Weil spaces and closed tangent structure

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Overview

$\boxed{1}$

$$\begin{tikzcd}
\text{Tangent categories} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathcal{W}_1\text{-actegories}^* \arrow[dl, leftrightarrow] \\
& \mathcal{W}_1\text{-presheaf cats}^* &
\end{tikzcd}$$

$\boxed{2}$ Embedding theorem

$\boxed{3}$ Closed and representable tangent categories

Tangent categories and Weil algebras á la Leung

Tangent categories have a tangent functor

$$\mathbb{X} \xrightarrow{T} \mathbb{X}$$

for which TM is thought of as the object of vectors tangent to M.

There are also pullbacks of T

$$\mathbb{X} \xrightarrow{T_n} \mathbb{X}$$

for which T_nM is thought of n-vectors anchored at the same point of M.

Tangent categories and Weil algebra á la Leung

Of course the tangent functor can be iterated

$$\mathbb{X} \xrightarrow{T^n} \mathbb{X}$$

for which $T^n M$ is thought of as the n-dimensional singular microcubes anchored at a point of M.

The pullbacks may also be iterated

$$\mathbb{X} \xrightarrow{T_{n_1}} \mathbb{X} \cdots \mathbb{X} \xrightarrow{T_{n_k}} \mathbb{X}$$

for which $T_{n_k}(\cdots T_{n_1} M \cdots)$ is thought of as ...

Tangent categories and Weil algebras á la Leung

Nodes $\equiv$ tangent functor Edges $\equiv$ pullback along p.

Graph	Tangent categories
$\boxed{t_1}$	TM
$\begin{tikzcd} \boxed{t_1} \arrow[r, dash] & \boxed{t_2} \end{tikzcd}$	T_2M
$\begin{tikzcd} \boxed{t_1} \arrow[rr, dash] \arrow[dr, dash] & & \boxed{t_2} \arrow[dl, dash] \\ & \boxed{t_3} & \end{tikzcd}$	T_3M
$\begin{tikzcd} \boxed{t_1} \arrow[d, dash] & \\ & \boxed{t_3} \\ \boxed{t_2} & \end{tikzcd}$	$T_2(TM)$

Tangent categories and Weil algebras á la Leung

¿ What about?

$$
\begin{tikzcd}
\boxed{t_1} \arrow[r, dash] & \boxed{t_2} \arrow[d, dash] \\
& \boxed{t_3}
\end{tikzcd}
$$

Here one can add in the $t_1 - t_2$ part, or the t_2–t_3 part, but not the $t_1 - t_3$ part.

Tangent categories and Weil algebras á la Leung

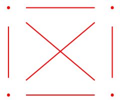
The graphs that allow defining valid tangent structure are in correspondence with *Weil*$_1$ *algebras*.

¿ What are Weil$_1$ algebras?

Let R be a ring (or cancellative rig, like $\mathcal{N}$). Form the Lawvere theory out of

$$W^n := R[x_1, \ldots, x_n]/(x_i x_j)_{1 \leqslant i \leqslant j \leqslant n}$$

W^4 :



and augmented maps.

Weil$_1$ algebras, or $\mathcal{W}_1$ is the closure of the above to

$$R[x_1, \ldots, x_n]/I \otimes R[y_1, \ldots, y_m]/J := R[x_1, \ldots, y_m]/(I \cup J)$$

in augmented R algebras.

$G_1 \otimes G_2 : \boxed{G_1} \qquad \boxed{G_2}$

Transvere limits in $\mathcal{W}_1$

Proposition (Leung)

$\mathcal{W}_1$ is an FL theory.

In $\mathcal{W}_1$ the following are limits:

1. *W^n is the n fold product of $R[x]/(x^2)$;*
2. *If $\lim_i V_i$ is a limit, then for any A*

$$A \otimes \lim_i V_i \simeq \lim_i A \otimes V_i;$$

3. *The following square*

$$\begin{tikzcd} a + bx + cy \arrow[r, mapsto] \arrow[d, mapsto] & a + bx + cyz \arrow[r, mapsto] & a + bx + cxz \arrow[d, mapsto] \\ a \arrow[rr, mapsto] & & a + 0x \end{tikzcd}$$

is a pullback

Transverse limits: Limits constructed starting with 1 or 3, and inductively applying 2 are called transverse.

