The Representability of Partial Recursive Functions in Arithmetical Theories and Categories

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First-order theories

- With equality
- $\Gamma \vdash \varphi$ satisfying the rules for intuitionistic sequent calculus
- Logical axioms:
  - For all theories, decidability of equality (DE):

$$x \neq y \vee x = y$$

  - To obtain classical theories, the excluded middle (EM): $\neg\varphi \vee \varphi$, for all formulas φ

The arithmetical theory M

- Let $\mathscr{L}_M$ be the first-order language with 0, S, $\cdot$, $+$.
- Let $S^n(0)$ be the n$^{\text{th}}$ numeral, denoted $\overline{n}$.
- Let $x < y$ abbreviate $(\exists w)(x + S(w) = y)$.
- Let $(\exists! y)\varphi(\mathbf{x}, y)$ abbreviate

$$(\exists y)\varphi(\mathbf{x}, y) \wedge (\forall y)(\forall z)(\varphi(\mathbf{x}, y) \wedge \varphi(\mathbf{x}, z) \Rightarrow y = z).$$

The arithmetical theory M

Definition

M is a theory over $\mathscr{L}_M$ with the nonlogical axioms

(M1) $S(x) \neq 0$

(M2) $S(x) = S(y) \Rightarrow x = y$

(M3) $x + 0 = x$

(M4) $x + S(y) = S(x + y)$

(M5) $x \cdot 0 = 0$

(M6) $x \cdot S(y) = (x \cdot y) + x$

(M7) $x \neq 0 \Rightarrow (\exists y)(x = S(y))$

(M8) $x < y \vee x = y \vee y < x$

We consider an arbitrary *arithmetical theory* T, i.e. a consistent r.e. extension of M.

Recursive functions, brief overview

- Primitive recursive: basic functions; closed under subsitution (S) and primitive recursion (PR)
- Total recursive: basic functions; closed under (S), (PR), and total μ
- Partial recursive: basic functions; closed under (S), (PR), and partial μ

Representability of total functions

Definition

A function $f : \mathbb{N}^k \to \mathbb{N}$ is *numeralwise representable* in T as a total function if there exists a formula $\varphi(\mathbf{x}, y)$ satisfying

(a) for all $\mathbf{m}, n \in \mathbb{N}^{k+1}$, if $f(\mathbf{m}) = n$, then $\vdash \varphi(\overline{\mathbf{m}}, \overline{n})$

(b) for all $\mathbf{m} \in \mathbb{N}^k$, $\vdash (\exists! y)\varphi(\overline{\mathbf{m}}, y)$

f is *strongly representable* in T as a total function if there exists a formula $\varphi(\mathbf{x}, y)$ satisfying (a) and

(b)' $\vdash (\exists! y)\varphi(\mathbf{x}, y)$

Representability of partial functions

Definition

For $f : \mathbb{N}^k \dashrightarrow \mathbb{N}$ and $\varphi(\mathbf{x}, y)$ consider the conditions

(P1) for all $\mathbf{m}, n \in \mathbb{N}^{k+1}$, $f(\mathbf{m}) \simeq n$ iff $\vdash \varphi(\overline{\mathbf{m}}, \overline{n})$

(P2) for all $\mathbf{m} \in \mathbb{N}^k$, $\vdash \varphi(\overline{\mathbf{m}}, y) \wedge \varphi(\overline{\mathbf{m}}, z) \Rightarrow y = z$

(P3) $\vdash \varphi(\mathbf{x}, y) \wedge \varphi(\mathbf{x}, y) \Rightarrow y = z$

(P4) $\vdash (\exists! y)\varphi(\mathbf{x}, y)$

For $f : \mathbb{N}^k \dashrightarrow \mathbb{N}$, if there exists $\varphi(\mathbf{x}, y)$ in T such that

- (P1) and (P2) hold, f is *numeralwise representable* in T as a partial function
- (P1) and (P3) hold, f is *type-one representable* in T
- (P1) and (P4) hold, f is *strongly representable* in T as a partial function

Representability theorems for partial recursive functions

Theorem (I)

Let T be any arithmetical theory. All partial recursive functions are type-one representable in T.

