

Cartan Calculus for Sector Forms

Ben MacAdam

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Background: Sector Forms

- Sector forms were introduced by J.E. White
- Rather than consider the Whitney sum of the tangent bundle $T_n(M)$, consider the *iterated tangent bundle* $T^n(M)$
- Alternating case
 - ▶ Synthetic Differential Geometry: Minguez, Kock, Lavendhomme, Nishimura
 - ▶ Abstract manifold categories: Bertam
- Cruttwell and Lucyshyn-Wright studied sector forms in a tangent category, emphasizing its simplicial structure.

Motivation: Cartan Calculus

- The Cartan calculus for differential forms is fundamental in Hamiltonian mechanics
- Alternating sector forms and the Cartan Calculus have been studied in SDG by Minguez, Lavendhomme, and Nishimura
- Tangent categories have much less logic than a model of SDG, so developing the theory requires new techniques and leads to some new realizations.
- Want to develop the Cartan calculus in a manner that emphasizes the simplicial structure developed by Cruttwell and Lucyshyn-Wright

Tangent Categories

A **tangent category with negatives** is a category with an additive bundle functor

$$p : T \Rightarrow id, 0 : id \Rightarrow T, + : T_p \times_p T \Rightarrow T, (-) : T \Rightarrow T$$

Along with a canonical flip $c : T^2 \Rightarrow T^2$ and vertical lift $T \rightarrow T^2$.

A differential object is an abelian group where

$$T(A) \cong A \oplus A \quad \text{in } Ab(\mathbb{X})$$

so that $A \xrightleftharpoons[\lambda]{0} T(A) \xrightleftharpoons[\kappa]{p} A$ - κ is a *vertical connection*.

Lie Bracket

The vertical lift satisfies the following universal property in a tangent category with negatives

$$\begin{array}{ccccc}
 X & & & & \\
 \downarrow \exists! \{f\} & \searrow f & & & \\
 T(M) & \xrightarrow{\ell} & T^2(M) & \begin{array}{c} \xrightarrow{T(p)} \\ \xrightarrow{\quad p \quad} \\ \xrightarrow{\quad p \circ 0 \quad} \end{array} & T(M)
 \end{array}$$

The *Lie bracket* of vector fields is defined:

$$[x, y] := \{xT(y) - yT(x)c\}$$

Definition (White)

An E -valued sector form on M is an n -fold linear map from M into a differential object E .

$$\omega : T^n(M) \rightarrow E$$

so that for all $1 \leq i \leq n+1$:

$$T^{i-1}(\ell) c_i T(\omega) = \omega \lambda$$

where $c_i := T^i(c) c_{i-1}$, $c_1 = id$

Denote the set of E -valued n -sector forms on M as $\Psi^n(M, E)$.

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Where does the notion of n -fold linearity condition come from?

Linear Morphisms

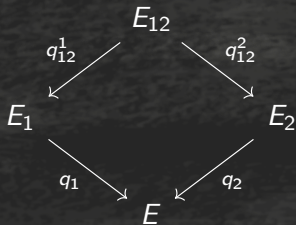
Definition (Linear Morphism of Differential Bundles)

Recall that a differential bundle is an additive bundle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ has an associated *lift* $\lambda : E \rightarrow T(E)$. A linear morphism of differential bundles preserves the respective lifts:

$$\begin{array}{ccc}
 T(E) & \xrightarrow{T(\gamma)} & T(E') \\
 \lambda \uparrow & & \uparrow \lambda' \\
 E & \xrightarrow{\gamma} & E' \\
 q \downarrow & & \downarrow q' \\
 B & \xrightarrow{f} & B'
 \end{array}$$

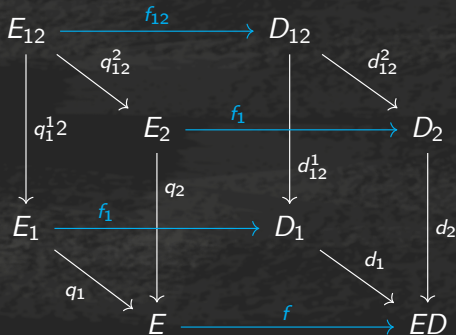
N-fold linearity

An n -fold differential bundle is a differential bundle in the category of $n - 1$ differential bundles, e.g. a 2-fold differential bundle is a diagram:



Where each map is a differential bundle, and each parallel pair of maps is a morphism of differential bundles.