Tangent category

Tangent structure [Rosický, Cockett-Cruttwell,Leung]
A tangent structure on $\mathbb{X}$ is exactly a strong (iso) monoidal functor

$$\mathcal{W}_1 \longrightarrow [\mathbb{X}, \mathbb{X}]$$

that sends transverse limits to pointwise limits.

Observation

Tangent structures on $\mathbb{X}$ are in 1–1 correspondence with models of $\mathcal{W}_1$ in $[\mathbb{X}, \mathbb{X}]$ regarded as a limit sketch.

Weil prolongation as an actegory

Uncurrying Leung's theorem, a tangent structure is exactly an actegory

$$\mathbb{X} \times \mathcal{W}_1 \xrightarrow{\propto} \mathbb{X}$$

so natural isomorphisms

$$A \propto (U \otimes V) \simeq (A \propto U) \propto V \qquad A \propto R \simeq A$$

such that for every transverse limit $\lim_i V_i$ in $\mathcal{W}_1$

$$A \propto (\lim_i V_i) \simeq \lim_i (A \propto V_i)$$

For example

$$TM = M \propto R[x]/(x^2) \qquad T_2 M = M \propto R[x,y]/(x^2, xy, y^2)$$

This means $\mathcal{W}_1$ is a tangent category: $\propto \equiv \otimes$.

Functor categories involving a tangent category

When $\mathbb{Y}$ is a tangent category then $[\mathbb{X}, \mathbb{Y}]$ is too.

$$F \propto U : \mathbb{X} \longrightarrow \mathbb{Y}$$
$$M \mapsto F(M) \propto U$$

When $\mathbb{X}$ is a tangent category then $[\mathbb{X}, \mathbb{Y}]$ is not. ... but should be?

$$F \propto U : \mathbb{X} \longrightarrow \mathbb{Y}$$
$$M \mapsto F(M \propto U)$$

The **microlinear functors** do (definitionally) form a tangent category.

Microlinear Weil spaces

The category of Weil spaces [Bertram '14] is

$$[\mathcal{W}_1, \mathsf{Set}]$$

The category of microlinear Weil spaces then forms a tangent category, $\boxed{\text{M-}\mathcal{W}_1}$.

Our goal is to understand and then exploit M-$\mathcal{W}_1$.

Sketches of $1/\infty$

$\mathcal{W}_1$ is a finite limit sketch under transverse limits.

Proposition

For a finite limit sketch, $\mathrm{Mod}(T)$ *is a locally finitely presentable category.*

Observation

M-$\mathcal{W}_1$ *is locally finitely presentable hence complete and cocomplete.*

Sketches of $1/\infty$

Proposition (Kennison 1968)

M-$\mathcal{W}_1$ *is a reflective subcategory of* $\mathrm{Psh}(T)$ *containing all* $\mathcal{W}_1(U, \_)$.

This is a move that will help us to characterize M-$\mathcal{W}_1$ enriched categories.

M-$\mathcal{W}_1$ is self enriched

Observation

$\mathcal{Y}(U \otimes V) \simeq \mathcal{Y}(U) \times \mathcal{Y}(V)$

Proposition

For any Weil space, $M \propto U \simeq [\mathcal{Y}(U), M]$*.*

$$
\begin{aligned}
[\mathcal{Y}(U), M](X) &\simeq \mathrm{Nat}(\mathcal{Y}(U) \times \mathcal{Y}(X), M) \\
&\simeq \mathrm{Nat}(\mathcal{Y}(U \otimes X), M) \\
&\simeq M(U \otimes X) \equiv (M \propto U)(X)
\end{aligned}
$$