Theorem (II)

Let T be a ***classical*** *arithmetical theory. All partial recursive functions are strongly representable in T as partial functions.*

Consequences of the Existence Property (EP)

The Existence Property (EP)

For every formula φ in T and any variable x occurring free in φ,

$$\text{if} \vdash (\exists x)\varphi, \text{ then } \exists\, n \in \mathbb{N} \text{ such that} \vdash \varphi\left[\frac{\overline{n}}{x}\right].$$

The Kleene normal form theorem (alternate version)

Theorem (Kleene normal form)

For each $k \in \mathbb{N}$, $k > 0$, there exist primitive recursive functions $U : \mathbb{N} \to \mathbb{N}$ and $T_k : \mathbb{N}^{k+2} \to \mathbb{N}$ such that, for any partial recursive function $f : \mathbb{N}^k \dashrightarrow \mathbb{N}$, there exists a number $e \in \mathbb{N}$ such that

$$f(\mathbf{m}) \simeq U(\mu_n(T_k(e, \mathbf{m}, n) = 0))$$

for all $\mathbf{m} \in \mathbb{N}^k$.

The strong representability of primitive recursive functions in arithmetical theories

Theorem

Let T be any arithmetical theory. All primitive recursive functions are strongly representable in T as total functions.

Proof.

It suffices to express the basic functions and the recursion schemes (S) and (PR) by formulas in T. For example:

- $y = S(x)$ strongly represents the successor function.
- If $g, h : \mathbb{N} \to \mathbb{N}$ are primitive recursive and strongly representable by $\psi(y, z)$ and $\varphi(x, y)$, respectively, then

  $$(\exists y)(\varphi(x, y) \wedge \psi(y, z))$$

  strongly represents $f = g(h) : \mathbb{N} \to \mathbb{N}$.

$\square$

Representing functions obtained by partial minimisation

Lemma (1)

Let $g : \mathbb{N}^{k+1} \to \mathbb{N}$ $(k \geq 0)$ be a total function that is numeralwise representable in T as a total function, and let $f : \mathbb{N}^k \dashrightarrow \mathbb{N}$ be obtained from g by partial μ. Then, f is type-one representable in T.

Proof.

g is numeralwise representable in T by $\sigma(\mathbf{x}, y, z)$ and f is defined by

$$f(\mathbf{m}) \simeq \mu_n(g(\mathbf{m}, n) = 0).$$

Thus, f is type-one representable in T by the formula

$$\sigma(\mathbf{x}, y, 0) \wedge (\forall u)(u < y \Rightarrow \neg\sigma(\mathbf{x}, u, 0)).$$

$\square$

Weak representability of r.e. relations

Definition (Weak representability)

A relation $E \subseteq \mathbb{N}^k$ is *weakly representable* in T if there exists a formula $\psi(\mathbf{x})$ with exactly k free variables such that, for all $\mathbf{m} \in \mathbb{N}^k$,

$$E(\mathbf{m}) \text{ iff } \vdash \psi(\overline{\mathbf{m}}).$$

Lemma (2)

All k-ary r.e. relations on $\mathbb{N}$ $(k \geq 0)$ are weakly representable in T.

(long technical proof)

Proof of Theorem (I)

Theorem (I)

Let T be any arithmetical theory. All partial recursive functions are type-one representable in T.

Proof.

Let $f : \mathbb{N}^k \dashrightarrow \mathbb{N}$ be a partial recursive function.

$k = 0$: If f is the constant n in $\mathbb{N}$, take the formula $\overline{n} = y$. If f is completely undefined, take the formula $y = y \wedge 0 \neq 0$.

Proof of Theorem (I)

Proof (continued).

$k \geq 1$: By the Kleene normal form theorem, we obtain $U : \mathbb{N} \to \mathbb{N}$, $T_k : \mathbb{N}^{k+2} \to \mathbb{N}$, and $e \in \mathbb{N}$ such that

$$f(\mathbf{m}) \simeq U(\mu_n(T_k(e, \mathbf{m}, n) = 0)) \qquad \forall \mathbf{m} \in \mathbb{N}^k.$$

As T_k is primitive recursive, by Lemma 1 there exists a formula $\sigma(\mathbf{x}, z)$ that type-one represents the partial function given by

$$\mu_n(T_k(e, \mathbf{m}, n) = 0) \qquad \forall \mathbf{m} \in \mathbb{N}^k.$$

As U is primitive recursive, there exists a formula $\varphi(z, y)$ that strongly represents U as a total function. φ also type-one represents U.