A morphism of n -fold differential bundles determines a cubical complex - a morphism is *linear* when each face is a morphism of differential bundles. E.g. for 2-fold bundles:



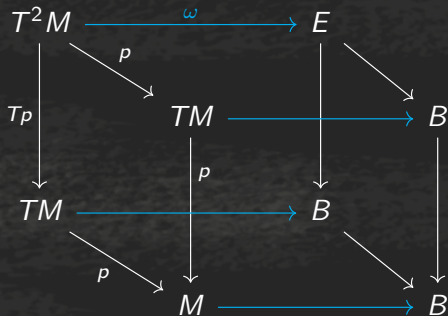
For each $n \in \mathbb{N}$, $M \in \mathbb{X}_0$, there is an n -fold differential bundle $T^n(M)$.

Given a differential bundle $\begin{array}{c} E \\ \downarrow q \\ B \end{array}$ there is an associated n -fold bundle

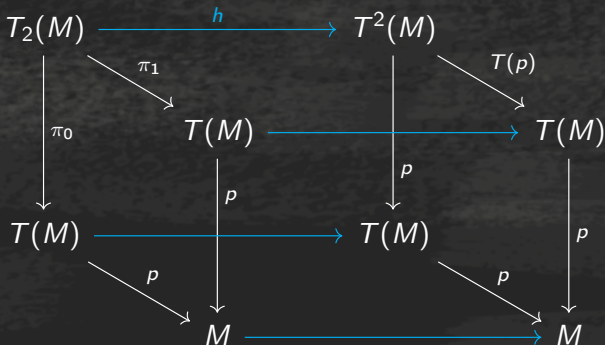
whose head is E with n copies of q and the rest of the edges are 1_B , e.g. for 2

$$\begin{array}{ccc} E & \xrightarrow{q} & B \\ q \downarrow & & \parallel \\ B & \xlongequal{\quad} & B \end{array}$$

A morphism $\omega : T^n(M) \rightarrow E$ is n -fold linear when ω is an n -fold linear morphism to the n -fold differential bundle associated to E .
 E.g. for $n = 2$ this would be



n-fold bundles give a natural characterization of a horizontal connection



This leads to a natural characterization of *higher order* horizontal connections.

Consider the cube associated to $T^3(M)$

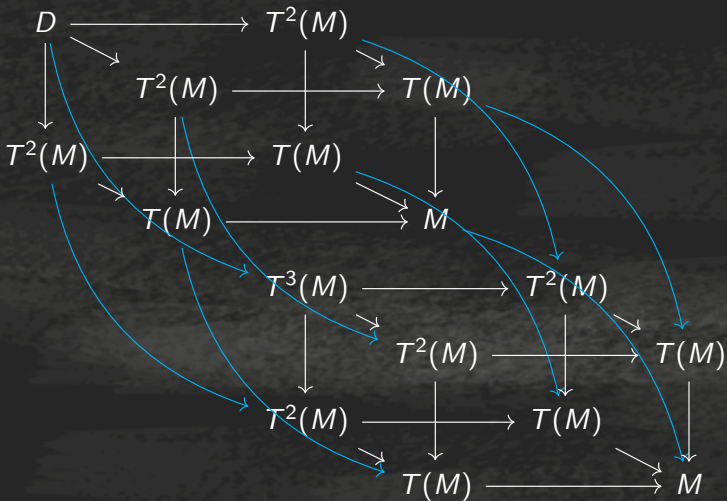
$$\begin{array}{ccccc}
 T^3(M) & \xrightarrow{p} & T^2(M) & & \\
 \downarrow T^2(p) & \searrow T(p) & \downarrow & \searrow p & \\
 & T^2(M) & \xrightarrow{T(p)} & T(M) & \\
 & \downarrow p & \downarrow T(p) & & \\
 T^2(M) & \xrightarrow{p} & T(M) & & \\
 & \searrow T(p) & \downarrow & \searrow p & \\
 & & T(M) & \xrightarrow{p} & M
 \end{array}$$

