A Kantorovich Monad for Ordered Spaces

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Probability monads

Idea [Giry, 1982]:

Spaces of random elements as formal convex combinations.

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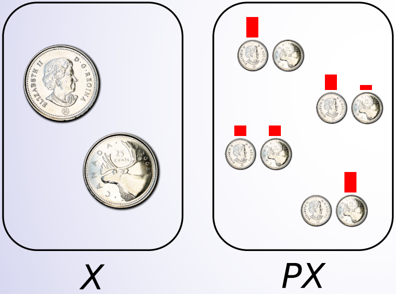

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- Functor $X \mapsto PX$

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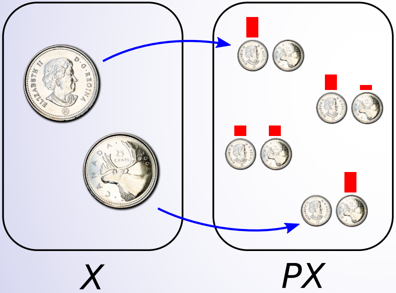

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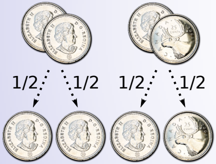

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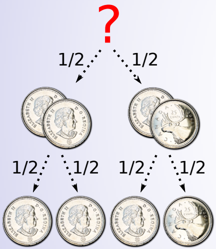

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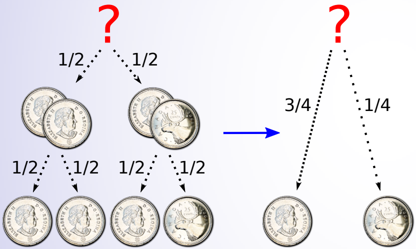

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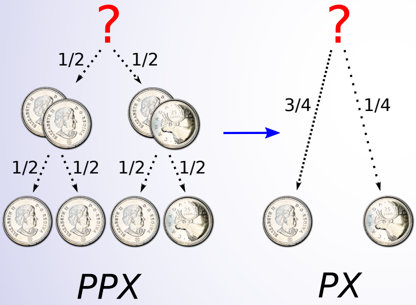

- Base category C
- Functor $X \mapsto PX$
- Unit $\delta : X \to PX$
- Composition
  $E : PPX \to PX$
  [Lawvere, 1962]

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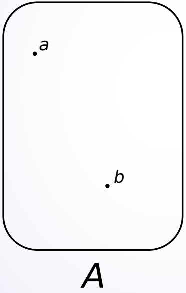

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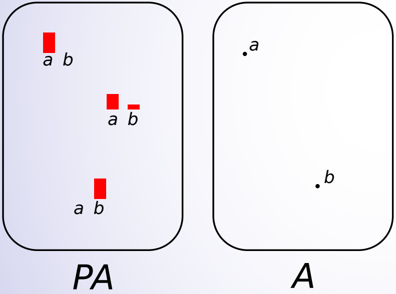

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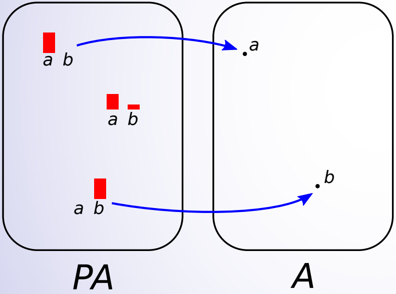

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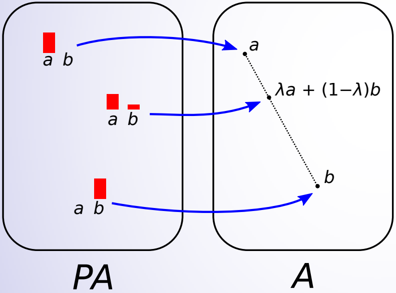

- Algebras $e : PA \to A$ are "convex spaces"
- Formal averages are mapped to actual averages

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Radon monad [Świrszcz, 1974]:

