series have such an operator which is used in the Connes-Kreimer approach to renormalization. Guo and Liu studied when a projection operator on Ribenboim power series is in fact a Rota-Baxter operator. Do these necessarily exist for finiteness monoids and their rings?
- Morita theory.

Going forward

Lately, we've been looking at categorical aspects of these generalized power series, which don't seem to have been much explored after Ribenboim's original work.

Today, I'll talk about **Morita equivalence**.

Definition

Two rings R and S are *Morita equivalent* if their categories of (left) modules are equivalent. (We note that the categories of left modules are equivalent if and only if the categories of right modules are equivalent.) We denote Morita equivalence by $R \approx S$.

Morita Equivalence: Examples

- Two commutative rings are Morita equivalent if and only if they are isomorphic.
- For any ring R, we have $R \approx M_n(R)$.

These were both well-known before Morita's work. Morita rephrased equivalence in terms of bimodules, which has allowed the ideas to be generalized via bicategories.

Morita's Theorem

Theorem

Suppose R and S are rings and $R \approx S$. Let $F \colon R - \mathrm{Mod} \to S - \mathrm{Mod}$ and $G \colon S - \mathrm{Mod} \to R - \mathrm{Mod}$ be functors inducing an equivalence. Then letting $P = F(R)$ and $Q = G(S)$, then

- $P = {}_S P_R$ *and* $Q = {}_R P_S$ *are faithfully balanced bimodules.*
- ${}_S P_R \cong Hom_S(Q, S) \cong Hom_R(Q, R)$
- ${}_R Q_S \cong Hom_S(P, S) \cong Hom_R(P, R)$
- $F \cong P \otimes_R -$
- $G \cong Q \otimes_S -$

Morita contexts I

Let C be a ring, let $e \in C$ be an idempotent and let $e' = 1 - e$ be its complementary idempotent. Then we obtain a decomposition of C as

$$C \cong e'Ce' \oplus e'Ce \oplus eCe' \oplus eCe$$

We arrange these into a 2×2 matrix as:

$$\begin{bmatrix} e'Ce' & e'Ce \\ eCe' & eCe \end{bmatrix}$$

We rename the entries of this matrix as follows:

$$\begin{bmatrix} B & M \\ N & A \end{bmatrix}$$

We have $M =_B M_A$ and that $N =_A N_B$. We have bimodule maps as follows:

$$f \colon M \otimes_A N \to B \qquad g \colon N \otimes_B M \to A$$

Morita contexts II

From the associativity of the multiplication in C, we conclude, for all $n_i \in N$ and $m_j \in M$

$$n_1 f(m_2 \otimes n_3) = g(n_1 \otimes m_2) n_3 \qquad f(m_1 \otimes n_2) m_3 = m_1 g(n_2 \otimes m_3)$$

We present the following two equivalent definitions of *Morita context*.

Definition (Version 1)

A *Morita context* between rings A and B consists of a ring C equipped with an idempotent e such that $A \cong eCe$ and $B \cong e'Ce'$ where $e' = 1 - e$.

Morita contexts III

Definition (Version 2)

A *Morita context* between rings A and B consists of bimodules $M =_B M_A$ and $N =_A N_B$ and bimodule maps

$$n_1 f(m_2 \otimes n_3) = g(n_1 \otimes m_2) n_3 \qquad f(m_1 \otimes n_2) m_3 = m_1 g(n_2 \otimes m_3)$$

satisfying the previous associativity constraints.

Definition

The quotient rings:

$$\overline{C} = C/(e) \quad \text{and} \quad \overline{C}' = C/(e')$$

are the *Morita defects* of C.

Morita contexts IV

Theorem

Let (C, e) be a Morita context between A and B. The following are equivalent:

- *A and B are Morita equivalent via (C, e).*
- *Both Morita defects are 0.*
- *The maps f and g above are isomorphisms.*

One can form the Morita context ring in any additive category. It is always a ring. These are called *formal matrix rings*.