Proof of Theorem (I)

Proof (continued).

By Lemma 2, there exists a formula $\eta(\mathbf{x})$ that weakly represents the r.e. domain D_f of f. Then, f is type-one representable in T by the formula $\theta(\mathbf{x}, y)$ defined by

$$\eta(\mathbf{x}) \wedge (\exists z)(\sigma(\mathbf{x}, z) \wedge \varphi(z, y)).$$

Indeed,

- (P3) for θ follows from (P3) for σ and φ.
- For (P1), since η weakly represents D_f, we only have to consider inputs on which f is defined. Hence, we can show that $\vdash \theta(\overline{\mathbf{m}}, \overline{p})$ implies $f(\mathbf{m}) \simeq p$ by (P3) for θ and the fact that $\vdash \overline{f(\mathbf{m})} = \overline{p}$ iff $f(\mathbf{m}) = p$.

$\square$

Exact separability of r.e. relations

Definition (Exact separability)

Two relations $E, F \subseteq \mathbb{N}^k$ are *exactly separable* in T if there exists a formula $\psi(\mathbf{x})$ in T with exactly k free variables such that, for all $\mathbf{m} \in \mathbb{N}^k$,

$$E(\mathbf{m}) \text{ iff } \vdash \psi(\overline{\mathbf{m}})$$
$$F(\mathbf{m}) \text{ iff } \vdash \neg\psi(\overline{\mathbf{m}})$$

Lemma (3)

Let T be a ***classical*** *arithmetical theory. Any two disjoint k-ary r.e. relations on $\mathbb{N}$ $(k \geq 0)$ are exactly separable in T.*

(long technical proof)

Proof of Theorem (II)

Theorem (II)

*Let T be a **classical** arithmetical theory. All partial recursive functions are strongly representable in T as partial functions.*

Proof.

Let $f : \mathbb{N}^k \dashrightarrow \mathbb{N}$ be a partial recursive function.

$k = 0$: If f is completely undefined, let G be a closed undecidable formula in T and take

$$(y = 0 \Rightarrow \neg G) \wedge (y \neq 0 \Rightarrow G) \wedge y < \overline{2}$$

Proof of Theorem (II)

Proof (continued).

$k \geq 1$: Let $n_0, n_1 \in \mathbb{N}$ be distinct. By Lemma 3, we obtain a formula $\sigma(\mathbf{x})$ that exactly separates $f^{-1}(\{n_0\})$ and $f^{-1}(\{n_1\})$. By Theorem (I), we obtain a formula $\varphi(\mathbf{x}, y)$ that type-one represents f. Consider

$$\psi(\mathbf{x}) \underset{\text{def}}{\equiv} (\exists z)\varphi(\mathbf{x}, z) \wedge \neg\varphi(\mathbf{x}, \overline{n_0}) \wedge \neg\varphi(\mathbf{x}, \overline{n_1})$$

$$\begin{aligned} \theta(\mathbf{x}, y) \underset{\text{def}}{\equiv} & (\psi(\mathbf{x}) \wedge \varphi(\mathbf{x}, y)) \vee (\neg\psi(\mathbf{x}) \wedge \sigma(\mathbf{x}) \wedge y = \overline{n_0}) \\ & \vee(\neg\psi(\mathbf{x}) \wedge \neg\sigma(\mathbf{x}) \wedge y = \overline{n_1}). \end{aligned}$$

By (EM), $\vdash (\neg\psi(\mathbf{x}) \wedge \neg\sigma(\mathbf{x})) \vee (\neg\psi(\mathbf{x}) \wedge \sigma(\mathbf{x})) \vee \psi(\mathbf{x})$, from which (P4) follows. (P1) is obtained by cases. $\square$

Representability of total recursive functions

Corollary (of Theorem (I))

Let T be an arithmetical theory. All total recursive functions are numeralwise representable in T as total functions.

