Cospans and Symmetric Lenses

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ABSTRACT

We characterize those symmetric d-lenses which are representable as cospans of d-lenses. Such a symmetric d-lens must have unique corrs per pair of objects and satisfy two other technical conditions. When the d-lens is also “least change” then the corresponding cospan consists of c-lenses.

KEYWORDS

Symmetric lens, cospan, universality

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1 INTRODUCTION

The body of this paper presents a collection of results that together solve a characterisation problem. With the limitations of space, and the need for precise mathematical argument, the sections following this one are succinct and mathematically detailed, so it's particularly appropriate to give some indication of the importance of the problem here, and of the possibilities its solution opens in the future work section. The problem we address is determining when a symmetric lens may be “represented” by a cospan of asymmetric lenses. The succeeding sections are laid out so as to provide a solution to that problem, and to show exactly where each condition needed for the characterisation of such symmetric lenses is used in the arguments.

Cospans of lenses have been important since before lenses were named. In consultancy work we determined that cospans of what are now called c-lenses were particularly valuable in constructing interoperations between legacy systems [5]. Remarkably often we were able to construct such cospans, yet it is easy to show that not all bidirectional transformations when presented as symmetric lenses can in fact arise from cospans of asymmetric lenses. Further analysis revealed that a substantial part of the practical value of cospans of lenses came from what would now be called a cyber security problem – if the bidirectional transformation between two organisations can be represented by a cospan of asymmetric lenses, then the organisations are much more likely to agree to the work because they can better manage the security of their own systems. (A study of these kinds of cyber security issues, and of the relevance of a cospan solution, is given in [12].)

But how can we tell if a particular bidirectional transformation can indeed be decomposed into a cospan of asymmetric lenses? That is the question that this paper answers.

In Section 2 we provide definitions and necessary previous results. In particular, we begin by defining symmetric d-lenses which we continue to call *fb-lenses* to emphasize their propagation operations, and the notion of equivalent fb-lenses. We also consider asymmetric d-lenses and their special case, c-lenses. The notion of representation of fb-lenses by either a span or cospan of asymmetric lenses is made precise and we update the definition of compatibility relation for an fb-lens from [10].

Section 3 has our main results. First we study two equivalence relations defined from a compatibility relation on an fb-lens and use them to define the category at the base of the cospan of d-lenses that will represent the fb-lens. Next we define the Get functors for the cospan and, after adding additional requirements, we define the Puts. Finally, we show that the cospan we have defined represents the original fb-lens. Finally, we show that if the fb-lens is also “least change” (respectively, is “cartesian”), then the representing d-lenses have pre-cartesian Get functors (respectively, are c-lenses).

2 DELTA LENSES

In this section we collect the definitions and results from previous work that we need for our main results. We begin with definitions of the symmetric and asymmetric delta lenses that we have already studied extensively. We assume the reader is familiar with the terminology of basic category theory for Computer Science as found in, for example, Pierce's

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[13]. We will usually use bold-face for categories $\mathbf{X}, \mathbf{Y}, \ldots$, upper case for objects $X, Y, \ldots$ and lower case for arrows $x, y, \ldots$ and write id_X for an identity arrow. The class of objects of $\mathbf{X}$ is denoted $|\mathbf{X}|$, and its category of arrows is denoted $\mathbf{X}^2$. Pullbacks are denoted using the usual fibred product notation.

The concept of a symmetric version of delta lenses was first introduced by Diskin and colleagues [3]. We have used the following definition in a series of articles [7–9]. The idea is that the categories $\mathbf{X}$ and $\mathbf{Y}$ are model spaces: the objects are particular models and the arrows specify updating processes. The "corrs" are witnesses of the consistency of a model in $\mathbf{X}$ and a model in $\mathbf{Y}$. The propagation operations restore consistency: when a model on one side is consistent with a second model on the other and is updated to a new state, the propagation operation specifies an update of the second model to a consistent state witnessed by a new corr.