M-$\mathcal{W}_1$ is self enriched

Proposition

When M is microlinear, then for any Weil space X, $[X, M]$ is microlinear. Hence M-$\mathcal{W}_1$ *is an exponential ideal.*

$$[X, M] \propto \lim_i V_i \simeq [\lim_i V_i, [X, M]]$$
$$\simeq [X, M \propto \lim_i V_i] \simeq \lim_i([X, M] \propto V_i)$$

Corollary

The reflector $\mathrm{Psh}(\mathcal{W}_1) \xrightarrow{L} \text{M-}\mathcal{W}_1$ *preserves products, and the product functor $M \times \_$ is cocontinuous.*

Corollary

M-$\mathcal{W}_1$ *is a monoidally reflective subcategory of* $\mathrm{Psh}(\mathcal{W}_1)$.

M-$\mathcal{W}_1$ is self enriched

Proposition

M-$\mathcal{W}_1$ categories form a reflective subcategory of $\mathsf{Psh}(\mathcal{W}_1)$ *categories, and similarly for those categories that admit powers by representables.*

Corollary

Let $\mathbb{X}$ *be a* $\mathsf{Psh}(\mathcal{W}_1)$ *enriched category, such that for each* A, B, $\mathbb{X}(A, B)(\_)$ *is in* M-$\mathcal{W}_1$, *then* $\mathbb{X}$ *is in fact a* M-$\mathcal{W}_1$ *enriched category.*

Enriched Characterization of Tangent Structure

When $\mathbb{X}$ is a tangent category, every homset gives rise to a functor

$$\mathbb{X}(A, B)(\_) := \mathbb{X}(A, B \propto \_) : \mathcal{W}_1 \longrightarrow \mathsf{Set}$$

This functor preserves transverse limits.

Proposition (Garner)

A tangent category is exactly a category enriched in M-$\mathcal{W}_1$ *that admits powers by representable functors.*

$$A \pitchfork \mathcal{Y}(U) = A \propto U$$

M-$\mathcal{W}_1$ has self enriched limits

Proposition

M-$\mathcal{W}_1$ *is complete and cocomplete as an* M-$\mathcal{W}_1$ *category.*

This follows from Kelly chapter 3. M-$\mathcal{W}_1$ is enriched in M-$\mathcal{W}_1$ and has limits, colimits, and copowers.

We use this to setup enriched limits and and colimits on enriched presheaf categories, and to characterize enriched natural transformations.

M-$\mathcal{W}_1$ is a model of SDG part 1

Proposition

The tangent functor is representable: that is $TM \simeq [D, M]$ for some D.

Proof.

Take $D = \mathcal{Y}(R[x]/(x^2))$. From a previous result, $M \propto U \simeq [\mathcal{Y}(U), M]$ for all U, and hence $TM \simeq [\mathcal{Y}(R[x]/x^2), M]$ □

M-$\mathcal{W}_1$ is a model of SDG part 2

Proposition

M-$\mathcal{W}_1$ *has a ring of line type, that satisfies the Kock-Lawvere axiom.*

In a complete, representable tangent category, one may form the ring of homotheties mirroring a move of Kolář. In SMan homotheties of the tangnet bundle is up to isomorphism $\mathbb{R}$.

M-$\mathcal{W}_1$ is a model of SDG part 2

Proposition

M-$\mathcal{W}_1$ *has a ring of line type, that satisfies the Kock-Lawvere axiom.*

Start with

$$\begin{tikzcd} R_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[D, D]} \arrow[d, "p"] \\ 1 \arrow[r, "0"'] & D \end{tikzcd}$$

Using equalizers obtain a subobject, which is a ring

Embedding theorem for tangent categories

Proposition (Garner)

Let $\mathbb{X}$ be a tangent category. The M-$\mathcal{W}_1$*-category*

$$[\mathbb{X}^{op}, \mathsf{M}\text{-}\mathcal{W}_1]$$

is a representable tangent category.

Corollary (Garner)

Every tangent category embeds into a representable tangent category.