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- The map $\delta : X \to PX$ assigns to x the Dirac delta measure δ_x.
- The map $E : PPX \to PX$ gives the average:
$$(E\mu) : A \mapsto \int_{PX} p(A)\, d\mu(p).$$

Probability monads

Kantorovich monad [van Breugel, 2005, Fritz and Perrone, 2017]:

- Given a complete metric space X, PX is the set of Radon probability measures of finite first moment, equipped with the *Wasserstein distance*, or *Kantorovich-Rubinstein distance*, or *earth mover's distance*:

$$d_{PX}(p,q) = \sup_{f:X\to\mathbb{R}} \left| \int_X f(x)\, d(p-q)(x) \right|$$

- Functorial and monad structure are analogous, where the morphisms are the *short maps*.

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- If X is compact, PX is compact [Villani, 2009].

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Kantorovich	CMet	Closed convex subsets of Banach spaces [b]

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The stochastic order

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Constructing an order on PX entending the order on X.

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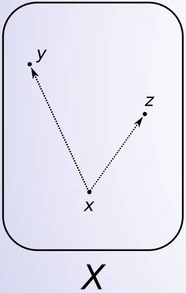

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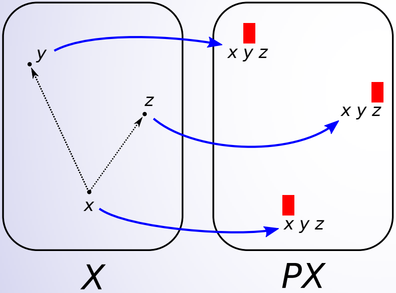

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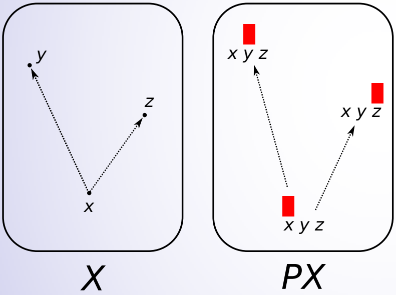

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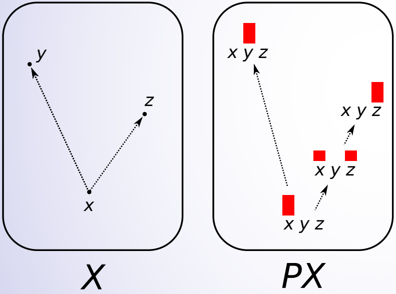

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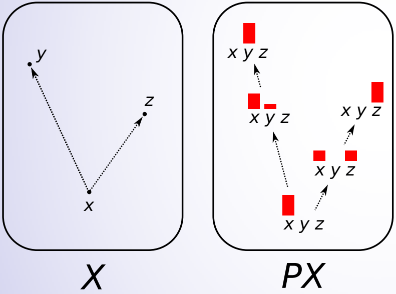

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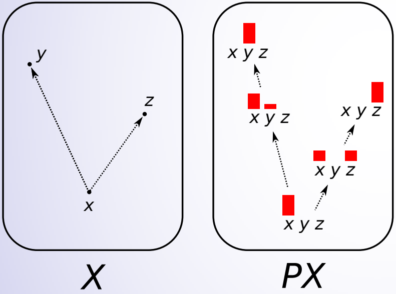

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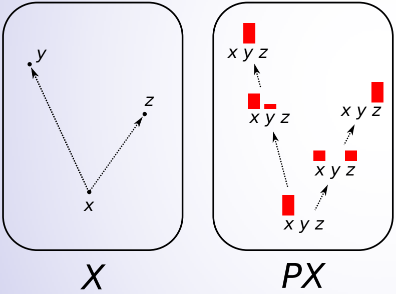

- "Extending the order by convexity"
- "Moving the mass upwards"
- "Larger measure to upper sets"

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Theorem [Strassen, 1965]:

Let X be a Polish space equipped with a closed preorder. Let p, q be Radon probability measures on X. Then the following are equivalent:

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Definition:

We say that $p \leq q$ in the *usual stochastic order*.