Definition 2.1. Let $\mathbf{X}$ and $\mathbf{Y}$ be categories. An *fb-lens* from $\mathbf{X}$ to $\mathbf{Y}$ is a 4-tuple $L = (\delta_\mathbf{X}, \delta_\mathbf{Y}, \mathrm{f}, \mathrm{b}) : \mathbf{X} \longleftrightarrow \mathbf{Y}$ specified as follows. The data $\delta_\mathbf{X}, \delta_\mathbf{Y}$ are functions with a common domain R for a span of sets

$$\delta_\mathbf{X} : |\mathbf{X}| \longleftarrow R \longrightarrow |\mathbf{Y}| : \delta_\mathbf{Y}$$

An element of R is called a *corr*. For r in R, if $\delta_\mathbf{X}(r) = X$, $\delta_\mathbf{Y}(r) = Y$, the corr is denoted $r : X \leftrightarrow Y$, or sometimes just $r : X - Y$. The data f and b are operations called *forward* and *backward propagation*:

$$\mathrm{f} : Arr(\mathbf{X}) \times_{|\mathbf{X}|} R \longrightarrow Arr(\mathbf{Y}) \times_{|\mathbf{Y}|} R$$

$$\mathrm{b} : Arr(\mathbf{Y}) \times_{|\mathbf{Y}|} R \longrightarrow Arr(\mathbf{X}) \times_{|\mathbf{X}|} R$$

where the pullbacks ensure that if $\mathrm{f}(x, r) = (y, r')$, we have $d_0(x) = \delta_\mathbf{X}(r)$, $d_1(y) = \delta_\mathbf{Y}(r')$ and similarly for b. We *also require* that $d_0(y) = \delta_\mathbf{Y}(r)$ and $\delta_\mathbf{X}(r') = d_1(x)$, and the similar equations for b.

Furthermore, we require that both propagations respect both the identities and composition in $\mathbf{X}$ and $\mathbf{Y}$, so that we have:

$$r : X \leftrightarrow Y \Rightarrow \mathrm{f}(\mathrm{id}_X, r) = (\mathrm{id}_Y, r) \text{ and } \mathrm{b}(\mathrm{id}_Y, r) = (\mathrm{id}_X, r)$$

and

$$\mathrm{f}(x, r) = (y, r'), \mathrm{f}(x', r') = (y', r'') \Rightarrow \mathrm{f}(x'x, r) = (y'y, r'')$$

and

$$\mathrm{b}(y, r) = (x, r'), \mathrm{b}(y', r') = (x', r'') \Rightarrow \mathrm{b}(y'y, r) = (x'x, r'')$$

It will eventually be important for us that every model state in $\mathbf{X}$ or $\mathbf{Y}$ is consistent with at least one state on the other side, and we define:

Definition 2.2. An *fb-lens* $L = (\delta_\mathbf{X}, \delta_\mathbf{Y}, \mathrm{f}, \mathrm{b})$ is called δ *surjective* if both $\delta_\mathbf{X}$ and $\delta_\mathbf{Y}$ are surjective functions.

Notation. We will denote the pair $\mathrm{f}(x, r)$ by $(\mathrm{f}_a(x, r), \mathrm{f}_c(x, r))$ and similarly for b.

We also need to recall the definition of the *asymmetric* version of delta lens ([2, 6]) which we will usually abbreviate to *d-lens*. We refer the reader to those articles for the "comma category" notation $(G, 1_\mathbf{X})$ used in:

Definition 2.3. An *asymmetric delta lens (d-lens)* from $\mathbf{S}$ to $\mathbf{X}$ is a pair (G, P) where $G : \mathbf{S} \longrightarrow \mathbf{X}$ is a functor (the "Get") and $P : |(G, 1_\mathbf{X})| \longrightarrow |\mathbf{S}^2|$ is a function (the "Put") and the data for $x : G(S) \longrightarrow X$ and $x' : G(S') \longrightarrow X'$ satisfy:

(i) d-PutInc: the domain of $P(S, x)$ is S
(ii) d-PutId: $P(S, \mathrm{id}_{G(S)}) = \mathrm{id}_S$
(iii) d-PutGet: $G(P(S, x)) = x$
(iv) d-PutPut: if S' is the codomain of $P(S, x)$ (and hence $G(S') = X$) then $P(S, x'x) = P(S', x')P(S, x)$.

