Universal Updates for Symmetric Lenses

Michael Johnson
CoACT, Departments of Mathematics and Computing
Macquarie University
and the Optus Macquarie University Cyber Security Hub
Robert Rosebrugh
Department of Mathematics and Computer Science
Mount Allison University

Abstract

Asymmetric c-lenses are the special cases of asymmetric d-lenses (also called delta lenses) whose updates satisfy a universal property which in many applications ensures "least-change". There has therefore been hope that symmetric c-lenses might characterize those symmetric d-lenses which satisfy a similar universal property. This paper begins an analysis of symmetric c-lenses and their relationship to symmetric d-lenses and explains why the authors do not expect symmetric c-lenses, that is, equivalence classes of spans of c-lenses, to be central to developing universal properties for symmetric lenses. Instead, we consider cospans of c-lenses and show that they generate symmetric c-lenses with an appropriate universal property. That property is further analysed and used to motivate proposed generalisations to obtain universal, least-change, properties for symmetric d-lenses. In addition we explore how to characterise the symmetric d-lenses that arise from cospans of c-lenses among all symmetric d-lenses.

1 Introduction

Bidirectional transformations maintain consistency between two different data sources.

Over the last several years there has been a significant exploration of bidirectional transformations and possible *least-change* properties, see [8] and works cited there. Least-change properties are of interest because in developing a bidirectional transformation there are usually many choices in defining consistency restorers [26] or Put operations [28], and optimal, or at least good, choices are likely to be those that make smallest or fewest changes. When one data source is changed, we would like the other to be changed as little as possible, but it must be changed in some way that restores consistency.

One proposal for least-change asymmetric lenses has been c-lenses [16] (but see also [15] and works cited there, and [14]). While c-lenses were defined independently of delta lenses [5], they turn out to be special cases of delta lenses — they are those delta lenses whose Puts satisfy a particular universal property (presented in detail later). In short, the universal property says that every possible value that a particular Put could take factors through the value of the Put that the c-lens does take. In other words, the c-lens Put is minimal among all the possible Puts. In a very real sense, one can see that the value of the Put specified by the c-lens makes changes that are

In: R. Eramo, M. Johnson (eds.): Proceedings of the Sixth International Workshop on Bidirectional Transformations (Bx 2017), Uppsala, Sweden, April 29, 2017, published at http://ceur-ws.org

less than or isomorphic to any other possible d-lens Put with the same parameters. The Puts of c-lenses are least-change Puts.

Of course, many applications of bidirectional transformations depend upon using symmetric lenses [6, 9].

Recall that in an asymmetric lens one data source, the slave data source, can be completely reconstructed from the other data source, the master data source. Thus the only interesting updates arise when a change is made to the slave data source and the master data source needs to be updated to restore consistency. That update is the one called the Put, and as we've noted, c-lenses provide a universal update — an update that satisfies the universal condition outlined above and presented in more detail in Section 2.

In a symmetric lens, when a change is made to one of the data sources we seek a corresponding update to the other data source to restore consistency. In non-trivial symmetric cases, neither data source is derivable from the other, and a change in either data source requires an update of the other data source to restore consistency. Usually, there will be many possible choices for such updates. It is natural to ask what universal conditions we might put on these symmetric lens updates. How do we make a "best" choice?

The authors have carried out detailed studies on the relationships between various kinds of asymmetric lenses and the corresponding kinds of symmetric lenses [17, 18, 20, 21]. In all cases the symmetric lenses can be constructed as equivalence classes of spans of asymmetric lenses of the corresponding kind. As a result, we have been asked a number of times about symmetric c-lenses. Such things should be equivalence classes of spans of asymmetric c-lenses, and since asymmetric c-lenses have universal updates, our interlocutors have hoped that symmetric c-lenses might be symmetric lenses with universal updates. This paper addresses those enquiries.

When asked, we've suspected that, and correspondingly reported in lectures our suspicion that, symmetric c-lenses do not in fact provide universal updates. The source of the difficulty is that in a span of c-lenses the master data source for both c-lenses is the peak (also called the head) of the span. The reason symmetric lenses are *equivalence classes of* spans of corresponding asymmetric lenses is because the actual nature of the peak of the span is unimportant for the symmetric lens — the peak must be able to accommodate enough information that it can mediate the update in both directions, but peaks that could accommodate more information while formally different may, with appropriately similar asymmetric lenses, amount to the same symmetric lens. As long as the updates mediated through the peak are the same, the symmetric lenses are the same, even if the peaks are different.

Now the reader will see why we've suspected that c-lenses do not provide universal updates for symmetric lenses: because the universal properties occur in the peak of the span, and what happens in the peak of the span is unimportant except in as much as it must have the power to mediate the updates between the data sources, it seems likely that symmetric c-lenses are not particularly special among symmetric d-lenses.

In Section 3 we look in more detail at the relationship between c-lenses and d-lenses. We show that every d-lens has, as we would expect, a free presentation — it is a "quotient" of a "free" d-lens. But more, the "free" d-lenses are in fact c-lenses. We also see how we can, in some cases, use this to prove that a given span of d-lenses is equivalent to some span of c-lenses. We remain unsure as to whether there are spans of d-lenses that are not equivalent to any span of c-lenses.

Interestingly, cospans play a crucial role in the argument in Section 3. In Section 4 we give an explicit construction of a symmetric delta lens from a cospan of d-lenses. Former work has focused on spans rather than cospans. As Section 5 shows, taking the cospan approach has implications for universal updates: A cospan of c-lenses does, unlike the span case, generate a symmetric d-lens whose updates automatically satisfy a universal property. This addresses in part the title of this paper, and calls out for further generalisation.

So, what about more general symmetric delta lenses (those that don't arise from cospans of c-lenses)? What would it mean to ask for their updates to be universal, whether they corresponded to cospans of d-lenses that aren't c-lenses, or indeed were just an arbitrary symmetric delta lens? Motivated by the cospan of c-lenses case, we explore in Section 6 what extra structure might be required to talk about universality, and then in the following section we look at how that extra structure interacts with our attempt to characterize those symmetric lenses which do correspond to cospans of asymmetric lenses. Section 7 begins the analysis of symmetric lenses which arise from cospans, providing a necessary condition which suggests that symmetric lenses which can be represented by cospans are relatively uncommon.

We end this introduction with a quick remark about state spaces which will hopefully limit possible confusion. On one level of analysis, bidirectional transformations are frequently between systems whose state spaces have the property that every state can be updated in at least one way to every other state. In the authors' opinions, this kind of representation can obscure important information. Instead, we prefer, whenever possible, to have state spaces whose arrows have semantic relevance. For example, an arrow might indicate an increase in the information order (so in a database example, an arrow would indicate an insert — more information is provided by the state at the target of the arrow than was provided by the state at the source of the arrow). It will be easier for readers to think of state spaces like the information order state space just described when trying to understand least-change updates and proposed universal updates for symmetric lenses.

Of course, none of this denies that there are updates which take place along *reductions* in the information order (deletions in the case of databases). These updates can be analysed in the op-state space, or equivalently, using notation from Section 2, using the monad L rather than the monad R. And finally, again of course, there are mixed updates. The machinery for dealing all at once with updates along a chosen order like the information order, updates along the reversed order, and mixes of the two, was presented in [19]. This means that the behavioural analysis of simple every-state-can-be-updated-to-every-other-state state spaces can be recovered using the techniques of [19], so it makes sense to confine our analysis in what follows to updates in the positive direction along a chosen order. The information order provides an ideal example and it may be helpful to keep it in mind while reading this paper.

2 Background and notation: d-lenses and c-lenses

In this section we introduce notation and review some material from earlier work. Much of this section, and the next section, assumes some knowledge of relatively sophisticated category theory including monads and fibrations. Readers with less background in those areas might like to read the notational conventions presented in the next paragraph and then skip to Section 4. The remainder of the paper, beginning there, is intended to be comprehensible for readers with a background in lenses and basic category theory.