Ordered Wasserstein spaces

Definition:

Let X be a complete metric space with a closed preorder. We call *Wasserstein space* the space PX of Radon probability measures of finite first moment, equipped with the *Wasserstein distance*, or *Kantorovich-Rubinstein distance*, or *earth mover's distance*:

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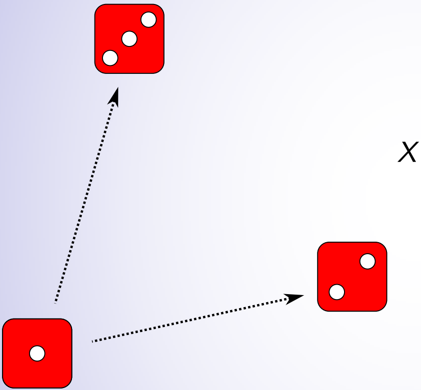
Theorem:

If X is compact and partially ordered, PX is partially ordered.

Ordered Wasserstein spaces



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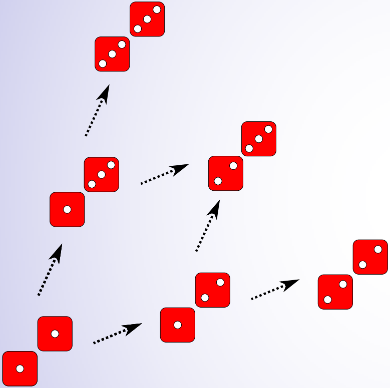

$$X \xrightarrow{i_{1,2}} X_2$$

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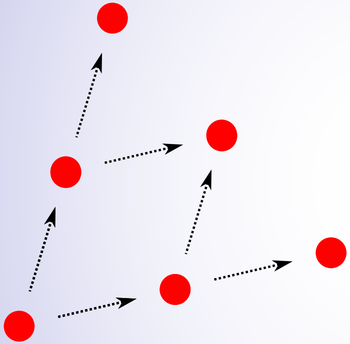

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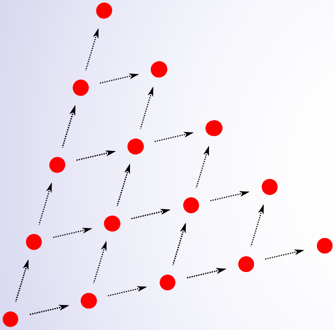

$$X \xrightarrow{i_{1,2}} X_2 \xrightarrow{i_{2,4}} X_4$$

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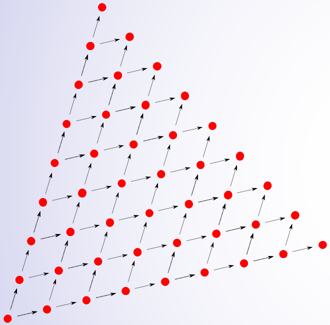

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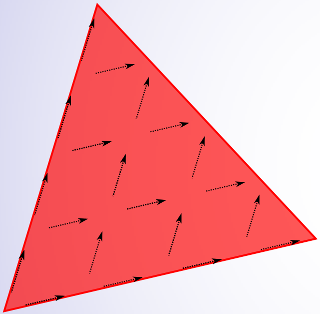

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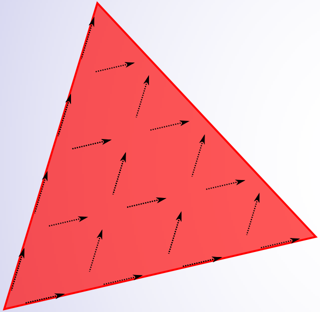

$$\begin{tikzcd} X \arrow[r, "i_{1,2}"] \arrow[drr, "i_1"'] & X_2 \arrow[r, "i_{2,4}"] \arrow[dr, "i_2"] & X_4 \arrow[r] \arrow[d, "i_4"] & \dots \\ & & PX & \end{tikzcd}$$

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- Maps $e : PA \to A$ (integration), where A is a convex space.