Delete the apex, and replace it with the pullback D :

$$\begin{array}{ccccc}
 D & \xrightarrow{\pi_0} & T^2(M) & & \\
 \downarrow \pi_2 & \searrow \pi_1 & \downarrow & \searrow p & \\
 & T^2(M) & \xrightarrow{T(p)} & T(M) & \\
 & \downarrow p & \downarrow T(p) & \downarrow p & \\
 T^2(M) & \xrightarrow{p} & T(M) & & \\
 & \searrow T(p) & \downarrow p & \searrow p & \\
 & T(M) & \xrightarrow{p} & M &
 \end{array}$$

Note there is a canonical map $\langle p, T(p), T^2(p) \rangle : T^3(M) \rightarrow D$
 A horizontal connection is a section h of $\langle p, T(p), T^2(p) \rangle$ that induces an 3-fold linear morphism of differential bundles.

These maps can be troublesome to work with.



It's most convenient to treat this as an extension property for a Kan complex.

Definition

An augmented symmetric cosimplicial object is a functor

$$X : \text{FinCard} \rightarrow \mathbb{X}$$

where FinCard is the full subcategory of Set whose objects are finite cardinals.

A cosimplicial object is presented by:

$$\begin{aligned}\epsilon_i^n &: n+1 \rightarrow n & 1 \leq i \leq n \\ \delta_i^n &: n \rightarrow n+1 & 1 \leq i \leq n+1 \\ \sigma_i^n &: n \rightarrow n & 1 \leq i \leq n-1\end{aligned}$$

Satisfying:

$$\begin{aligned}\epsilon_i \epsilon_j &= \epsilon_{j+1} \epsilon_i & i \leq j \\ \delta_i \delta_j &= \delta_i \delta_{j+1} & i \leq j \\ \delta_i \epsilon_j &= \begin{cases} \epsilon_{j-1} \delta_i & i < j \\ \delta_i \epsilon_j = 1 & i = j, j+1 \\ \delta_i \epsilon_j = \epsilon_j \delta_{i-1} & i > j+1 \end{cases}\end{aligned}$$

Theorem (Cruttwell and Lucyshyn-Wright)

The E -valued sector forms on M form an augmented symmetric cosimplicial abelian group.

The structure maps are given by:

- The codegeneracy: $\epsilon_i^n(\omega) := T^{i-1}(\ell)\omega$
- The symmetry: $\sigma_i^n(\omega) := T^{i-1}(c)\omega$
- The generating coface map: $\delta_1^n(\omega) = T(\omega)\hat{p}$

The coface maps are generated by:

$$\delta_i^n(\omega) := \sigma_i \circ \cdots \circ \sigma_1 \circ \delta_1^n(\omega)$$

for $i > 1$.

Definition (**unshuffles**)

An (m, n) -shuffle map is an $n + m$ permutation σ so that:

- $0 \leq i < j < m : \sigma(i) \leq \sigma(j)$
- $m \leq i < j < (m + n) : \sigma(i) \leq \sigma(j)$

an (n, m) -unshuffle is a permutation so that σ^{-1} is an (n, m) shuffle. Denote the n, m unshuffles $U(n, m)$

Definition (Unshuffle operator)

The unshuffle operator on $n + m$ sector forms is define

$$uS_{n,m}(\omega) := \sum_{\gamma \in U(n,m)} \text{sgn}(\gamma) \cdot \sigma_{\gamma}(\omega)$$

where we treat σ_{γ} is the operator defined by the permutation γ .