We assume the reader is familiar with basic notions of category theory as found, for example, in [2] or [27]. Our notation is mostly quite standard; for example we use bold face for the names of categories, capitals for objects and functors and lower case (often Greek) for morphisms. Some specific items follow. We will use d_0 and d_1 for the domain and codomain operators on morphisms and 1_X (and occasionally id_X) to denote identity morphisms. We denote the set of objects of a category $\mathbf{X}$ by $|\ \mathbf{X}\ |$ and the set of arrows by $Arr(\mathbf{X})$. We write $\mathbf{X^2}$ for the so-called "arrow category" of $\mathbf{X}$. An object A of $\mathbf{X^2}$ is an arrow of $\mathbf{X}$ denoted $A = A_f : A_0 \longrightarrow A_1$. An arrow in $\mathbf{X^2}$ from the object A to another object $B = B_f : B_0 \longrightarrow B_1$ is a pair of arrows

$$g = (g_0 : A_0 \longrightarrow B_0, g_1 : A_1 \longrightarrow B_1)$$

satisfying $g_1 A_f = B_f g_0$. A diagram of the form $X \longleftarrow Y \longrightarrow Z$ is called a *span* with head Y, and dually $X \longrightarrow Y \longleftarrow Z$ is a *cospan.* If the cospan $X \longrightarrow Y \longleftarrow Z$ has a pullback span $X \longleftarrow P \longrightarrow Z$, we will sometimes denote the pullback object P by $X \times_Y Z$.

The following notion is standard but we review it to set our notation and remind the reader. For functors $\mathbf{S} \xrightarrow{G} \mathbf{V} \xleftarrow{H} \mathbf{T}$ with common codomain $\mathbf{V}$, their "comma category" is denoted (G, H). Its objects are triples (S, T, a) where S and T are objects of $\mathbf{S}$ and $\mathbf{T}$, and $a : G(S) \longrightarrow H(T)$ is a morphism of $\mathbf{V}$. A morphism from (S, T, a) to (S', T', a') is two morphisms $S \longrightarrow S'$ and $T \longrightarrow T'$ making the obvious square in $\mathbf{V}$ commute. The comma category has projection functors to $\mathbf{S}$ and $\mathbf{T}$ and a transformation γ denoted as follows:

$$\begin{tikzcd} & (G,H) \arrow[dl, "L_G H"'] \arrow[dr, "R_H G"] \arrow[d, phantom, "\overset{\gamma}{\to}"] & \\ \mathbf{S} \arrow[dr, "G"'] & {} & \mathbf{T} \arrow[dl, "H"] \\ & \mathbf{V} & \end{tikzcd}$$

Where possible we suppress subscripts on projections. Note that $\gamma_{(S,T,a)}$ is just a. When $H = 1_{\mathbf{V}}$ the objects of $(G, 1_{\mathbf{V}})$ are formally pairs (S, α) with S an object of $\mathbf{S}$ and $\alpha : GS \longrightarrow V$ an arrow of $\mathbf{V}$ (whose codomain is arbitrary). An arrow $(S, \alpha) \longrightarrow (S', \alpha')$ where $\alpha' : GS' \longrightarrow V'$ is a pair (σ, φ) where $\sigma : S \longrightarrow S', \varphi : V \longrightarrow V'$ and they satisfy $\varphi\alpha = \alpha' G(\sigma)$.

We now recall the definition of an *asymmetric* delta lens ([5, 16]) which we will usually abbreviate to d-lens.

Definition 1 *An* asymmetric delta lens (d-lens) *from* $\mathbf{S}$ *to* $\mathbf{V}$ *is a pair* (G, P) *where* $G : \mathbf{S} \longrightarrow \mathbf{V}$ *is a functor (the "Get") and* $P : |(G, 1_{\mathbf{V}})| \longrightarrow |\mathbf{S^2}|$ *is a function (the "Put") and the data for* $\alpha : G(S) \longrightarrow V$ *and* $\beta : G(S') \longrightarrow V'$ *satisfy:*

(i) d-PutInc: the domain of $P(S, \alpha)$ *is* S

(ii) d-PutId: $P(S, \mathrm{id}_{G(S)}) = \mathrm{id}_S$

(iii) d-PutGet: $G(P(S,\alpha)) = \alpha$

(iv) d-PutPut: if S' *is the codomain of* $P(S,\alpha)$ *(and hence* $G(S') = V$*) then* $P(S,\beta\alpha) = P(S',\beta)P(S,\alpha)$*.*

The comma category (G,H) has a universal property that we use to establish some further notation needed below. Explicitly, given a triple consisting of two functors and a natural transformation ($K : \mathbf{X} \longrightarrow \mathbf{S}, L : \mathbf{X} \longrightarrow \mathbf{T}, \varphi : GK \longrightarrow HL$), there is a unique $F : \mathbf{X} \longrightarrow (G,H)$ (satisfying certain properties). When $H = 1_{\mathbf{V}}$ we can define the functor η_G corresponding to the triple $(1_{\mathbf{S}}, G, 1_G)$. We can also iterate the comma category construction and its projections, as in the right hand diagram:

$$\begin{tikzcd}
& \mathbf{S} \arrow[ddl, bend right, "1_{\mathbf{S}}"'] \arrow[ddr, bend left, "G"] \arrow[d, dashed, "\eta_G"] & \\
& (G,1_{\mathbf{V}}) \arrow[dl, "L1_{\mathbf{V}}"'] \arrow[dr, "RG"] \arrow[dd, phantom, "\xrightarrow{\alpha}"] & \\
\mathbf{S} \arrow[dr, "G"'] & & \mathbf{V} \arrow[dl, "1_{\mathbf{V}}"] \\
& \mathbf{V} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (RG,1_{\mathbf{V}}) \arrow[dl, "L_{RG}1_{\mathbf{V}}"'] \arrow[dr, "RRG"] \arrow[dd, phantom, "\xrightarrow{\beta}"] & \\
(G,1_{\mathbf{V}}) \arrow[dr, "RG"'] & & \mathbf{V} \arrow[dl, "1_{\mathbf{V}}"] \\
& \mathbf{V} &
\end{tikzcd}$$

Then we can define the functor corresponding to the triple $(L_G1_{\mathbf{V}} \cdot L_{RG}1_{\mathbf{V}}, RRG, \beta(\alpha L_{RG}1_{\mathbf{V}}))$, and we denote it μ_G:

$$\begin{tikzcd}
& (RG,1_{\mathbf{V}}) \arrow[ddl, bend right, "L_G1_{\mathbf{V}}L_{RG}1_{\mathbf{V}}"'] \arrow[ddr, bend left, "RRG"] \arrow[d, dashed, "\mu_G"] & \\
& (G,1_{\mathbf{V}}) \arrow[dl, "L1_{\mathbf{V}}"'] \arrow[dr, "RG"] \arrow[dd, phantom, "\xrightarrow{\alpha}"] & \\
\mathbf{S} \arrow[dr, "G"'] & & \mathbf{V} \arrow[dl, "1_{\mathbf{V}}"] \\
& \mathbf{V} &
\end{tikzcd}$$

The assignment $G \mapsto RG$ defines (on objects) the functor part of a monad R on $\mathbf{cat}/\mathbf{V}$. The η_G and μ_G just defined are the unit and multiplication of the monad at an object G. Similarly, $H \mapsto LH = L_{1_{\mathbf{V}}}H : (1_{\mathbf{V}}, H) \longrightarrow \mathbf{V}$ defines a monad L on $\mathbf{cat}/\mathbf{V}$.

The following is the original definition of c-lens. It amounts to an algebra for the monad R. As such, it connects to a large body of work on opfibrations. The formulation of a (split) opfibration as an algebra for the monad R was first achieved by Street in [29]. Algebras for L are the split fibrations. The recognition that lenses are algebras for monads (or in some cases for "semi-monads"), thus ultimately connecting back to Street's work, appeared in [22, 16, 23], and see also [7].

