Cartan Calculus for Sector Forms

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Background: Sector Forms

- Sector forms were introduced by J.E. White
- Rather than consider the Whitney sum of the tangent bundle $T_n(M)$, consider the *iterated tangent bundle* $T^n(M)$
- Alternating case
  - Synthetic Differential Geometry: Minguez, Kock, Lavendhomme, Nishimura
  - Abstract manifold categories: Bertam
- Cruttwell and Lucyshyn-Wright studied sector forms in a tangent category, emphasizing its simplicial structure.

Motivation: Cartan Calculus

- The Cartan calculus for differential forms is fundamental in Hamiltonian mechanics
- Alternating sector forms and the Cartan Calculus have been studied in SDG by Minguez, Lavendhomme, and Nishimura
- Tangent categories have much less logic than a model of SDG, so developing the theory requires new techniques and leads to some new realizations.
- Want to develop the Cartan calculus in a manner that emphasizes the simplicial structure developed by Cruttwell and Lucyshyn-Wright

Tangent Categories

A **tangent category with negatives** is a category with an additive bundle functor

$$p : T \Rightarrow id, 0 : id \Rightarrow T, + : T_p \times_p T \Rightarrow T, (-) : T \Rightarrow T$$

Along with a canonical flip $c : T^2 \Rightarrow T^2$ and vertical lift $T \to T^2$.

A differential object is an abelian group where

$$T(A) \cong A \oplus A \quad \text{in } Ab(\mathbb{X})$$

so that $A \underset{\lambda}{\overset{0}{\rightrightarrows}} T(A) \underset{\kappa}{\overset{p}{\rightrightarrows}} A$ - κ is a *vertical connection*.

Lie Bracket

The vertical lift satisfies the following universal property in a tangent category with negatives

$$\begin{tikzcd} X \arrow[d, dotted, "\exists!\{f\}"'] \arrow[dr, "f"] & & \\ T(M) \arrow[r, "\ell"] & T^2(M) \arrow[r, shift left=2, "T(p)"] \arrow[r, "p" description] \arrow[r, shift right=2, "pp0"'] & T(M) \end{tikzcd}$$

The *Lie bracket* of vector fields is defined:

$$[x, y] := \{xT(y) - yT(x)c\}$$

Definition (White)

An *E-valued sector form on M* is an n-fold linear map from M into a differential object E.

$$\omega : T^n(M) \to E$$

so that for all $1 \leq i \leq n+1$:

$$T^{i-1}(\ell) c_i T(\omega) = \omega\lambda$$

where $c_i := T^i(c) c_{i-1}$, $c_1 = id$

Denote the set of E-valued n-sector forms on M as $\Psi^n(M, E)$.

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Where does the notion of n-fold linearity condition come from?

Linear Morphisms

Definition (Linear Morphism of Differential Bundles)

Recall that a differential bundle is an additive bundle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ has an associated *lift* $\lambda : E \to T(E)$. A linear morphism of differential bundles preserves the respective lifts:

$$\begin{tikzcd} T(E) \arrow[r, "T(\gamma)"] & T(E') \\ E \arrow[u, "\lambda"] \arrow[r, "\gamma"] \arrow[d, "q"'] & E' \arrow[u, "\lambda'"'] \arrow[d, "q'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

N-fold linearity

An n-fold differential bundle is a differential bundle in the category of $n-1$ differential bundles, e.g. a 2-fold differential bundle is a diagram:

$$\begin{tikzcd} & E_{12} \arrow[dl, "q_{12}^1"'] \arrow[dr, "q_{12}^2"] & \\ E_1 \arrow[dr, "q_1"'] & & E_2 \arrow[dl, "q_2"] \\ & E & \end{tikzcd}$$

Where each map is a differential bundle, and each parallel pair of maps is a morphism of differential bundles.