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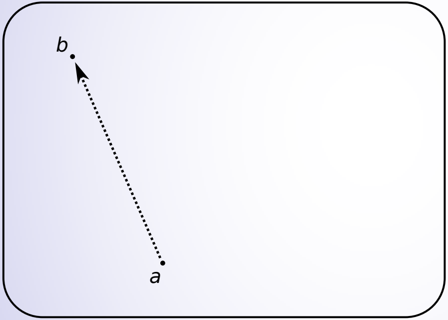

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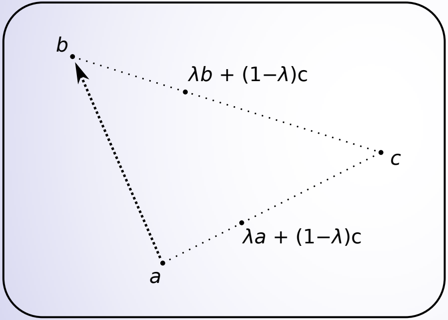

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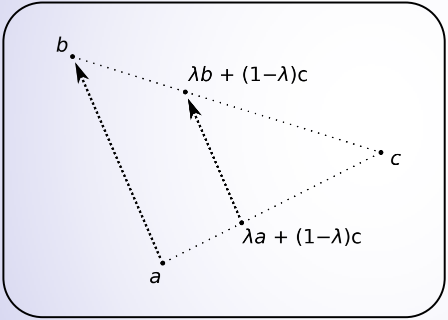

$$a \leq b$$
$$\Downarrow$$
$$\lambda a + (1-\lambda)c$$
$$\leq$$
$$\lambda b + (1-\lambda)c$$

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Radon	KOHaus	Compact convex subsets of locally convex top. vector spaces w. closed pos. cone [a]
Kantorovich	KOMet (CPOMet)	Compact convex subsets of Banach spaces w. closed pos. cone (closed subsets, wedge)

[a] [Keimel, 2008]

Higher structure

Pointwise order: $f \leq g : X \to Y$ iff for every $x \in X$, $f(x) \leq g(x)$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

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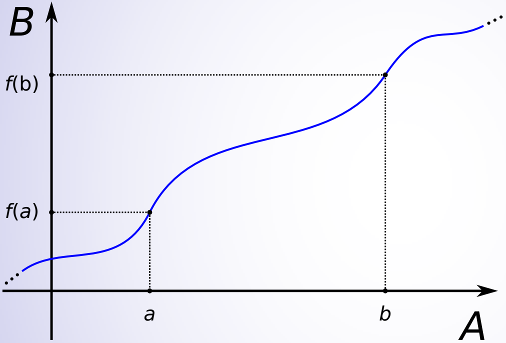
Proposition:

Let $f \leq g : X \to Y$. Then $Pf \leq Pg : PX \to PY$.

Corollary:

CPOMet and KOMet are strict 2-categories, and P a strict 2-monad.

Higher structure



$$\begin{tikzcd} PA \arrow[r, "Pf"] \arrow[d, "e"'] & PB \arrow[d, "e"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

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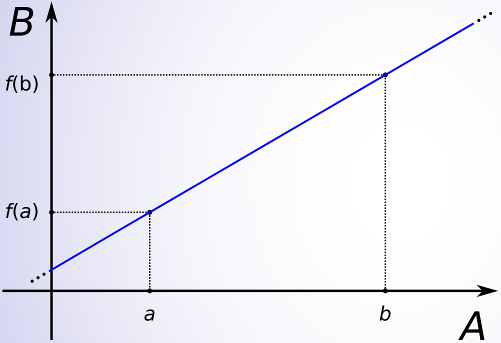

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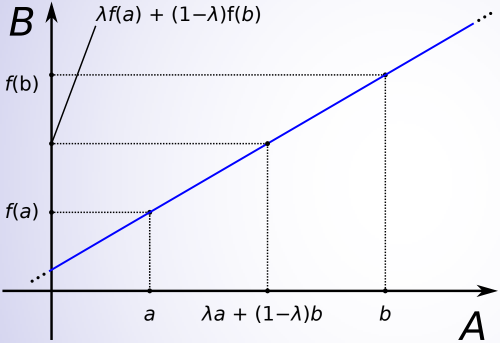