We recall two constructions of fb-lenses from d-lenses. First, (see [9], p15) from a span $(G_L, P_L) : \mathbf{X} \longleftarrow \mathbf{S} \longrightarrow \mathbf{Y} : (G_R, P_R)$ of d-lenses we construct an fb-lens whose corrs are the objects of $\mathbf{S}$, and whose forward propagation is defined by applying first P_L then G_R. Backward propagation is similar. On the other hand, (see Construction 8 of [10]) from a *co*span $(G_L, P_L) : \mathbf{X} \longrightarrow \mathbf{V} \longleftarrow \mathbf{Y} : (G_R, P_R)$ of d-lenses we can also construct an fb-lens. Its corrs are the pairs (X, Y) of objects of $\mathbf{X}$ and $\mathbf{Y}$ which are matching in the sense that $G_L(X) = G_R(Y)$ and its forward propagation is defined by applying first G_L then P_R. In [9] we worked through examples of the first construction. The second construction will be important to us in this article and here is an example.

Example 2.4. We denote by **set** the category whose objects are *finite* sets and whose arrows are functions between them. As a category of models, the model states (objects) of **set** are each just a set, which can be thought of as the state of a single entity. An arrow of **set** is a function which updates one state (entity set) to another.

We also consider another category of model states, $\mathbf{set}^2$. An object (or model state) X of $\mathbf{set}^2$ is a function $X_f : X_0 \longrightarrow X_1$ between sets X_0 and X_1. The object X has two entity sets, X_0 and X_1, and one constraint specified by the function X_f. In a category of models, X_0 might be the current state of a Persons entity, X_1 that of an Addresses entity, and X_f the assignment of a person to an address. In another category of models, a model $Y_f : Y_0 \longrightarrow Y_1$, Y_0 might be the state of an Addresses entity, Y_1 that of a Cities entity, and Y_f the assignment of an address to a city. An arrow in $\mathbf{set}^2$ from the object X to another object (model state) X' is a pair of functions $x = (x_0, x_1)$ between corresponding entity sets which are compatible with the respective constraints in the sense that $X'_f x_0 = x_1 X_f$.

In [9] we defined two distinct d-lenses from $\mathbf{set}^2$ to **set** which we briefly review.

The first d-lens (G_1, P_1) has as its Get the "codomain" functor $G_1 : \mathbf{set}^2 \longrightarrow \mathbf{set}$ which sends an object X with $X_f : X_0 \longrightarrow X_1$ of $\mathbf{set}^2$ to the set $G_1(X) = X_1$ and sends an arrow to its second factor.

The first Put, P_1 is defined as follows. Consider any set X_1' and any function, say $x_1 : X_1 \longrightarrow X_1'$, from $G_1(X) = X_1$ to X_1'. We require $P_1(X, x_1)$ to be an arrow from X. Its codomain X' is defined to be the model with function $X_f' = x_1 X_f : X_0' := X_0 \longrightarrow X_1'$. Then the arrow $P_1(X, x_1)$ is the pair (id_{X_0}, x_1) which satisfies $x_1 X_f = X_f' \mathrm{id}_{X_0}$.

The second d-lens (G_0, P_0) has as its Get the "domain" functor, $G_0 : \mathbf{set}^2 \longrightarrow \mathbf{set}$ which sends an object Y with $Y_f : Y_0 \longrightarrow Y_1$ of $\mathbf{set}^2$ to the set $G_0(Y) = Y_0$ and sends an arrow to its first factor.

The second Put, P_0 has a more interesting definition. Start with a model Y and any function, say $y_0 : Y_0 \longrightarrow Y_0'$, from $G_0(Y) = Y_0$ to Y_0'. The codomain of $P_0(Y, y_0)$ has to be an object Y' of $\mathbf{set}^2$ whose function has the domain Y_0'. We define Y' to be the object whose function is the bottom arrow in the $\mathbf{set}$ pushout of Y_f along y_0:

$$\begin{array}{ccc} Y_0 & \xrightarrow{Y_f} & Y_1 \\ {\scriptstyle y_0}\downarrow & + & \downarrow{\scriptstyle y_1} \\ Y_0' & \xrightarrow[Y_f']{} & Y_1' \end{array}$$

Now we define $P_0(Y, y_0)$ to be the arrow in $(\mathbf{set}^2)$ from Y to Y' defined by the pair of functions $P_0(Y, y_0) = (y_0, y_1)$.