A morphism of n-fold differential bundles determines a cubical complex - a morphism is *linear* when each face is a morphism of differential bundles. E.g. for 2-fold bundles:

$$\begin{tikzcd}
E_{12} \arrow[rr, "f_{12}"] \arrow[dr, "q^2_{12}"] \arrow[dd, "q^1_12"'] & & D_{12} \arrow[dr, "d^2_{12}"] \arrow[dd, "d^1_{12}" near end] & \\
& E_2 \arrow[rr, "f_1" near start] \arrow[dd, "q_2" near start] & & D_2 \arrow[dd, "d_2"] \\
E_1 \arrow[rr, "f_1" near start] \arrow[dr, "q_1"'] & & D_1 \arrow[dr, "d_1"] & \\
& E \arrow[rr, "f"] & & ED
\end{tikzcd}$$

For each $n \in \mathbb{N}, M \in \mathbb{X}_0$, there is an n-fold differential bundle $T^n(M)$.

Given a differential bundle $\begin{array}{c} E \\ \downarrow^{q} \\ B \end{array}$ there is an associated n-fold bundle whose head is E with n copies of q and the rest of the edges are 1_B, e.g. for 2

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[d, "q"'] & B \arrow[d, equal] \\ B \arrow[r, equal] & B \end{tikzcd}$$

A morphism $\omega : T^n(M) \to E$ is n-fold linear when ω is an n-fold linear morphism to the n-fold differential bundle associated to E. E.g. for $n = 2$ this would be

$$\begin{tikzcd}
T^2M \arrow[rr, "\omega"] \arrow[dr, "p"] \arrow[dd, "Tp"'] & & E \arrow[dr] \arrow[dd] & \\
& TM \arrow[rr] \arrow[dd, "p"] & & B \arrow[dd] \\
TM \arrow[rr] \arrow[dr, "p"'] & & B \arrow[dr] & \\
& M \arrow[rr] & & B
\end{tikzcd}$$

n-fold bundles give a natural characterization of a horizontal connection

$$
\begin{tikzcd}
T_2(M) \arrow[rr, "h"] \arrow[dr, "\pi_1"] \arrow[dd, "\pi_0"] & & T^2(M) \arrow[dr, "T(p)"] \arrow[dd, "p"] & \\
& T(M) \arrow[rr] \arrow[dd, "p"] & & T(M) \arrow[dd, "p"] \\
T(M) \arrow[rr] \arrow[dr, "p"] & & T(M) \arrow[dr, "p"] & \\
& M \arrow[rr] & & M
\end{tikzcd}
$$

This leads to a natural characterization of *higher order* horizontal connections.
Consider the cube associated to $T^3(M)$

$$
\begin{tikzcd}
T^3(M) \arrow[rr, "p"] \arrow[dr, "T(p)"] \arrow[dd, "T^2(p)"] & & T^2(M) \arrow[dr, "p"] \arrow[dd, "T(p)" near start] & \\
& T^2(M) \arrow[rr, "T(p)" near start] \arrow[dd, "p" near start] & & T(M) \arrow[dd, "p"] \\
T^2(M) \arrow[rr, "p" near start] \arrow[dr, "T(p)"'] & & T(M) \arrow[dr, "p"] & \\
& T(M) \arrow[rr, "p"] & & M
\end{tikzcd}
$$

Delete the apex, and replace it with the pullback D:

$$
\begin{tikzcd}
D \arrow[rr, "\pi_0"] \arrow[dr, "\pi_1"] \arrow[dd, "\pi_2"'] & & T^2(M) \arrow[dr, "p"] \arrow[dd, "T(p)" near start] & \\
& T^2(M) \arrow[rr, "T(p)" near start] \arrow[dd, "p" near start] & & T(M) \arrow[dd, "p"] \\
T^2(M) \arrow[rr, "p" near start] \arrow[dr, "T(p)"'] & & T(M) \arrow[dr, "p"] & \\
& T(M) \arrow[rr, "p"] & & M
\end{tikzcd}
$$

Note there is a canonical map $\langle p, T(p), T^2(p)\rangle : T^3(M) \to D$
A horizontal connection is a section h of $\langle p, T(p), T^2(p)\rangle$ that induces an 3-fold linear morphism of differential bundles.