The exterior derivative:

$$\partial(\omega) := \sum_{i=1}^{n+1} (-1)^{i-1} \delta_i^n(\omega)$$

is precisely $\partial(\omega) = us_{1,n}(\delta_1^n(\omega))$

Theorem (Cruttwell and Lucyshyn-Wright)

$(\Psi^\bullet(M, E), \partial)$ is a cochain complex, the alternating forms form a subcomplex.

Remark

Lavendhomme and Minguez use the antisymmetrization operator.

$$A(\omega) = \frac{1}{n!} \sum_{\gamma \in S_n} \text{sgn}(\gamma) \sigma_\gamma(\omega)$$

Tensor Product of Forms

Suppose we have a differential object R with a multiplication

$$\cdot : R \times R \rightarrow R$$

so that

$$T(\cdot)\kappa = \langle \kappa, p \rangle \cdot + \langle p, \kappa \rangle \cdot$$

The tensor product of forms was introduced by Minguez.

$$\omega \otimes \gamma := \langle T^m(p^m)\omega, p^n\gamma \rangle \cdot$$

Then we have that:

$$\delta_1^{n+m}(\omega \otimes \gamma) = \langle T^{m+1}(p^m)\delta_1^n(\omega), T(p^n)\delta_1^m(\gamma) \rangle$$

Definition (Wedge Product for forms)

$\omega \in \Psi^n(M, R), \gamma \in \Psi^m(M, R)$ is

$$\omega \wedge \gamma := us_{n,m}(\omega \otimes \gamma) \in \Psi^{m+n}(M, R)$$

Theorem

Let $\omega \in \Psi^n(M, R), \gamma \in \Psi^m(M, R)$

1. $\partial(\omega \wedge \gamma) = \partial(\omega) \wedge \gamma + (-1)^n \omega \wedge \partial(\gamma)$
2. $\omega \wedge \gamma = (-1)^n \gamma \wedge \omega$
3. ω, γ alternating, then $\omega \wedge \gamma$ is alternating.

Theorem

For a linear ring R , the complex of sector forms $\Psi^\bullet(M, R)$ is a differential graded algebra, and the alternating sector forms form a subalgebra.

Remark

Minguez and Lavendhomme use the antisymmetrization operator.

$$\omega \wedge \gamma := \frac{1}{(n+m)!} \sum_{\nu \in S_{n+m}} \text{sgn}(\nu) \cdot \sigma_{\nu}(\omega \otimes \gamma)$$

this has some disadvantages:

- *Requires $\mathbb{N}$ invertible*
- *Sends a pair of sector forms to an alternating sector form.*

The Interior Product

Definition

Given a vector field $x \in \chi(M)$, $\omega \in \Psi^n(M, E)$ define the interior product:

$$i_x(\omega) := T^n(x)\omega$$

Definition

Given a vector field $x \in \chi(M)$, $\omega \in \Psi^n(M, E)$ define the Lie derivative

$$L_x(\omega) := i_x(\delta_{n+1}^n(\omega))$$

Proposition

The Lie derivative induces a cochain endomorphism

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & \Psi^k(M, E) & \xrightarrow{\partial} & \Psi^{k+1}(M, E) & \xrightarrow{\partial} & \dots \\ & & \downarrow L_x & & \downarrow L_x & & \\ \dots & \xrightarrow{\partial} & \Psi^k(M, E) & \xrightarrow{\partial} & \Psi^{k+1}(M, E) & \xrightarrow{\partial} & \dots \end{array}$$

The Cartan Calculus is a collection of relationships on the commutator of the operators i, L, ∂ .

Theorem (Cartan Calculus)

1. *If ω is an alternating form: $[i_x, i_y] = 0$.*
2. $[L_x, L_y] = L_{[x, y]}$
3. $[L_x, i_y] = i_{[x, y]}$
4. *Cartan's Homotopy Formula: $L_x = (-1)^n \cdot [\partial, i_x]$*

Proof.