Corollary (of Theorem (II))

*Let T be a **classical** arithmetical theory. All total recursive functions are strongly representable in T as total functions.*

Classifying categories

- Given a theory T, we construct a classifying category $\mathscr{C}(T)$:
  - objects: formulas of T
  - morphisms: equivalence classes of provably functional relations between formulas
- For a general theory T, $\mathscr{C}(T)$ is regular.
- If T is an intuitionistic arithmetical theory, we claim that in $\mathscr{C}(T)$:
  - there is at least a weak NNO;
  - numerals are standard;
  - 1 is projective and indecomposable.

Work in progress

For an arithmetical theory T:

1. Consider the formulas representing recursive functions in $\mathscr{C}(T)$ (for all possible variations). What sub-categories do we obtain?
2. Construct a partial map category associated with $\mathscr{C}(T)$ and show it is a Turing category.
3. Ultimately, we want to consider partial recursive functionals of higher type using a notion of λ-calculus with equalisers and a construction of the free CCC with equalisers.

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Kleene Equality

To deal with partialness, we use Kleene Equality. If e_1 and e_2 are two expressions on $\mathbb{N}$ that may or may not be defined, then

$$e_1 \simeq e_2 \text{ iff } (e_1, e_2 \text{ are defined and equal}) \\ \text{OR } (e_1, e_2 \text{ are undefined}).$$

For example, if $f : \mathbb{N}^k \dashrightarrow \mathbb{N}$ is a partial function and $\mathbf{m}, n \in \mathbb{N}^{k+1}$,

$$f(\mathbf{m}) \not\simeq n \text{ iff } (f(\mathbf{m}) \text{ is defined but not equal to } n) \\ \text{OR } (f(\mathbf{m}) \text{ is undefined}).$$

Consequences of the Existence Property (EP)

If T is classical:

- G is a closed undecidable formula in T, $f : \mathbb{N}^k \dashrightarrow \mathbb{N}$ the completely undefined function.

  $$\varphi(\mathbf{x}, y) \underset{\text{def}}{\equiv} \mathbf{x} = \mathbf{x} \wedge (y = 0 \Rightarrow \neg G) \wedge (y \neq 0 \Rightarrow G) \wedge y < \overline{2}$$

  strongly represents f in T as a partial function.
- If T were to satisfy EP, then $\vdash G$ or $\vdash \neg G$, a contradiction.

Consequences of the Existence Property (EP)

If T is intuitionistic:

- Let $f : \mathbb{N} \dashrightarrow \mathbb{N}$ be a partial function undefined at $m \in \mathbb{N}$.
- Suppose that there exists $\varphi(x, y)$ satisfying (P1) and (P4).
- By (P4), $\vdash (\exists y)\varphi(\overline{m}, y)$.
- By EP, there exists $n \in \mathbb{N}$ such that $\vdash \varphi(\overline{m}, \overline{n})$.
- By (P1), $f(m) \simeq n$, and so $f(m)$ is defined. Contradiction.

So, strong representability of partial functions doesn't make sense and Theorem (II) fails.

A technical lemma

Lemma

Let $E_1 \subseteq \mathbb{N}^k$ *and* $E_2 \subseteq \mathbb{N}^{k+j}$ *(*$k, j \geq 0$*) be r.e. relations. There exists a formula* $\varphi(\mathbf{x}, \mathbf{u})$ *in* T *with* $k + j$ *free variables such that, for all* $\mathbf{m} \in \mathbb{N}^k$ *and* $\mathbf{p} \in \mathbb{N}^j$*,*

if $E_1(\mathbf{m})$ *and* $\neg E_2(\mathbf{m}, \mathbf{p})$*, then* $\vdash \varphi(\overline{\mathbf{m}}, \overline{\mathbf{p}})$
if $\neg E_1(\mathbf{m})$ *and* $E_2(\mathbf{m}, \mathbf{p})$*, then* $\nvdash \varphi(\overline{\mathbf{m}}, \overline{\mathbf{p}})$*.*

A technical lemma

Proof (idea).