Definition 2 [23] *A* c-lens *from* $\mathbf{S}$ *to* $\mathbf{V}$ *is a pair* (G,P) *where* $\mathbf{S} \xrightarrow{G} \mathbf{V}$ *and* $(G,1_{\mathbf{V}}) \xrightarrow{P} \mathbf{S}$ *are functors satisfying*

i) c-PutGet: $GP = RG$

ii) c-GetPut: $P\eta_G = 1_{\mathbf{S}}$

iii) c-PutPut: $P\mu_G = P(P,1_{\mathbf{V}})$

or equivalently but diagrammatically, the following commute:

$$\begin{tikzcd}
\mathbf{S} \arrow[r, "\eta_G"] \arrow[dr, "1_{\mathbf{S}}"'] & (G,1_{\mathbf{V}}) \arrow[d, "P"] \arrow[dr, "RG"] & \\
& \mathbf{S} \arrow[r, "G"'] & \mathbf{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
(RG,1_{\mathbf{V}}) \arrow[r, "(P,1_{\mathbf{V}})"] \arrow[d, "\mu_G"'] & (G,1_{\mathbf{V}}) \arrow[d, "P"] \\
(G,1_{\mathbf{V}}) \arrow[r, "P"'] & \mathbf{S}
\end{tikzcd}$$

We recall from [16] that a c-lens can be seen to be a d-lens which satisfies a universal property. The Put for a c-lens returns just an object S' of $\mathbf{S}$ for an object (S,α) of its domain, while that for a d-lens returns an arrow whose domain is S. The functoriality enjoyed by a c-lens means that its Put can be extended to return an arrow (called "opcartesian") from (S,α) to S' which makes it a d-lens. The universal property of the opcartesian arrow for (S,α), say $\alpha^*: S \longrightarrow S'$, is the following: for any $\gamma: S \longrightarrow S''$ in $\mathbf{S}$ such that $G(\gamma) = \beta\alpha : G(S) \longrightarrow G(S'')$, there is a unique $\gamma' : S' \longrightarrow S''$ such that $G(\gamma') = \beta$ and $\gamma = \gamma'\alpha^*$. Thus, a c-lens can be seen to be a d-lens whose Put satisfies just this additional (least-change) property.

We now turn to spans of d-lenses. Such spans represent symmetric d-lenses, but, as noted in Section 1, different spans may represent the same symmetric d-lens. The equivalence that we need on spans of d-lenses is generated by functors Φ satisfying the following conditions.

Definition 3 [18] *Suppose that in the diagram*

$$\begin{tikzcd} & \mathbf{S}' \arrow[dl, "{(G'_L,P'_L)}"'] \arrow[dr, "{(G'_R,P'_R)}"] \arrow[d, "\Phi"] & \\ \mathbf{X} & \mathbf{S} \arrow[l, "{(G_L,P_L)}"] \arrow[r, "{(G_R,P_R)}"'] & \mathbf{Y} \end{tikzcd}$$

the top and bottom spans are spans of d-lenses and Φ is a functor. Then Φ is said to satisfy conditions (E) *if:*

(1) $G_L\Phi = G'_L$ and $G_R\Phi = G'_R$,

(2) Φ is surjective on objects, and

(3) whenever $\Phi S' = S$, we have both

$$P_L(S, G_LS \xrightarrow{\alpha} X) = \Phi P'_L(S', G'_LS' \xrightarrow{\alpha} X) \quad \textit{and} \quad P_R(S, G_RS \xrightarrow{\beta} Y) = \Phi P'_R(S', G'_RS' \xrightarrow{\beta} Y).$$

Definition 4 [18] *Define $\equiv_{Sp}$ to be the equivalence relation on spans of d-lenses from $\mathbf{X}$ to $\mathbf{Y}$ which is generated by functors Φ satisfying conditions (E).*

3 Symmetric d-lenses and symmetric c-lenses

We turn now to a further study of functors like Φ and an initial exploration of when spans of d-lenses might be $\equiv_{Sp}$-equivalent to spans of c-lenses. We present some useful results, but further study will be required to fully answer the question, raised in the introduction, of just how special are symmetric c-lenses among symmetric d-lenses.

Let $\mathbf{S} \xrightarrow{G} \mathbf{V}$ be a functor. We write $(G^i, 1_{\mathbf{V}})$ for the subcategory of $(G, 1_{\mathbf{V}})$ with the same objects as $(G, 1_{\mathbf{V}})$ but whose morphisms from say $(S, \alpha : GS \longrightarrow V)$ to $(T, \alpha' : GT \longrightarrow V')$ require that $S = T$ and are (rather than commutative squares as they are in the full comma category) commutative triangles of the form:

$$\begin{tikzcd} & GS \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ V \arrow[rr, "\beta"'] & & V' \end{tikzcd}$$

so that necessarily $\alpha' = \beta\alpha$. Composition is by juxtaposition of triangles. In what follows, we will sometimes refer to morphisms of this form as the *triangular morphisms*. We remark that if we write $\mathbf{S}^i$ for the (discrete) category with the same objects as $\mathbf{S}$ and only identity arrows, and G^i for the composite of G with the inclusion of $\mathbf{S}^i$ in $\mathbf{S}$, then $(G^i, 1_{\mathbf{V}})$ is actually the comma category which the notation suggests. We write $RG^i : (G^i, 1_{\mathbf{V}}) \longrightarrow \mathbf{V}$ for the functor whose value on an object $(S, \alpha : GS \longrightarrow V)$ is the object V, and whose value on an arrow $\beta : (S, \alpha) \longrightarrow (S, \alpha')$ as above is just $\beta : V \longrightarrow V'$. It is easy to see that, RG^i is functorial. Once again, this notation is consistent with that of the previous section.

Suppose further that $\mathbf{S} \xrightarrow{(G,P)} \mathbf{V}$ is a d-lens. We next define a functor $\Phi : (G^i, 1_{\mathbf{V}}) \longrightarrow \mathbf{S}$. On an object $(S, \alpha : GS \longrightarrow V)$ of $(G^i, 1_{\mathbf{V}})$ define $\Phi(S, \alpha)$ to be the codomain, $d_1P(S, \alpha) = T$ say, of $P(S, \alpha) : S \longrightarrow T$. Suppose $(S, \alpha' : GS \longrightarrow T')$ is another object and $\beta : (S, \alpha) \longrightarrow (S, \alpha')$ is an arrow in $(G^i, 1_{\mathbf{V}})$, so $\alpha' = \beta\alpha$. Notice that, since $GP(S, \alpha) = \alpha$, we have $GT = V$. Moreover, $\Phi(S, \alpha' : GS \longrightarrow V') = d_1P(S, \alpha') = T'$ say, so that $GT' = V'$. We define Φ on a morphism β as above by $\Phi(\beta : (S, \alpha) \longrightarrow (S, \alpha')) = P(T, \beta)$. This is meaningful since (G, P)

is a d-lens so that $P(S, \alpha') = P(S, \beta\alpha) = P(T, \beta)P(S, \alpha)$. Thus the domain of $\Phi(\beta)$ is the codomain of $P(S, \alpha)$ which is $\Phi(S, \alpha)$ while the codomain of $\Phi(\beta)$ is the codomain of $P(S, \alpha')$ which is $\Phi(S, \alpha')$. That Φ is functorial now also follows from a similar argument.

There is a "one-sided" version of the conditions (E) of Definition 3:

Definition 5 *Suppose that in the diagram*

$$\begin{tikzcd} \mathbf{S}' \arrow[d, "\Psi"'] \arrow[dr, "(G'P')"] & \\ \mathbf{S} \arrow[r, "(G,P)"'] & \mathbf{V} \end{tikzcd}$$

both (G, P) and (G', P') are d-lenses and Ψ is a functor. Then Ψ is said to satisfy conditions (E_1) *if:*

(1) $G\Psi = G'$,

(2) Ψ is surjective on objects, and

(3) whenever $\Psi S' = S$, we have $P(S, GS \xrightarrow{\alpha} V) = \Psi P'(S', G'S' \xrightarrow{\alpha} V)$

Notice that the left hand side of the last equation is $P(\Psi S', G\Psi S' \xrightarrow{\alpha} V)$.

Lemma 6 *Let $\mathbf{S} \xrightarrow{(G,P)} \mathbf{V}$ be a d-lens. With the notation introduced above, consider the diagram*

$$\begin{tikzcd} (G^i, 1_{\mathbf{V}}) \arrow[d, "\Phi"'] \arrow[dr, "RG^i"] & \\ \mathbf{S} \arrow[r, "(G,P)"'] & \mathbf{V} \end{tikzcd}$$

It follows that $G\Phi = RG^i$ and RG^i is a c-lens. Moreover Φ satisfies conditions (E1).