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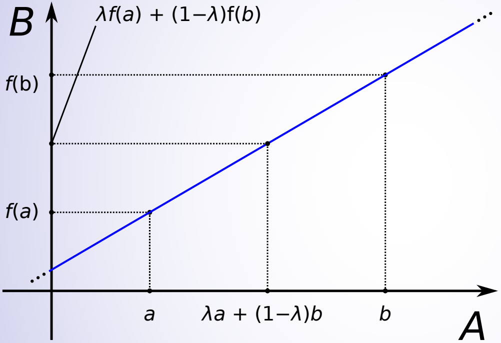

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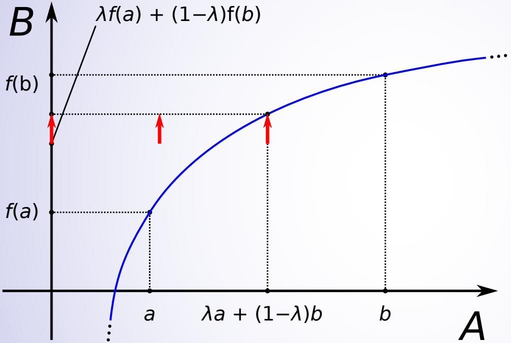

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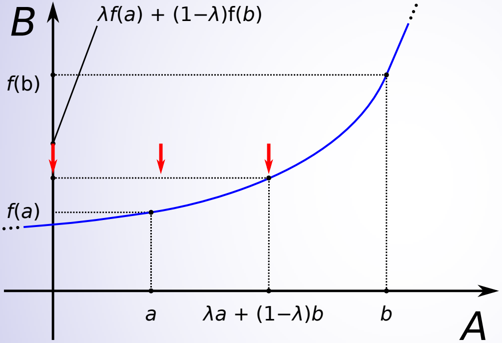

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Higher structure

Theorem:
Let A be a P-algebra. Consider $\mathbb{R}$ with its usual order. Let $f : A \to \mathbb{R}$ be short and monotone. Then:

- f is *affine* if and only if it is a *strict* P-morphism;
- f is *concave* if and only if it is a *lax* P-morphism;
- f is *convex* if and only if it is an *oplax* P-morphism;

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This allows to define concave and convex function between general ordered vector spaces, giving a categorical characterization.

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Remark:

For $f, g : \mathbb{R} \to \mathbb{R}$ convex, $g \circ f$ may not be convex.

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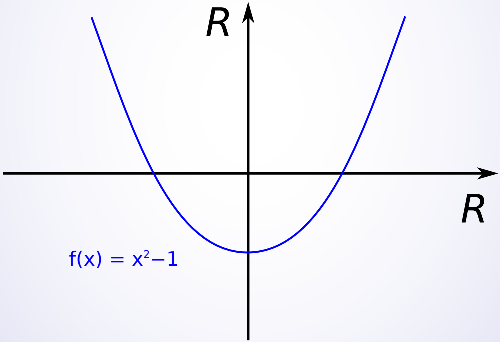

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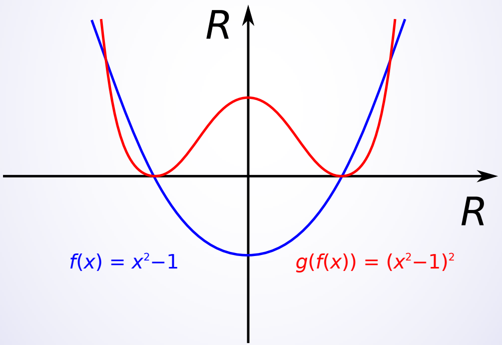

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This is just composition in a category (of oplax morphisms).

The exchange law

Idea:
Stronger compatibility condition between metric and order.