These maps can be troublesome to work with.

$$
\begin{tikzcd}
D \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddddrr, bend right] & & T^2(M) \arrow[dr] \arrow[dd] \arrow[ddddrr, bend left] & & & \\
& T^2(M) \arrow[rr, crossing over] \arrow[dd] \arrow[ddddrr, bend right] & & T(M) \arrow[dd] \arrow[ddddrr, bend left] & & \\
T^2(M) \arrow[rr, crossing over] \arrow[dr] \arrow[ddddrr, bend right] & & T(M) \arrow[dr] \arrow[ddddrr, bend left] & & & \\
& T(M) \arrow[rr] \arrow[ddddrr, bend right] & & M \arrow[ddddrr, bend left] & & \\
& & T^3(M) \arrow[rr] \arrow[dr] \arrow[dd] & & T^2(M) \arrow[dr] \arrow[dd] & \\
& & & T^2(M) \arrow[rr, crossing over] \arrow[dd] & & T(M) \arrow[dd] \\
& & T^2(M) \arrow[rr, crossing over] \arrow[dr] & & T(M) \arrow[dr] & \\
& & & T(M) \arrow[rr] & & M
\end{tikzcd}
$$

It's most convenient to treat this as an extension property for a Kan complex.

Definition

An augmented symmetric cosimplicial object is a functor

$$X : \mathsf{FinCard} \to \mathbb{X}$$

where FinCard is the full subcategory of Set whose objects are finite cardinals.

A cosimplicial object is presented by:

$$\begin{array}{lcccl} \epsilon_i^n : & n+1 & \to & n & 1 \le i \le n \\ \delta_i^n : & n & \to & n+1 & 1 \le i \le n+1 \\ \sigma_i^n : & n & \to & n & 1 \le i \le n-1 \end{array}$$

Satisfying:

$$\begin{array}{lcll} \epsilon_i \epsilon_j & = & \epsilon_{j+1}\epsilon_i & i \le j \\ \delta_j \delta_i & = & \delta_i \delta_{j+1} & i \le h \\ \delta_i \epsilon_j & = & \begin{cases} \epsilon_{j-1}\delta_i & i < j \\ \delta_i \epsilon_j = 1 & i = j, j+1 \\ \delta_i \epsilon_j = \epsilon_j \delta_{i-1} & i > j+1 \end{cases} & \end{array}$$

Theorem (Cruttwell and Lucyshyn-Wright)

The E-valued sector forms on M form an augmented symmetric cosimplicial abelian group.

The structure maps are given by:

- The codegeneracy: $\epsilon_i^n(\omega) := T^{i-1}(\ell)\omega$
- The symmetry: $\sigma_i^n(\omega) := T^{i-1}(c)\omega$
- The generating coface map: $\delta_1^n(\omega) = T(\omega)\hat{p}$

The coface maps are generated by:

$$\delta_i^n(\omega) := \sigma_i \circ \cdots \circ \sigma_1 \circ \delta_1^n(\omega)$$

for $i > 1$.

Definition (**un**shuffles)

An (m, n)-shuffle map is an $n + m$ permutation σ so that:

- $0 \leq i < j < m : \sigma(i) \leq \sigma(j)$
- $m \leq i < j < (m + n) : \sigma(i) \leq \sigma(j)$

an (n, m)-unshuffle is a permutation so that σ^{-1} is an (n, m) shuffle. Denote the n, m unshuffles $U(n, m)$

Definition (Unshuffle operator)

The unshuffle operator on $n + m$ sector forms is define

$$us_{n,m}(\omega) := \sum_{\gamma \in U(n,m)} sgn(\gamma) \cdot \sigma_\gamma(\omega)$$

where we treat σ_γ is the operator defined by the permutation γ.