(Adapted from the case for $j = 0$ in [S]) Let $E_1 \subseteq \mathbb{N}^k$ and $E_2 \subseteq \mathbb{N}^{k+j}$ $(k, j \geq 0)$ be r.e. relations. There exist primitive recursive relations $F_1 \subseteq \mathbb{N}^{k+1}$, $F_2 \subseteq \mathbb{N}^{k+j+1}$ such that, for all $\mathbf{m} \in \mathbb{N}^k$ and $\mathbf{p} \in \mathbb{N}^j$,

$$E_1(\mathbf{m}) \text{ iff } \exists n \in \mathbb{N} \text{ s.t. } F_1(\mathbf{m}, n)$$
$$E_2(\mathbf{m}, \mathbf{p}) \text{ iff } \exists n \in \mathbb{N} \text{ s.t. } F_2(\mathbf{m}, \mathbf{p}, n).$$

We obtain formulas $\psi_1(\mathbf{x}, y)$ and $\psi_2(\mathbf{x}, \mathbf{u}, y)$ that numeralwise represent F_1 and F_2, respectively, in T. Then, $\varphi(\mathbf{x}, \mathbf{u})$ given by

$$(\exists y)(\psi_1(\mathbf{x}, y) \wedge (\forall z)(z \leq y \Rightarrow \neg\psi_2(\mathbf{x}, \mathbf{u}, z))$$

is the required formula. $\square$

Weak representability of r.e. relations (proof)

Proof (Lemma 2).

$k = 0$: $\mathbb{N}^0 = \{*\}$ is weakly representable by $0 = 0$ and $\varnothing$ is weakly representable by $0 \neq 0$.

$k \geq 1$: Let $E \subseteq \mathbb{N}^k$, let $\mathbf{x}, y$ be $k+1$ distinct fixed variables. T has an associated Gödel numbering where $\ulcorner\psi\urcorner$ denotes the Gödel number of ψ and γ_n is the formula with Gödel number n. Then, we can construct a primitive recursive function $g : \mathbb{N}^{k+1} \to \mathbb{N}$ such that

$$g(\mathbf{m}, n) = \begin{cases} \ulcorner\gamma_n\left[\frac{\overline{\mathbf{m}}}{\mathbf{x}}, \frac{\overline{n}}{y}\right]\urcorner & \text{if } \gamma_n \text{ exists} \\ n & \text{otherwise} \end{cases}$$

$\square$

Weak representability of r.e. relations (proof)

Proof (continued).

Since T is an r.e. theory, $D \subseteq \mathbb{N}^{k+1}$ given by

$$D(\mathbf{m}, n) \text{ iff } GTHM_T(g(\mathbf{m}, n)) \text{ iff } \vdash \gamma_n \left[\frac{\overline{\mathbf{m}}}{\mathbf{x}}, \frac{\overline{n}}{y}\right]$$

is an r.e. relation. By the technical lemma, we obtain $\varphi(\mathbf{x}, y)$ in T such that, for all $\mathbf{m}, n \in \mathbb{N}^{k+1}$,

$$\text{if } E(\mathbf{m}) \text{ and } \nvdash \gamma_n \left[\frac{\overline{\mathbf{m}}}{\mathbf{x}}, \frac{\overline{n}}{y}\right], \text{ then } \vdash \varphi(\overline{\mathbf{m}}, \overline{n})$$

$$\text{if } \neg E(\mathbf{m}) \text{ and } \vdash \gamma_n \left[\frac{\overline{\mathbf{m}}}{\mathbf{x}}, \frac{\overline{n}}{y}\right], \text{ then } \nvdash \varphi(\overline{\mathbf{m}}, \overline{n}).$$

$\square$

Weak representability of r.e. relations (proof)

Proof (continued).

Let $p = \ulcorner \varphi(\mathbf{x}, y) \urcorner$. Then, $\gamma_p = \varphi$ and so, for all $\mathbf{m} \in \mathbb{N}^k$,

$$\begin{aligned} &\text{if } E(\mathbf{m}) \text{ and } \nvdash \varphi(\overline{\mathbf{m}}, \overline{p}), \text{ then } \vdash \varphi(\overline{\mathbf{m}}, \overline{p}) \\ &\text{if } \neg E(\mathbf{m}) \text{ and } \vdash \varphi(\overline{\mathbf{m}}, \overline{p}), \text{ then } \nvdash \varphi(\overline{\mathbf{m}}, \overline{p}). \end{aligned}$$

It follows that, for all $\mathbf{m} \in \mathbb{N}^k$,

$$E(\mathbf{m}) \text{ iff } \vdash \varphi(\overline{\mathbf{m}}, \overline{p}),$$

and so $\varphi(\mathbf{x}, \overline{p})$ weakly represents E in T. $\square$