Proof. First, from the description of Φ on an object $\alpha : GS \longrightarrow V$ above, it is immediate that $G\Phi(S, \alpha) = V = RG^i(S, \alpha)$. Similarly, on an arrow β, we have $G\Phi(\beta) = GP(\beta) = \beta = RG^i(\beta)$.

Next we show that RG^i is a c-lens, that is a split op-fibration. Thus, for an object $(S, \alpha : GS \longrightarrow V)$ of $(G^i, 1_{\mathbf{V}})$ and an arrow $\beta : RG^i(S, \alpha) \longrightarrow V'$ of $\mathbf{V}$, we need to provide an op-cartesian arrow in $(G^i, 1_{\mathbf{V}})$. But $RG^i(S, \alpha) = V$, so we have $\beta : V \longrightarrow V'$ in $\mathbf{V}$. For the op-cartesian arrow in $(G^i, 1_{\mathbf{V}})$ we take $\beta : (S, \alpha) \longrightarrow (S, \beta\alpha)$. For use below, the Put for the d-lens structure on RG^i will be denoted P' and we have just defined $P'((S, \alpha), \beta) = \beta : (S, \alpha) \longrightarrow (S, \beta\alpha)$ To see that this definition works, suppose further that $\gamma : (S, \alpha) \longrightarrow (S, \alpha'' : GS \longrightarrow V'')$ is an arrow of $(G^i, 1_{\mathbf{V}})$ such that $\gamma = RG^i(\gamma)$ factors as $\gamma = \gamma'\beta$ in $\mathbf{V}$ (and refer to the diagram below). The required arrow of $(G^i, 1_{\mathbf{V}})$ from $(S, \beta\alpha) \longrightarrow (S, \alpha'')$ is γ', which is indeed an arrow since $\alpha'' = \gamma\alpha = \gamma'(\beta\alpha)$. It is unique in making the upper triangle below commute since $RG^i(\gamma') = \gamma'$.

$$\begin{tikzcd} (G^i, 1_{\mathbf{V}}) \arrow[dd, dashed, "RG^i"'] & (S, \alpha) \arrow[rr, "\gamma"] \arrow[dr, "\beta"'] & & (S, \alpha'') \\ & & (S, \beta\alpha) \arrow[ur, "\gamma'"'] & \\ \mathbf{V} & V \arrow[rr, "\gamma"] \arrow[dr, "\beta"'] & & V'' \\ & & V' \arrow[ur, "\gamma'"'] & \end{tikzcd}$$

Finally, we show that Φ satisfies conditions (E_1). We already have that $G\Phi = RG^i$. To see that Φ is surjective on objects we note that for any object S in $\mathbf{S}$, we have $S = d_1 1_S = d_1 P(S, 1_{GS}) = \Phi(S, 1_{GS})$. Next, suppose that for (S, α) in $(G^i, 1_{\mathbf{V}})$, we have $\Phi(S, \alpha) = T(= d_1 P(S, \alpha))$. We need to show that $P(T, \beta : GT \longrightarrow V') = \Phi P'((S, \alpha), \beta : G^i(S, \alpha) \longrightarrow V')$. Now as noted in the previous paragraph, $P'((S, \alpha), \beta : G^i(S, \alpha) \longrightarrow V')$ is the arrow $\beta : (S, \alpha) \longrightarrow (S, \beta\alpha)$ in $(G^i, 1_{\mathbf{V}})$, and Φ of it was defined to be $P(T, \beta)$. This completes the proof. ∎

It may be worth remarking that the conditions (E_1) on Φ are, as noted in [21], equivalent to the requirement that Φ be a surjective on objects homomorphism between the algebras (RG^i, P') and (G, P) for the semi-monad R^i whose algebras, when they satisfy an extra "identity" condition, are d-lenses (see [16]). Anyway, Φ is a morphism of d-lenses.

In fact, in a sense that we won't explore in full here because of space limitations, Φ may be seen as the free presentation of the algebra (G, P).

In brief: recall that for any monad (T, m, η) the free algebra on A is $m_A : TTA \longrightarrow TA$. Thus the free c-lens on G is $\mu_G : RRG \longrightarrow RG$. It turns out that the image of μ_G includes only triangular morphisms (in other words, all the opcartesian morphisms for the free c-lens are triangular). So μ_G restricts to $\mu_G^i : R^i R^i G \longrightarrow R^i G$, where R^i is, as above, the semi-monad that takes G to $R^i G = RG^i$. Furthermore, the "free" d-lens (inverted commas because we are now in the semi-monad case) might be defined as $\mu_G^i : R^i R^i G \longrightarrow R^i G$. It is a d-lens since μ_G, and therefore μ_G^i, satisfy the extra "identity" condition from [16] and μ_G^i, the restriction of μ_G, is the same as the multiplication for the semi-monad defined in [16].

It is interesting to note that the "free" d-lens is in fact a c-lens. Furthermore, Φ presents the d-lens as a surjective on objects homomorphic image of a free c-lens.

In unpublished work we have explored lens structures on Φ because, referring to the diagram below in which the diamond is a pullback of functors, when the Φ_L and Φ_R are c-lenses, the pullback projections of Φ_L and Φ_R would be c-lenses, the composites of c-lenses are c-lenses, so the top would be a span of c-lenses, and since the pullbacks and composites of functors satisfying (E1) satisfy (E1), the middle diamond would show that the upper span of c-lenses is $\equiv_{Sp}$-equivalent to the lower span of d-lenses. Of course Φ need not have a c-lens, or indeed d-lens, structure because we can construct examples where Φ is not surjective on arrows while it is surjective on objects. It remains important to explore when spans of d-lenses are $\equiv_{Sp}$-equivalent to spans of c-lenses..

$$
\begin{tikzcd}
 & & \mathbf{T} \arrow[dl, "\Phi'_R"'] \arrow[dr, "\Phi'_L"] & & \\
 & (G_L^i, 1) \arrow[dl, "RG_L^i"'] \arrow[dr, "\Phi_L"] & & (G_R^i, 1) \arrow[dl, "\Phi_R"'] \arrow[dr, "RG_R^i"] & \\
\mathbf{V} & & \mathbf{S} \arrow[ll, "{(G_L,P_L)}"] \arrow[rr, "{(G_R,P_R)}"'] & & \mathbf{W}
\end{tikzcd}
$$

As it happens, a weaker condition than c-lens structures on the cospan (Φ_L, Φ_R) will suffice to find an $\equiv_{Sp}$-equivalent span of c-lenses for a given span of d-lenses. The cospan might be a half-duplex interoperability cospan [4], and that would be enough to give Φ'_R and Φ'_L c-lens structures and hence, as above, to provide a span of c-lenses which is $\equiv_{Sp}$-equivalent to the given spans of d-lenses. This connects with old work on enterprise interoperations which we will return to in Section 8.

In the remainder of this paper we turn to a detailed, but in category theoretic requirements more elementary, study of *co*spans of lenses and the implications that they have for universal updates for symmetric lenses.

4 Symmetric delta lenses and cospans of d-lenses

In symmetric lenses the two data sources are in some sense peers. Neither can generally be used to reconstruct the other, and the two operations are no longer called Put and Get (with the implication that the Get operation is straightforward while the Put operation needs to deal with the complications of many possible choices), and so they are typically given more neutral names like Left and Right or Forwards and Backwards.

So far we have looked at symmetric lenses via equivalence classes of spans of asymmetric lenses. This has been appropriate because in the absence of a direct definition of symmetric c-lenses, we can still study them via the "symmetrising" span construction applied to asymmetric c-lenses.

We turn now to a deeper study of symmetric d-lenses and revert to a more traditional definition that highlights the two operations. The symmetric lenses that we will use are called fb-lenses (the f and b standing for Forwards and Backwards). They are based upon the symmetric delta lenses of Diskin et al [5].