The homotopy formula is immediate:

$$\begin{aligned} i_x(\partial(\omega)) - \partial(i_x(\omega)) &= i_x(u_{1,n}(\delta_1^n(\omega))) - u_{1,n-1}(i_x(\delta_1^n(\omega))) \\ &= (-1)^n i_x \delta_{n+1}^n(\omega) = (-1)^n L_x(\omega) \end{aligned}$$

□

To prove the other two formulas we need the following:

Lemma

For any $f : X \rightarrow T^2(M)$ so that $\{f\}$ is well defined:

$$T^k(\{f\}) = \{T^{k+1}(f) c_{k+1} T(c_{(k+1)})\} c_{k+1}^{-1}$$

where $c_0 = 1, c_i := T^{i-1}(c) c_{i-1}$

$[L_x, L_y]$:

Proof.

Expand the the left term to $T^k(\{xT(y) - yT(x)c\})\delta_{n+1}^n(\omega)$

Using the above lemma, find:

$$\{T^k(xT(y) - yT(x)c)c_{(k)}T(c_{(k)})\}T(\omega)\kappa$$

Pull the $T(\omega)\kappa$ inside the brackets:

$$\{T^k(xT(y) - yT(x)c)c_{(k)}T(c_{(k)})T(T(\omega)\kappa)\}$$

Now the codomain has a vertical connection κ , so we have:

$$T^k(xT(y) - yT(x)c)c_{(k)}T(c_{(k)})T(T(\omega)\kappa)\kappa$$

Some manipulation will yield $L_x(L_y(\omega)) - L_y(L_x(\omega))$

□

Conclusion

- Sector forms emphasize cosimplicial structure, rather than the cochain complex.
- The unshuffle operator gives a canonical construction of cochain structures from simplicial.
- We get a differential graded algebra of sector forms, and a differential graded algebra of singular forms.
- We get all the maps of the Cartan Calculus, satisfying analogous identities.
- This lets you formalize basic Hamiltonian mechanics in a tangent category, using a “symplectic sector form”

Further Work

- Using n -fold linearity, it is simple to generalize this so that the codomain is a differential bundle with a vertical connection. There are various Nijenhuis calculi for differential forms on vector bundles that can be developed using differential bundle valued sector forms.
- Cartan's calculus is about the interaction with $\chi(M)$ and $\Psi^\bullet(M, R)$. What about higher *sector fields*?
- The collection of n -sectors $\chi^n(M)$ is a semi-simplicial set - higher order connections in the sense of Bertram may be naturally interpreted as a *extension* property, yielding a Kan complex.
- Classical Field Theory is may be studied using n -plectic manifolds, so this should be possible using n -plectic sector forms and the Cartan calculus.



Bertram, Wolfgang (2008). *Differential geometry, Lie groups and symmetric spaces over general base fields and rings*. American Mathematical Soc.



Cartan, Henri (1950). "La transgression dans un groupe de Lie et dans un espace fibré principal". In: *Colloque de topologie (espaces fibrés)*. Bruxelles, pp. 57–71.



Cockett, J Robin B and Geoff SH Cruttwell (2014). "Differential structure, tangent structure, and SDG". In: *Applied Categorical Structures* 22.2, pp. 331–417.



Cruttwell, GSH and Rory BB Lucyshyn-Wright (2018). "A simplicial foundation for differential and sector forms in tangent categories". In: *Journal of Homotopy and Related Structures*, pp. 1–59.



Lavendhomme, René (1996). "Basic Concepts of Synthetic Differential Geometry". In:



Minguez, MC (1985). "Cálculo diferencial sintético y su interpretacion en modelos de preheces". In: *Publ, Sem, Mat, Serie II, Sec 2*.



Nishimura, Hirokazu (1997). "Synthetic hamiltonian mechanics". In: *International Journal of Theoretical Physics* 36.1, pp. 259–279.