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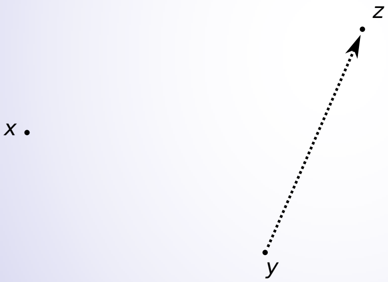

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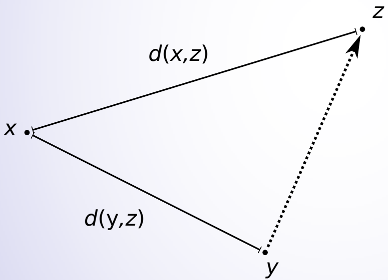

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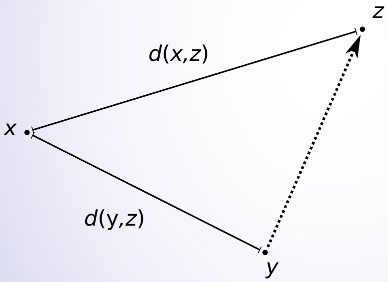

In LMSs:

$$y \leq z \;\Rightarrow\; d(x,z) \leq d(x,y)$$

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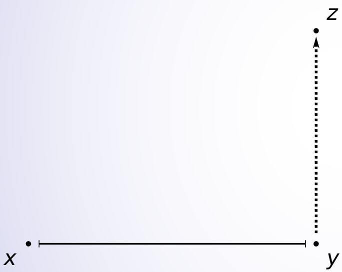

Given $y \leq z$ and x,

The exchange law

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$$\begin{tikzcd} w \arrow[r, dash] & z \\ x \arrow[u, dotted] \arrow[r, dash] & y \arrow[u, dotted] \end{tikzcd}$$

Given $y \leq z$ and x, $\exists w \geq x$ such that $d(w, z) \leq d(x, y)$.

The exchange law

Proposition:

Let X be an ordered metric space satisfying the exchange law. Then the Lawvere metric induced by the metric and the order is given by:

$$d'(p, q) = \sup_f \int f \, dp - \int f \, dq$$

where f varies between short monotone functions $X \to \mathbb{R}^+$.

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where f varies between short monotone functions $X \to \mathbb{R}^+$.

Corollary (cfr. [Hiai et al., 2018])

$p \leq q$ if and only if for every short monotone map $f : X \to \mathbb{R}^+$,

$$\int f\,dp \leq \int f\,dq.$$

References

Fritz, T. and Perrone, P. (2017).
A Probability Monad as the Colimit of Finite Powers.
Submitted. arXiv:1712.05363.

Giry, M. (1982).
A Categorical Approach to Probability Theory.
In *Categorical aspects of topology and analysis*, volume 915 of *Lecture Notes in Mathematics*.

Hiai, F., Lawson, J., and Lim, Y. (2018).
The stochastic order of probability measures on ordered metric spaces.
Journal of Mathematical Analysis and Applications, 464(1):707–724.

Jones, C. and Plotkin, J. D. (1989).
A Probabilistic Powerdomain of Evaluations.
Proceedings of the Fourth Annual Symposium of Logics in Computer Science.

Keimel, K. (2008).
The monad of probability measures over compact ordered spaces and its Eilenberg-Moore algebras.
Topology Appl., 156(2):227–239.

Lawvere, W. (1962).
The category of probabilistic mappings.
Available at https://ncatlab.org/nlab/files/lawvereprobability1962.pdf.

Strassen, V. (1965).
The existence of probability measures with given marginals.
Annals of Mathematical Statistics, 36(2423–439).

Świrszcz, T. (1974).
Monadic functors and convexity.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 22.

van Breugel, F. (2005).
The Metric Monad for Probabilistic Nondeterminism.
Available at http://www.cse.yorku.ca.

Villani, C. (2009).
Optimal transport: old and new, volume 338 of *Grundlehren der mathematischen Wissenschaften*.
Springer.

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