Definition 7 [21] *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be categories. An* fb-lens *from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is given by a 4-tuple* $M = (\delta_\mathbf{X}, \delta_\mathbf{Y}, \mathsf{f}, \mathsf{b}) : \mathbf{X} \longleftrightarrow \mathbf{Y}$ *specified as follows. The data* $\delta_\mathbf{X}, \delta_\mathbf{Y}$ *are functions which come equipped with a common domain* R

and form a span of sets

$$\delta_{\mathbf{X}} : |\mathbf{X}| \longleftarrow R \longrightarrow |\mathbf{Y}| : \delta_{\mathbf{Y}}$$

An element r of R is called a corr. *For r in R, if $\delta_{\mathbf{X}}(r) = X, \delta_{\mathbf{Y}}(r) = Y$ the corr is denoted $r : X \leftrightarrow Y$, or sometimes even just $r : X - Y$. The data* f *and* b *are operations called* forward *and* backward propagation*:*

$$\mathsf{f} : Arr(\mathbf{X}) \times_{|\mathbf{X}|} R \longrightarrow Arr(\mathbf{Y}) \times_{|\mathbf{Y}|} R$$

$$\mathsf{b} : Arr(\mathbf{Y}) \times_{|\mathbf{Y}|} R \longrightarrow Arr(\mathbf{X}) \times_{|\mathbf{X}|} R$$

where the pullbacks ensure that if $\mathsf{f}(x,r) = (y,r')$, we have $d_0(x) = \delta_{\mathbf{X}}(r)$, $d_1(y) = \delta_{\mathbf{Y}}(r')$ and similarly for b*. We also require that $d_0(y) = \delta_{\mathbf{Y}}(r)$ and $\delta_{\mathbf{X}}(r') = d_1(x)$, and the similar equations for* b*.*

Furthermore, we require that both propagations respect both the identities and composition in $\mathbf{X}$ and $\mathbf{Y}$, so that we have:

$$r : X \leftrightarrow Y \quad \textit{implies} \quad \mathsf{f}(\mathrm{id}_X, r) = (\mathrm{id}_Y, r) \quad \textit{and} \quad \mathsf{b}(\mathrm{id}_Y, r) = (\mathrm{id}_X, r)$$

and

$$\mathsf{f}(x,r) = (y,r') \ \textit{and}\ \mathsf{f}(x',r') = (y',r'') \quad \textit{imply} \quad \mathsf{f}(x'x,r) = (y'y,r'')$$

and

$$\mathsf{b}(y,r) = (x,r') \ \textit{and}\ \mathsf{b}(y',r') = (x',r'') \quad \textit{imply} \quad \mathsf{b}(y'y,r) = (x'x,r'')$$

If $\mathsf{f}(x,r) = (y,r')$ and $\mathsf{b}(y',r) = (x',r'')$, we display instances of the propagation operations as:

$$\begin{tikzcd} X \arrow[r, dash, "r"] \arrow[d, "x"'] \arrow[dr, phantom, "\overset{\mathsf{f}}{\longrightarrow}"] & Y \arrow[d, dashed, "y"] \\ X' \arrow[r, dash, dashed, "r'"'] & Y' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dash, "r"] \arrow[d, dashed, "x'"'] \arrow[dr, phantom, "\overset{\mathsf{b}}{\longleftarrow}"] & Y \arrow[d, "y'"] \\ X' \arrow[r, dash, dashed, "r''"'] & Y' \end{tikzcd}$$

The main purpose of this section is to present the following construction of an fb lens from a cospan of d-lenses.

Construction 8 *To get an fb-lens from a cospan of d-lenses:*

$$\mathbf{X} \xrightarrow{(G_L,P_L)} \mathbf{V} \xleftarrow{(G_R,P_R)} \mathbf{Y}$$

Define $(\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathsf{f}, \mathsf{b}) : \mathbf{X} \longleftrightarrow \mathbf{Y}$ by

- *Let the set of corrs be $\{(X,Y) \mid G_L X = G_R Y = V\}$ (in diagrams we will often label the corr (X,Y) with the corresponding object V)*
- *Take $\delta_{\mathbf{X}}$ and $\delta_{\mathbf{Y}}$ to be the projections from the set of corrs to the objects of $\mathbf{X}$ and of $\mathbf{Y}$ respectively, and*
- *Let $\mathsf{f}(\alpha,(X,Y)) = (P_R(Y, G_L(\alpha)), (X',Y'))$ as in the diagram*

$$\begin{tikzcd} X \arrow[r, dash, "V"] \arrow[d, "\alpha"'] & Y \arrow[d, "P_R(Y,G_L(\alpha))"] \\ X' \arrow[r, dash, "V'"'] & Y' \end{tikzcd}$$

where $Y' = d_1 P_R(Y, G_L(\alpha))$ and since $G_R(P_R(Y, G_L(\alpha))) = G_L(\alpha)$ we set $V' = d_1 G_L(\alpha)$.

The definition of b *is similar.* ∎

It is easy to see that the fb-lens just constructed is the one which, under the equivalence between fb-lenses and spans of d-lenses [18], corresponds to the span of d-lenses obtained by pulling back the given cospan. We remind the reader that lenses "pullback" in the sense that there is a canonical lens structure on the pullback of the Get functors as was shown in [23], Proposition 4.2, for c-lenses, and [21], Proposition 5, for d-lenses.

Notice that, in Construction 8, because G_L and G_R define a cospan of object functions, there is *at most one corr* for given X and Y. That corr, when it exists, is determined by the object V of $\mathbf{V}$ which both X and Y map to. As before, in diagrams we will usually label a corr (X, Y) by $V = G_L X = G_R Y$. In a sense, V "witnesses" the consistency relationship between X and Y [25] and such V provide the most convenient labels for the tops and bottoms of forward or backward propagation squares.

A cospan similarly provides a relationship between the *arrows* of $\mathbf{X}$ and the *arrows* of $\mathbf{Y}$, and that relationship on arrows is consistent with the corrs. We will see in the next section that that relationship, ignored until now, is an important ingredient in specifying universal updates for symmetric lenses.

5 Universality and cospans of c-lenses

We begin with a definition.

Definition 9 *In the cospan of d-lenses*

$$\mathbf{X} \xrightarrow{(G_L, P_L)} \mathbf{V} \xleftarrow{(G_R, P_R)} \mathbf{Y}$$

arrows α of $\mathbf{X}$ and β of $\mathbf{Y}$ are called compatible *if $G_L(\alpha) = G_R(\beta)$.*

Next, if the lenses in the cospan, (G_L, P_L) and (G_R, P_R), are not just d-lenses but in fact c-lenses, then there is a universal property for the corresponding forward propagation f constructed in the previous section.

Suppose given $\alpha : X \longrightarrow X'$ and V representing the corr (X, Y). Recall that $\mathsf{f}(\alpha, (X, Y))$ has two components, an arrow of $\mathbf{Y}$ and a corr relating X' and the codomain of the arrow of $\mathbf{Y}$ so as to form the square shown (in which the new corr has been left unlabelled).

$$\begin{tikzcd}
X \arrow[r, dash, "V"] \arrow[d, "\alpha"'] & Y \arrow[d, "\mathsf{f}(\alpha,(X,Y))_0"] \\
X' \arrow[r, dash] & Y'
\end{tikzcd}$$

We distinguish the two components of $\mathsf{f}(\alpha, (X, Y))$ by subscripting with 0 for the first component (as shown on the right hand side of the square above) and 1 for the second component. Recall that we write d_0 and d_1 for the operations which give the domain and codomain of an arrow (thus for example $d_0\alpha = X$).

Now we state the universal property satisfied by $\mathsf{f}(\alpha, (X, Y))$.

Proposition 10 *Suppose given a cospan of c-lenses $\mathbf{X} \xrightarrow{(G_L, P_L)} \mathbf{V} \xleftarrow{(G_R, P_R)} \mathbf{Y}$ and $\alpha : X \longrightarrow X'$ along with V representing the corr (X, Y). For any $\beta : Y \longrightarrow Y''$ compatible with α there is a unique $\beta' : Y' \longrightarrow Y''$ with $\beta = \beta' \mathsf{f}(\alpha, (X, Y))_0$ and $G_R(\beta') = 1_{G_L X'}$.*

$$\begin{tikzcd}
X \arrow[r, dash, "V"] \arrow[d, "\alpha"'] & Y \arrow[d, "\mathsf{f}(\alpha,(X,Y))_0"] \arrow[dd, bend right, "\beta"'] \\
X' \arrow[r, dash] \arrow[dr, dash] & Y' \arrow[d, "\beta'"] \\
& Y''
\end{tikzcd}$$

Proof. The proof is a routine application of the universal property of the c-lens (G_R, P_R): Since $G_L(\alpha)$ is an arrow of $\mathbf{V}$ with domain $V = G_R Y$ we can use the c-lens Put to calculate $P_R(Y, G_L(\alpha))$. But by Construction 8 the value of that Put is precisely the first component of $\mathsf{f}(\alpha, (X, Y))$ so the latter has the same universal property as the former, and the claimed universal property is satisfied by the former. ∎

In other words, among all those arrows with domain Y which are compatible with α, the forward propagation of α and (X, Y) constructed from the cospan of c-lenses (that is $\mathsf{f}(\alpha, (X, Y))_0$) is the least-change one. All other possible compatible updates factor through that one and do so via an arrow (β' in the picture) which is compatible with the identity on X'.