The exterior derivative:

$$\partial(\omega) := \sum_{i=1}^{n+1} (-1)^{i-1} \delta_i^n(\omega)$$

is precisely $\partial(\omega) = us_{1,n}(\delta_1^n(\omega))$

Theorem (Cruttwell and Lucyshyn-Wright)

$(\Psi^\bullet(M,E), \partial)$ is a cochain complex, the alternating forms form a subcomplex.

Remark

Lavendhomme and Minguez use the antisymmetrization operator.

$$A(\omega) = \frac{1}{n!} \sum_{\gamma \in S_n} sgn(\gamma) \sigma_\gamma(\omega)$$

Tensor Product of Forms

Suppose we have a differential object R with a multiplication

$$\cdot : R \times R \to R$$

so that

$$T(\cdot)\kappa = \langle \kappa, p \rangle \cdot + \langle p, \kappa \rangle \cdot$$

The tensor product of forms was introduced by Minguez.

$$\omega \otimes \gamma := \langle T^m(p^m)\omega, p^n\gamma \rangle \cdot$$

Then we have that:

$$\delta_1^{n+m}(\omega \otimes \gamma) = \langle T^{m+1}(p^m)\delta_1^n(\omega), T(p^n)\delta_1^m(\gamma) \rangle$$

Definition (Wedge Product for forms)
$\omega \in \Psi^n(M, R), \gamma \in \Psi^m(M, R)$ is

$$\omega \wedge \gamma := us_{n,m}(\omega \otimes \gamma) \in \Psi^{m+n}(M, R)$$

Theorem
Let $\omega \in \Psi^n(M, R), \gamma \in \Psi^m(M, R)$

1. $\partial(\omega \wedge \gamma) = \partial(\omega) \wedge \gamma + (-1)^n \omega \wedge \partial(\gamma)$
2. $\omega \wedge \gamma = (-1)\gamma \wedge \omega$
3. ω, γ *alternating, then* $\omega \wedge \gamma$ *is alternating.*

Theorem
For a linear ring R, the complex of sector forms $\Psi^\bullet(M, R)$ is a differential graded algebra, and the alternating sector forms form a subalgebra.

Remark

Minguez and Lavendhomme use the antisymmetrization operator.

$$\omega \wedge \gamma := \frac{1}{(n+m)!} \sum_{\nu \in S_{n+m}} sgn(\nu) \cdot \sigma_\nu(\omega \otimes \gamma)$$

this has some disadvantages:

- *Requires $\mathbb{N}$ invertible*
- *Sends a pair of sector forms to an alternating sector form.*

The Interior Product

Definition
Given a vector field $x \in \chi(M)$, $\omega \in \Psi^n(M, E)$ define the interior product:

$$i_x(\omega) := T^n(x)\omega$$

Definition
Given a vector field $x \in \chi(M)$, $\omega \in \Psi^n(M, E)$ define the Lie derivative

$$L_x(\omega) := i_x(\delta^n_{n+1}(\omega))$$

Proposition
The Lie derivative induces a cochain endomorphism

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & \Psi^k(M,E) \arrow[r, "\partial"] \arrow[d, "L_x"] & \Psi^{k+1}(M,E) \arrow[r, "\partial"] \arrow[d, "L_x"] & \dots \\
\dots \arrow[r, "\partial"] & \Psi^k(M,E) \arrow[r, "\partial"] & \Psi^{k+1}(M,E) \arrow[r, "\partial"] & \dots
\end{tikzcd}$$

The Cartan Calculus is a collection of relationships on the commutator of the operators i, L, ∂.