Next we define the forward and backward propagations for the fb-lens L constructed from the cospan:

$$(G_1, P_1) : \mathbf{set}^2 \longrightarrow \mathbf{set} \longleftarrow \mathbf{set}^2 : (G_1, P_1)$$

First note that a corr is a pair (X, Y) such that $X_1 = G_1(X) = G_0(Y) = Y_0$. In the interpretations above this is a matching set of Addresses. The forward propagation for an update $x = (x_0, x_1) : X \longrightarrow X'$ and a corr (X, Y) is the arrow $y = (y_0, y_1) : Y \longrightarrow Y'$ and the corr (X', Y') defined by $y_0 := x_1$ and where y_1 and Y_f' are the co-projections to the pushout of Y_f along y_0 as in

$$\begin{array}{ccc} X_0 & \xrightarrow{X_f} & X_1 = Y_0 \\ {\scriptstyle x_0}\downarrow & & \downarrow{\scriptstyle x_1} \\ X_0' & \xrightarrow[X_f']{} & X_1' \end{array} \quad \overset{\mathsf{f}}{\mapsto} \quad \begin{array}{ccc} Y_0 & \xrightarrow{Y_f} & Y_1 \\ {\scriptstyle y_0}\downarrow & + & \downarrow{\scriptstyle y_1} \\ X_1' = Y_0' & \xrightarrow[Y_f']{} & Y_1' \end{array}$$

In the interpretation, when the Names to Addresses state is updated with new addresses specified by x_1 then that propagates to an Addresses update in the other model category and the Cities are freely updated (by y_1) to accommodate the updated addresses.

The backward propagation for an update $y = (y_0, y_1) : Y \longrightarrow Y'$ and a corr (X, Y) is the arrow $x = (x_0, x_1) : X \longrightarrow X'$ and the corr (X', Y') where $x_1 := y_0$, $x_0 := \mathrm{id}_{X_0}$ and $X_f' := x_1 X_f$.

$$\begin{array}{ccc} X_0 & \xrightarrow{X_f} & X_1 \\ {\scriptstyle x_0}\downarrow & & \downarrow{\scriptstyle x_1} \\ X_0' & \xrightarrow[X_f']{} & X_1' = Y_0' \end{array} \quad \overset{\mathsf{b}}{\leftarrow\!\shortmid} \quad \begin{array}{ccc} X_1 = Y_0 & \xrightarrow{Y_f} & Y_1 \\ {\scriptstyle y_0}\downarrow & & \downarrow{\scriptstyle y_1} \\ Y_0' & \xrightarrow[Y_f']{} & Y_1' \end{array}$$

Interpreting this propagation is easy: In its codomain the Addresses update is simply composed with the original names to addresses mapping and the Names do not change.

There is a close relationship between fb-lenses and spans of d-lenses. An important result in [9] is the representation of an equivalence class of fb-lenses (related by equivalent behaviour) by an equivalence class of spans of asymmetric delta lenses. The representation is compatible with span and lens composition. We recall from [9] the equivalence relation on the fb-lenses from $\mathbf{X}$ to $\mathbf{Y}$.

Definition 2.5. Let $L = (\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathsf{f}, \mathsf{b})$ and $L' = (\delta_{\mathbf{X}}', \delta_{\mathbf{Y}}', \mathsf{f}', \mathsf{b}')$ be two fb-lenses (from $\mathbf{X}$ to $\mathbf{Y}$) with corrs $R_{\mathbf{XY}}, R_{\mathbf{XY}}'$. We say $L \equiv_{\mathrm{fb}} L'$ iff there is a relation σ from $R_{\mathbf{XY}}$ to $R_{\mathbf{XY}}'$ with the following properties:

(1) σ is compatible with the δ's, i.e. $r\sigma r'$ implies $\delta_{\mathbf{X}} r = \delta_{\mathbf{X}}' r'$ and $\delta_{\mathbf{Y}} r = \delta_{\mathbf{Y}}' r'$
(2) σ is total in both directions, i.e. for all r in $R_{\mathbf{XY}}$, there is r' in $R_{\mathbf{XY}}'$ with $r\sigma r'$ and conversely.
(3) for all r, r', x an arrow of $\mathbf{X}$, if $r\sigma r'$ and $\delta_{\mathbf{X}} r$ is the domain of x then the first components of $\mathsf{f}(x, r)$ and $\mathsf{f}'(x, r')$ are equal and the second components are σ related, i.e. $\mathsf{f}_a(x, r) = \mathsf{f}_a'(x, r')$ and $\mathsf{f}_c(x, r)\sigma \mathsf{f}_c'(x, r')$
(4) the corresponding condition for b, i.e. for all r, r', y an arrow of $\mathbf{Y}$, if $r\sigma r'$ and $\delta_{\mathbf{X}} r$ is the domain of x then $\mathsf{b}_a(y, r) = \mathsf{b}_a'(y, r')$ and $\mathsf{b}_c(y, r)\sigma \mathsf{b}_c'(y, r')$

The sense of *representation* we have in mind is the following.