There is of course a corresponding universal property for the back propagation b which we leave to the reader to formulate.

Remark 11 The proposition is about *co*spans of c-lenses. Meanwhile, spans of c-lenses similarly determine relations that correspond to the usual corrs on the objects of $\mathbf{X}$ and $\mathbf{Y}$ and that could be called compatibility relations on the arrows of $\mathbf{X}$ and $\mathbf{Y}$, but there does not seem to be a similar universal property using those relations, nor would we expect there to be one. There is an image in $\mathbf{Y}$ of the universal property that holds in the peak of the span, but since $\mathbf{Y}$ may have many more arrows than those that appear in that image, the result is not universal and cannot be expressed independently of reference to the peak of the span.

Remark 12 In fact, there is a stronger version of Proposition 10. That proposition has not used the full power of the c-lens universal property for P_R. We have not yet investigated the implications of the stronger result since Proposition 10 suffices for investigations of least-change updates. (For readers who are familiar with the description of these properties in terms of cartesian arrows, the condition in Proposition 10 corresponds to precartesianness, while the stronger condition that we have not yet investigated corresponds to full cartesianness.)

6 Universality and symmetric d-lenses

Having revealed the universal property of a symmetric lens that arises from a cospan of c-lenses, we turn now to more general symmetric lenses $\mathbf{X} \leftrightarrow \mathbf{Y}$, not just those that arise from cospans of c-lenses, and ask when we might be able to describe an fb-lens as being "least change". We would need a universal property for each operation (f and b), and the preceding section suggests that the statement of such a universal property will depend not just on corrs relating objects of $\mathbf{X}$ and objects of $\mathbf{Y}$, but also on a "compatibility" relation between the arrows of $\mathbf{X}$ and the arrows of $\mathbf{Y}$.

We will proceed supposing that we want universal properties as close as possible to those discovered in Section 5.

We already have some examples of compatible arrows. In any forward or backward propagation square

$$\begin{tikzcd} X \arrow[r, dash, "r"] \arrow[d, "x"'] \arrow[dr, phantom, "\overset{\mathsf{f}}{\longrightarrow}"] & Y \arrow[d, dashed, "y"] \\ X' \arrow[r, dash, dashed, "r'"'] & Y' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dash, "r"] \arrow[d, dashed, "x"'] \arrow[dr, phantom, "\overset{\mathsf{b}}{\longleftarrow}"] & Y \arrow[d, "y"] \\ X' \arrow[r, dash, dashed, "r''"'] & Y' \end{tikzcd}$$

the arrows x and y need to be compatible — after all, the universal property will say something like, in the first case, the arrow y is minimal *among all the arrows of* $\mathbf{Y}$ *which are compatible with* x, and similarly for the minimality of x among the arrows of $\mathbf{X}$ which are compatible with y in the second case.

So, we expect a compatibility relation to include both the relations determined by the f-squares and the b-squares.

Let's check our intuition here for a moment. A given arrow x of $\mathbf{X}$ will be compatible with an arrow y of $\mathbf{Y}$ which is obtained by forward propagating x along some corr r. Presumably this means that y, which makes changes in $\mathbf{Y}$, (1) makes changes that include the changes that x makes in as much as there is shared data between $\mathbf{X}$ and $\mathbf{Y}$, (2) makes no changes in $\mathbf{Y}$ which affect data shared with $\mathbf{X}$ beyond those that are made by x, and (3) may make further changes in $\mathbf{Y}$ which affect data only relevant to $\mathbf{Y}$. The third of these is why we expect there to be in general other arrows y' compatible with x and why we might seek a least-change choice from amongst them.

Of course, we could have two compatibility relations, one from arrows of $\mathbf{X}$ to arrows of $\mathbf{Y}$, and one from arrows of $\mathbf{Y}$ to arrows of $\mathbf{X}$. But if the relations are determined by a cospan as in Section 5, or indeed by a span as the corr relation is in symmetric lenses presented as spans of asymmetric lenses, then the two relations are essentially the same, each being merely the op-relation of the other. For simplicity for now we will study a single relation between the arrows of $\mathbf{X}$ and the arrows of $\mathbf{Y}$, and read it in the appropriate direction as required.

All this motivates the following definition:

Definition 13 *Let* $L = (\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathsf{f}, \mathsf{b})$ *be an fb-lens between* $\mathbf{X}$ *and* $\mathbf{Y}$ *with corrs* R*. A* compatibility relation *on* L *is a relation* C *between the arrows of* $\mathbf{X}$ *and the arrows of* $\mathbf{Y}$ *respecting the corrs (that is,* $\alpha\, C\, \beta$ *implies that* *there is a corr* $r : d_0(\alpha) \leftrightarrow d_0(\beta)$*, and similarly for* d_1*) and containing the relation on arrows given by the union of the f-squares and the b-squares. When a pair* (α, β) *of arrows is in the compatibility relation, that is when* $\alpha\, C\, \beta$*, we say that they are* compatible*.*

Now define a least-change fb-lens with compatibility relation to be one whose f and b satisfy the universal properties previously observed in cospans of c-lenses:

Definition 14 *An fb-lens* L *equipped with a compatibility relation* C *is called* least-change *if for any* $\alpha : X \longrightarrow X'$ *and corr* $r : X \leftrightarrow Y$ *it is the case that* $\mathsf{f}(\alpha, r)$ *satisfies the following universal property: For any* $\beta : Y \longrightarrow Y''$ *compatible with* α *there is a unique* $\beta' : Y' \longrightarrow Y''$ *with* $\beta = \beta' \mathsf{f}(\alpha, r)_0$ *and* $1_{X'}\, C\, \beta'$ *:*

$$\begin{tikzcd}
X \arrow[r, dash, "r"] \arrow[d, "\alpha"'] & Y \arrow[d, "\mathsf{f}(\alpha,r)_0"] \arrow[dd, bend right, "\beta"'] \\
X' \arrow[r, dash] \arrow[dr, dash] & Y' \arrow[d, "\beta'"] \\
& Y''
\end{tikzcd}$$

and similarly for the back propagation b*.*

It might be worth remarking that the corr in the diagram between X' and Y'' is not important, but it is guaranteed to exist because α and β are compatible, and the compatibility relation respects corrs (and indeed because $1_{X'}$ and β' are compatible and the compatibility relation respects corrs).

Note that being least-change depends upon conditions that an fb-lens with compatibility might or might not satisfy, rather than being a property derived from the c-lenses that make up a cospan as in Section 5 – a priori a least-change fb-lens might not even be representable as cospans of even general d-lenses. Among the important questions to be addressed in future work is the question of when are least-change fb-lenses representable as cospans of lenses, and among them, when are the cospan lenses c-lenses. Also, since fb-lenses are always representable as spans of d-lenses [21], it is interesting to ask when compatibility relations will correspond to the relations determined by the span as in Remark 11.

7 Cospan and span representations

We have now seen various symmetric lenses presented as spans of asymmetric lenses (Section 3) and cospans of asymmetric lenses (Construction 8) and directly via forward and backward operations (Definition 7). Now we briefly look at the interactions between these.

We know from earlier work that the various kinds of symmetric lenses can be equivalently presented in fb-style, or as equivalence classes of spans of corresponding asymmetric lenses (see [17] for set-based symmetric lenses [9], see [18] for delta-based symmetric lenses [6], see [20] for edit based symmetric lenses [10], and see [20] for the unified treatment of those three different kinds of symmetric lenses). We used the span representation for fb-lenses in the first part of this paper.