Theorem (Cartan Calculus)

1. *If ω is an alternating form:* $[i_x, i_y] = 0$.
2. $[L_x, L_y] = L_{[x,y]}$
3. $[L_x, i_y] = i_{[x,y]}$
4. *Cartan's Homotopy Formula:* $L_x = (-1)^n \cdot [\partial, i_x]$

Proof.
The homotopy formula is immediate:

$$\begin{aligned} i_x(\partial(\omega)) - \partial(i_x(\omega)) &= i_x(u_{1,n}(\delta_1^n(\omega))) - u_{1,n-1}(i_x(\delta_1^n(\omega))) \\ &= (-1)^n i_x \delta_{n+1}^n(\omega) = (-1)^n L_x(\omega) \end{aligned}$$

$\square$

To prove the other two formulas we need the following:

Lemma
For any $f : X \to T^2(M)$ so that $\{f\}$ is well defined:

$$T^k(\{f\}) = \{T^{k+1}(f) c_{k+1} T(c_{(k+1)})\} c_{k+1}^{-1}$$

where $c_0 = 1, c_i := T^{i-1}(c) c_{i-1}$

$[L_x, L_y]$:

Proof.
Expand the the left term to $T^k(\{xT(y) - yT(x)c\})\delta^n_{n+1}(\omega)$
Using the above lemma, find:

$$\{T^k(xT(y) - yT(x)c)c_{(k)}T(c_{(k)})\}T(\omega)\kappa$$

Pull the $T(\omega)\kappa$ inside the brackets:

$$\{T^k(xT(y) - yT(x)c)c_{(k)}T(c_{(k)})T(T(\omega)\kappa)\}$$

Now the codomain has a vertical connection κ, so we have:

$$T^k(xT(y) - yT(x)c)c_{(k)}T(c_{(k)})T(T(\omega)\kappa)\kappa$$

Some manipulation will yield $L_x(L_y(\omega)) - L_y(L_x(\omega))$ $\square$

Conclusion

- Sector forms emphasize cosimplicial structure, rather than the cochain complex.
- The unshuffle operator gives a canonical construction of cochain structures from simplicial.
- We get a differential graded algebra of sector forms, and a differential graded algebra of singular forms.
- We get all the maps of the Cartan Calculus, satisfying analogous identities.
- This lets you formalize basic Hamiltonian mechanics in a tangent category, using a "symplectic sector form"

Further Work

- Using n-fold linearity, it is simple to generalize this so that the codomain is a differential bundle with a vertical connection. There are various Nijenjuis calculi for differential forms on vector bundles that can be developed using differential bundle valued sector forms.
- Cartan's calculus is about the interaction with $\chi(M)$ and $\Psi^\bullet(M, R)$. What about higher *sector fields*?
- The collection of *n*-sectors $\chi^n(M)$ is a semi-simplicial set - higher order connections in the sense of Bertram may be naturally interpreted as a *extension* property, yielding a Kan complex.
- Classical Field Theory is may be studied using n-plectic manifolds, so this should be possible using n-plectic sector forms and the Cartan calculus.

Bertram, Wolfgang (2008). *Differential geometry, Lie groups and symmetric spaces over general base fields and rings*. American Mathematical Soc.

Cartan, Henri (1950). "La transgression dans un groupe de Lie et dans un espace fibré principal". In: *Colloque de topologie (espaces fibrés)*. Bruxelles, pp. 57–71.

Cockett, J Robin B and Geoff SH Cruttwell (2014). "Differential structure, tangent structure, and SDG". In: *Applied Categorical Structures* 22.2, pp. 331–417.

Cruttwell, GSH and Rory BB Lucyshyn-Wright (2018). "A simplicial foundation for differential and sector forms in tangent categories". In: *Journal of Homotopy and Related Structures*, pp. 1–59.

Lavendhomme, René (1996). "Basic Concepts of Synthetic Differential Geometry". In:

Minguez, MC (1985). "Cálculo diferencial sintético y su interpretacion en modelos de prehaces". In: *Publ, Sem, Mat, Serie II, Sec* 2.

Nishimura, Hirokazu (1997). "Synthetic hamiltonian mechanics". In: *International Journal of Theoretical Physics* 36.1, pp. 259–279.