Transverse limits in a tangent category

A **transverse limit** in a tangent category is a $\lim_i A_i$ such that

$$(\lim_i A_i) \propto V \simeq \lim_i (A_i \propto V)$$

Transverse limits include:

- Products $A \times B \propto W \simeq (A \propto W) \times (B \propto W)$;
- Pullback powers of p:

$$\begin{tikzcd} A \propto W^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \propto W \arrow[d, "p"] \\ A \propto W \arrow[r, "p"'] & A \end{tikzcd}$$

Transverse limits in a tangent category

When $\mathbb{X}$ is a tangent category and $\lim_i B_i$ is transverse, then for each $A \in \mathbb{X}$ and $U \in \mathcal{W}_1$:

$$\mathbb{X}(A, (\lim_i B_i) \propto U) \simeq \mathbb{X}(A, \lim_i (B_i \propto U)) \simeq \lim_i \mathbb{X}(A, B_i \propto U)$$

Hence

$$\mathbb{X}(A, \lim_i B_i)(\_) \simeq \lim_i \mathbb{X}(A, B_i)(\_) : \mathcal{W}_1 \longrightarrow \mathsf{Set}$$

is an enriched limit.

Observation

The enriched Yoneda embedding

$$B \mapsto \mathbb{X}(\_, B)(\_)$$

sends transverse limits to limits.

Closed tangent categories

A cartesian closed tangent category is **coherently closed** when the induced map ψ

$$\frac{A \times [A,B] \propto U \xrightarrow{\theta} (A \times [A,B]) \propto U \xrightarrow{\mathsf{ev} \propto U} B \propto U}{[A,B] \propto U \xrightarrow[\psi = \lambda(\theta; \mathsf{ev} \propto U)]{} [A, B \propto U]}$$

is invertible for each $U \in \mathcal{W}_1$. So that coherently closed

Closed tangent categories via M-$\mathcal{W}_1$

Proposition

When $\mathbb{X}$ is a coherently closed tangent category, then for each A, the M-$\mathcal{W}_1$*-functor $A \times \_$ has a* M-$\mathcal{W}_1$*-right adjoint $[A, \_]$.*

Corollary

The Yoneda embedding $A \mapsto \mathbb{X}(\_, A)(\_)$ preserves the internal hom. That is for each U,

$$\mathbb{X}(\_, [A, B])(U) \simeq [\mathbb{X}(\_, A)(U), \mathbb{X}(\_, B)(U)]$$

Endomorphisms of the tangent functor

In a complete coherently closed tangent category, the endomorphisms of the tangent bundle may be computed as

$$\mathsf{End}(T) := \int_c \mathbb{X}(Tc, Tc) \equiv \int_c [Tc, Tc]$$

We also have the subobject of tangent transformations:

$$\mathsf{Tan}(T) \rightarrowtail \mathsf{End}(T)$$

This can be built in stages by equalizers taking those transformations which are commutative additive bundle homomorphisms that commute with the lift.

In a representable tangent category, the tangent transformations can be simplified. Moreover, they always give the ring of line type.

Endomorphisms of the tangent functor

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We also have the subobject of tangent transformations:

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E.g.

$$\begin{tikzcd}
\mathrm{Hom}(T) \arrow[r, tail] & {[T, T]} \arrow[rr, "{[T,p]}"] \arrow[dr] & & {[T, 1]} \\
& & 1 \arrow[ur, "p"'] &
\end{tikzcd}$$

etc

Endomorphisms of the tangent functor

Lemma

Let $\mathbb{X}$ be a tangent category. The Yoneda embedding preserves the tangent functor.

As the Yoneda embedding preserves transverse limits and internal homs

Proposition

Let $\mathbb{X}$ be a complete, coherently closed tangent category where all limits are transverse. Then

$$\mathcal{Y}(\mathsf{Tan}(T)) \simeq \mathsf{Tan}(T) \simeq \mathcal{R}$$

Corollary

In a coherently closed, complete tangent category where all limits are transverse, there is a rig of line type.

Characterization representable tangent categories

Proposition

To have a complete representable tangent category is to have a coherently closed complete tangent category where all limits are transverse.