What about cospans?

We know from [17] and later work that lenses "pull-back", not in the category whose morphisms are lenses (which may not even have pullbacks), but rather one can pullback the Get functors in the category of categories and then there are canonical constructions of Put operations on the resulting functors. Thus, a cospan of lenses can be converted to a span of lenses.

Furthermore, the "pullback" operation (keeping the inverted commas to emphasise that it is as just described, and not the pullback in the category of lenses) respects Construction 8 so that the fb-behaviour of the resulting span of lenses is the same as the fb-behaviour of the cospan of lenses. And even more, the corr (Construction 8) and the compatibility relations (Definition 9) for the cospan are preserved, meaning that the resulting span determines a relation between the objects of $\mathbf{X}$ and the objects of $\mathbf{Y}$ and that relation is exactly the same as the corr relation for the cospan, and similarly that span of functors determines a relation between the arrows of $\mathbf{X}$ and the arrows of $\mathbf{Y}$ and that relation is exactly the same as the compatibility relation for the cospan).

We will say that a symmetric lens with compatibility relation is *represented* by a particular span or cospan of asymmetric lenses if the forwards and backwards propagations have the same effects and the corr and compatibility relations are the same. We have just seen that a cospan of lenses (whether d-lenses or c-lenses) and its "pullback" both represent the same symmetric lens with compatibility relation. Equivalent spans of d-lenses often also give examples of representations, but note that we need to check that the compatibility relations agree: a functor $\Phi : \mathbf{S} \longrightarrow \mathbf{S}'$ can satisfy conditions (E) without being surjective on arrows and so the compatibility relation tabulated by $\mathbf{S}$ might be strictly contained in the compatibility relation tabulated by $\mathbf{S}'$. It is important to note that equivalent spans of lenses may represent different symmetric lenses with compatibility relations.

Remember that the "pullback" operation shows us that every cospan of lenses can be represented by a span of lenses (the one obtained by "pulling back" the cospan).

We show now that the converse is not the case. Not every span of lenses can be represented by a cospan of lenses. We develop below a necessary condition for a span of d-lenses to be represented by a cospan of d-lenses. Furthermore, we will see that not all spans of d-lenses satisfy the necessary condition. Indeed, it is, relatively speaking, rare among spans of d-lenses.

We study briefly the compatibility relation on cospans of d-lenses. Recall, from Definition 9 that in a cospan of d-lenses the compatibility relation exists between an arrow of $\mathbf{X}$ and an arrow of $\mathbf{Y}$ exactly when they are both sent to the same arrow in, in the notation of the definition, $\mathbf{V}$. This tells us a lot about the structure of possible compatibility relations for cospans of d-lenses.

A relation R between two sets, A and B, is called *complete* if for every $a \in A$ and every $b \in B$ it is the case that $a\,R\,b$. This includes of course the case where one or both of A and B are empty, whence the empty relation is the only possible relation between A and B, and it is complete. In the cases where neither A nor B is empty, a complete relation is sometimes called "complete bipartite" because the graph of the relation, in which related elements are joined by an edge, is a complete bipartite graph with the parts being A and B.

Given relations R between two sets, A and B, and S between two sets C and D, the coproduct of R and S is the relation $R + S$ between $A + C$ and $B + D$ — two elements of the two disjoint unions $A + B$ and $C + D$ are $R + S$-related precisely if they are either R-related or S-related. Naturally this can be extended to coproducts of arbitrarily many relations including, if needed, of infinitely many relations.

Proposition 15 *In a cospan of d-lenses the compatibility relation (Definition 9) is a coproduct of complete relations.*

Proof. Suppose that the cospan of d-lenses is

$$\mathbf{X} \xrightarrow{(G_L,P_L)} \mathbf{V} \xleftarrow{(G_R,P_R)} \mathbf{Y}$$

Let C be the compatibility relation between $Arr(\mathbf{X})$ and $Arr(\mathbf{Y})$. It suffices to show for x and x' in $Arr(\mathbf{X})$ and y and y' in $Arr(\mathbf{Y})$ that if $x\,C\,y$, $x'\,C\,y$ and $x'\,C\,y'$ then $x\,C\,y'$. But this follows immediately since the first three relationships imply that all four arrows have the same image (under G_L or G_R as appropriate) in $\mathbf{V}$. ∎

Thus we have a necessary condition for a least-change fb-lens to be represented by a cospan of d-lenses: The compatibility relation must be a coproduct of complete relations. This is quite a strong limitation.

We briefly turn now to a degenerate case.

Consider a **set**-based symmetric lens [9] as an fb-lens between codiscrete categories [20]. Notice that whatever compatibility relation is taken, the f and b (or, in the original notation of [9], the *putl* and *putr*) satisfy the universal property making the lens least-change. This follows not simply from a least-change aspect of the lens, but rather from the degenerate nature of the state space: In a **set**-based lens every state can be converted to every other state in a unique way. It follows that all states are isomorphic and so any choice of update is "least-change" (at any rate, there is no lesser change!).

Proposition 16 *If an fb-lens is least-change (in particular if it is between codiscretes), and if its compatibility relation is a coproduct of complete relations, then it satisfies* $\mathsf{fbf} = \mathsf{f}$ *and* $\mathsf{bfb} = \mathsf{b}$.

The comparison of this result with the "$RLR = R$" property presented in [9], and with Anthony Anjorin's use of "stable squares" [1], squares which are both f-squares and b-squares at the same time, will be saved for future work.

8 Future work

In the first part of this paper, up to and including Section 3, we present initial results in our studies of when spans of d-lenses are equivalent to spans of c-lenses. The fundamental question of whether there are spans of d-lenses which are not equivalent to any span of c-lenses remains open, and is an important topic for future work. If, as seems possible, every span of d-lenses is equivalent to a span of c-lenses, then spans of c-lenses can be of no use in identifying universal updates for symmetric lenses. The question of when and how symmetric d-lenses with extra structure might have universal updates is taken up in the second part of the paper.

The second part of this paper begins a new endeavour. There are many open questions about cospans of asymmetric lenses, about symmetric lenses satisfying universal properties, and about the relationships between the two. We record here just a few of the issues.

Firstly the stronger universal property: Least change symmetric lenses are required to satisfy the universal property specified in Definition 14. That property was chosen because it exactly matches informal descriptions of least-change. But spans of asymmetric c-lenses satisfy a stronger universal property:

$$
\begin{tikzcd}
X \arrow[r, dash, "r"] \arrow[d, "\alpha"'] \arrow[dd, bend right, "\gamma"'] & Y \arrow[d, "\mathsf{f}(\alpha,r)_0"] \arrow[dd, bend right, "\beta"'] \\
X' \arrow[r, dash] \arrow[d, "\gamma'"] & Y' \arrow[d, "\beta'"] \\
X'' \arrow[r, dash] & Y''
\end{tikzcd}
$$

Given $\alpha : X \longrightarrow X'$ and $r : X \leftrightarrow Y$ then for any α-compatible $\beta : Y \longrightarrow Y''$, if there is a $\gamma : X \longrightarrow X''$ which is compatible with β and which factors through α via some arrow $\gamma' : X' \longrightarrow X''$ (see diagram), then there is a unique $\beta' : Y' \longrightarrow Y''$ with $\beta = \beta'\mathsf{f}(\alpha, r)_0$ and β' compatible with γ'. The reader can check easily that the universal property we have discussed up until now is the special case of this one obtained by taking $\gamma = \alpha$ and $\gamma' = 1_{X'}$.

We know from other applications that this universal property not only has the least-change property as a special case, but is strictly stronger (see for example the discussion about precartesian arrows in [3]). So, even among least change symmetric lenses there are some, including at least those that arise from cospans of c-lenses, which have even better properties. If the stronger properties were to prove important in practice there would be much future work to do in developing their theory and application.