This approach to Morita equivalence generalizes to most monoidal settings. B. Pecsi has given a definition of Morita context in terms of arbitrary bicategories. There are lots of interesting examples.

Abstract Morita theory

One can consider Morita theory in a number of settings.

- For inverse semigroups, there's been a great deal of work, by Talwar, Funk, Lawson, Steinberg and others. Morita equivalence can be framed in terms of *equivalence bimodules* or *enlargements*. There is also a topos-theoretic interpretation of Morita theory. (Two inverse semigroups are equivalent if certain categories of presheaves are equivalent.) Funk, Lawson and Steinberg show all the approaches are equivalent.

- For C^*-algebras, Morita equivalence can be expressed in terms of *imprimitivity bimodules* (Rieffel).

S-Posets

In particular, we can look at partially ordered monoids (pomonoids).

Definition

- If S is a pomonoid, then a poset P is an *S-poset* if equipped with a (right) action of S such that the resulting map $\cdot : P \times S \to P$ is monotone in each variable.
- If S, T are pomonoids, then a poset P is an *$S - T$-biposet* if equipped with a left action of S and a right action of T such that the resulting map $S \times P \times T \to P$ is monotone in each variable and $(sp)t = s(pt)$ for all $s \in S, p \in P, t \in T$.
- A *morphism* of S-posets is a monotone map which commutes with the S-action.
- We thus get a category denoted Pos_S.

Theorem

The category Pos_S *is complete, cocomplete and cartesian closed.*

Tensor product of S-posets

Given two biposets ${}_S M_T$ and ${}_T N_R$ for pomonoids S, T and R, one constructs the tensor product ${}_S M \otimes_T N_R$ in the evident way. One considers the cartesian product $M \times N$. Then consider the congruence generated by the set

$$\{((mt, n), (m, tn)) | m \in M, n \in N, t \in T\}$$

One can see that $m \otimes n \leq m' \otimes n'$ if and only if there exist elements $m_i \in M, n_i \in N, u_i, v_i \in T$ such that we have the following scheme:

$$\begin{array}{rcr} m \leq m_1 u_1 & & \\ m_1 v_1 \leq m_2 u_2 & & u_1 n \leq v_1 n_2 \\ & \vdots & \\ m_k v_k \leq m' & & u_k n_k \leq v_k n' \end{array}$$

The result will be an $S - R$ bimodule satisfying the usual properties.

The following results are due to Laan. They are typical of the general structure

Theorem

Let $F\colon \mathrm{Pos}_S \to \mathrm{Pos}_T$ be a Pos*-equivalence. Then*

- *There exists $P = {}_SP_T$ with $F \cong - \otimes_S P$.*
- *There exists $Q = {}_TQ_S$ with $F \cong Hom_S(Q, -)$ and the inverse of F given by $G \cong - \otimes_T Q$.*
- ${}_T(Q \otimes_S P)_T \cong T$

Morita theory for pomonoids II

A frequent question in all of these theories is what properties are preserved under Morita equivalence?

Theorem

Suppose $F\colon \mathrm{Pos}_S \to \mathrm{Pos}_T$ is a Pos*-equivalence. Let $A = A_S$ be an S-poset. Then*

- *A and $F(A)$ have isomorphic lattices of subobjects.*
- *A and $F(A)$ have isomorphic lattices of congruences.*
- *If A is flat, then so is $F(A)$.*

Analogous results hold for Morita equivalent rings, inverse semigroups, C^*-algebras, etc.

Morita theory for pomonoids III: Boring Theorem

Theorem

Let R and S be Morita equivalent rings. Let P and P' be Morita equivalent pomonoids. Then the rings $G(P, R)$ and $G(P', S)$ are Morita equivalent.

For example, if R and S are related by the bimodule $M = {}_S M_R$, then $G(P, M)$ is a bimodule relating $G(P, R)$ and $G(P, S)$, etc.