Similarly we restricted ourselves here to considering a single compatibility relation including both the relation determined by the f-squares and the relation determined by the b-squares. Having two independent compatibility relations is an easy extension of the work above, but guarantees, unless each happens to be the op-relation of the other, that the compatibility relation cannot arise from a span or cospan. Among our first questions is determining which least-change symmetric lenses arise from spans or cospans of asymmetric lenses, so in the first instance we are interested in the single compatibility relation (or equivalently, two op-related compatibility relations) case. After that work is completed we should study the more general case of a pair of compatibility relations, one from arrows of $\mathbf{X}$ to arrows of $\mathbf{Y}$ and including the relation determined by the f-squares, and one from arrows or $\mathbf{Y}$ to arrows of $\mathbf{X}$ and including the relation determined by the b-squares.

It is clear both from the simplicity of the situation in Section 5, and from earlier work of Johnson and Dampney [4, 11] and Lamo et al [24], that symmetric lenses arising from cospans are both special and very useful in applications. We should develop a good understanding of those special cases, and characterize those symmetric lenses which can be represented by cospans of lenses, or even better, by cospans of c-lenses. So far we have necessary conditions based on the structure of the corrs of a symmetric lens, and stronger necessary conditions based on the structure of the compatibility relations for symmetric lenses with compatibility relations, but there is more work to be done to achieve full characterizations.

We should also do further work on cospans of lenses including why they seem (1) to be rare among spans of asymmetric lenses, but (2) to suffice for many applications. One possible answer: Our prior work has mainly been in the database applications, and interoperating databases seem to naturally have compatibility relations which are unions of complete relations and that in turn implies that the relation meets the necessary conditions of the previous paragraph and can be captured by a cospan of functors. Furthermore our earlier work has usually included universal properties (so in the parlance of the present paper the symmetric lenses have been at least least-change lenses). In such situations we have proposals for how to enrich the cospan functors to possibly obtain a cospan of asymmetric lenses, and in some cases even of c-lenses. This could explain mathematically why in our earlier industrial work the rare situation of bidirectional transformations being represented by cospans of lenses did in fact obtain.

There are more basic open questions including the following: Can d-lenses with universality always be represented by cospans of c-lenses? And even more basically: What appropriate equivalence relations should be taken among cospans of lenses so that equivalent cospans generate the same symmetric lens?

There are many interesting opportunities for further work.

9 Conclusion

The results presented here open up new areas. The observation that *co*spans of c-lenses do satisfy universal properties yields immediately a class of symmetric lenses with universal updates, and motivates further proposals for more general "least-change" symmetric lenses. In addition, old work of Johnson and Dampney [4, 11] demonstrated that cospans of asymmetric bidirectional transformations (the work was before the introduction of lenses and so c-lenses and d-lenses had not yet been defined) could be used to solve industrial interoperability problems. It is particularly interesting that symmetric delta lenses that correspond to cospans of d-lenses seem to be rather rare among symmetric delta lenses, yet they sufficed to solve the problems that arose in practice, and we will investigate this further.

It is also worth noting that, as explained in Johnson's forthcoming Oxford Summer School lectures [13], and in [12], cospans of bidirectional transformations have particularly appealing cyber security properties and they substantially simplify the software engineering tasks required to achieve interoperability.

The remaining results presented here begin the detailed study of cospans of d-lenses, seeking to characterize them among symmetric delta lenses, and they lead to proposals for more generalised notions of symmetric delta lenses with universal updates.

10 Acknowledgements

The authors are grateful for the support of the Australian Research Council and the Centre of Australian Category Theory.

References

[1] Anjorin, A. (2017) Bx with Triple Graph Grammars. *Lectures to the Oxford Summer School on Bidirectional Transformations*, July, 2017. Written version in preparation.

[2] Barr, M. and Wells, C. (1995) *Category theory for computing science.* Prentice-Hall.

[3] Borceux, F. (1994) *Handbook of Categorical Algebra*, Vol 2. Cambridge University Press.

[4] Dampney, C.N.G and Johnson, M. (2001) Half-duplex interoperations for cooperating information systems. *Advances in Concurrent Engineering*, 565–571. See also [11].

[5] Zinovy Diskin, Yingfei Xiong, Krzysztof Czarnecki (2011) From State- to Delta-Based Bidirectional Model Transformations: the Asymmetric Case. *Journal of Object Technology* **10**, 1–25.

[6] Zinovy Diskin, Yingfei Xiong, Krzysztof Czarnecki, Hartmut Ehrig, Frank Hermann and Francesco Orejas (2011) From State- to Delta-Based Bidirectional Model Transformations: the Symmetric Case. *Lecture Notes in Computer Science* **6981**, 304–318.

[7] Gibbons, J. and Johnson, M. (2012) Relating algebraic and coalgebraic descriptions of lenses. *Electronic Communications of the EASST* **49**, 1–16.

[8] Gibbons, J. and Stevens, P. (2016) A theory of least change for bidirectional transformations. http://groups.inf.ed.ac.uk/bx/ and http://www.cs.ox.ac.uk/projects/tlcbx/ accessed, January 13, 2017.

[9] Hofmann, M., Pierce, B., and Wagner, D. (2011) Symmetric Lenses. ACM SIGPLAN Notices **46**, 371–384.

[10] Hofmann, M., Pierce, B., and Wagner, D. (2012) Edit Lenses. ACM SIGPLAN Notices **47**, 495–508.

[11] Johnson, M. (2007) Enterprise Software with Half-Duplex Interoperations. In Doumeingts, Mueller, Morel and Vallespir (eds), *Enterprise Interoperability: New Challenges and Approaches*, 521–530, Springer-Verlag. Revised and expanded version of [4]

[12] Johnson, M. (2016) Cyber Security and other New Applications of Bx. *Lecture to the Shonan Meeting on Bidirectional Transformations*, September 2017, NII Centre, Shonan, Japan.

[13] Johnson, M. (2017) Mathematical Foundations of Bidirectional Transformations. *Lectures to the Oxford Summer School on Bidirectional Transformations*, July, 2017. Written version in preparation.

[14] Johnson, M. and Rosebrugh, R. (2001) View updatatability based on the models of a formal specification. *Lecture Notes in Computer Science* **2021**, 534–549.

[15] Johnson, M. and Rosebrugh, R. (2007) Fibrations and universal view updatability. *Theoretetical Computer Science* **388**, 109–129.

[16] Johnson, M. and Rosebrugh, R. (2013) Delta lenses and fibrations. *Electronic Communications of the EASST* **57**, 18pp.

[17] Johnson, M. and Rosebrugh, R. (2014) Spans of lenses. *CEUR Proceedings* **1133**, 112–118.

[18] Johnson, M. and Rosebrugh, R. (2015) Spans of delta lenses. *CEUR Proceedings* **1396**, 1–15.

[19] Johnson, M. and Rosebrugh, R. (2015), Distributing Commas, and the Monad of Anchored Spans, CEUR Proceedings **1396**, 31–42.

[20] Johnson, M. and Rosebrugh, R. (2016) Unifying set-based, delta-based and edit-based lenses. *CEUR Proceedings* **1571**, 1–13.

[21] Johnson, M. and Rosebrugh, R. (2017) Symmetric delta lenses and spans of asymmetric delta lenses. *Journal of Object Technology*, to appear.

[22] Johnson, M., Rosebrugh, R. and Wood, R. J. (2010) Algebras and Update Strategies. *Journal of Universal Computer Science* **16**, 729–748.

[23] Johnson, M., Rosebrugh, R. and Wood, R. J. (2012) Lenses, fibrations and universal translations. *Mathematical Structures in Computer Science* **22**, 25–42.

[24] Lamo, Y., Mantz, F., Rutle, A. and de Lara, J. (2013) A declarative and bidirectional model transformation approach based on graph cospans. Proceedings of the 15th Symposium on Principles and Practice of Declarative Programming, 1–12.

[25] McKinna, J. (2016) Complements Witness Consistency. *CEUR Proceedings* **1571**, 90–94.

[26] Meertens, L. (1998) Designing constraint maintainers for user interaction. Unpublished manuscript http://www.kestrel.edu/home/people/meertens/pub/dcm.ps accessed January 13, 2017.

[27] Pierce, B. (1991) *Basic category theory for computer scientists.* MIT Press.

[28] Pierce, B. and Schmitt, A. (2003) Lenses and view update translation. Preprint.

[29] Street, R. (1974) Fibrations and Yoneda's lemma in a 2-category. *Lecture Notes in Mathematics* **420**, 104–133.