Definition 2.6. Let L be an fb-lens. A span $(G_L, P_L) : \mathbf{S} \longrightarrow \mathbf{X}$, $(G_R, P_R) : \mathbf{S} \longrightarrow \mathbf{Y}$ of d-lenses *represents* L iff up to a bijection of the sets of corrs, the construction above gives forwards and backwards propagations with the same actions as those of L. A cospan $(G_L, P_L) : \mathbf{X} \longrightarrow \mathbf{V}$, $(G_R, P_R) : \mathbf{Y} \longrightarrow \mathbf{V}$ of d-lenses *represents* L iff the analogous conditions hold.

We also recall from [9] that for a cospan $(G_L, P_L) : \mathbf{X} \longrightarrow \mathbf{V}$, $(G_R, P_R) : \mathbf{Y} \longrightarrow \mathbf{V}$ of d-lenses, the projection functors from the pullback in $\mathbf{cat}$ of the cospan are canonically the Gets for a span of d-lenses. As we have noted before, the pullback is *not* a pullback in a category of lenses.

Thus the "pullback" operation shows us that every cospan of d-lenses is associated with a span of d-lenses (the one obtained by "pulling back" the cospan) and both the span and cospan represent the same fb-lens.

In [10] we defined a notion of *compatibility relation* for fb-lenses derived from a consideration of cospans of d-lenses. For this article, that notion is refined as follows (essentially by adding conditions C1 and C3).

Definition 2.7. Let $L = (\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathrm{f}, \mathrm{b})$ be an fb-lens between $\mathbf{X}$ and $\mathbf{Y}$ with corrs R. A *compatibility relation* on L is a relation C between the arrows of $\mathbf{X}$ and the arrows of $\mathbf{Y}$ such that

- C0: $x\,C\,y$ implies that there exist corrs $r : d_0(x) \leftrightarrow d_0(y)$ and $r' : d_1(x) \leftrightarrow d_1(y)$ (say: "C respects corrs")
- C1: For any r, $\mathrm{id}_{\delta_{\mathbf{X}}(r)}\,C\,\mathrm{id}_{\delta_{\mathbf{Y}}(r)}$; and $x\,C\,y$ and $x'\,C\,y'$ implies $x'x\,C\,y'y$ whenever $x'x$ and $y'y$ are defined (C respects identities and composition).
- C2: if $d_0(x) = \delta_{\mathbf{X}}(r)$ then $x\,C\,\mathrm{f}_a(x,r)$ and if $d_0(y) = \delta_{\mathbf{Y}}(r)$ then $\mathrm{b}_a(y,r)\,C\,y$ (the sides of propagation squares are C related).
- C3: $x\,C\,y$ and $x'\,C\,y$ and $x'\,C\,y'$ implies $x\,C\,y'$

Condition C3 says that C is a *difunctional relation* [14] and [1] p. 200. We have previously called such a relation a coproduct of complete bipartite relations. See also [15].

Proposition 2.8. *Let L be the fb-lens constructed from the cospan $(G_L, P_L) : \mathbf{X} \longrightarrow \mathbf{S} \longleftarrow \mathbf{Y} : (G_R, P_R)$ of d-lenses. Then $C = \{(x,y) | G_L(x) = G_R(y)\}$ is a compatibility relation.*

Proof. The required corrs for C0 are given by $(d_0(x), d_0(y))$ and $(d_1(x), d_1(y))$. For identities (as in C1), if $G_L(X) = G_R(Y)$, then $G_L(\mathrm{id}_{d_0(X)}) = \mathrm{id}_{G_L(X)} = \mathrm{id}_{G_R(Y)} = G_R(\mathrm{id}_{d_0(Y)})$. For the composition, functoriality of the Gets also suffices. For C2, if r is the pair $(d_0(x), d_0(y))$, we have $\mathrm{f}_a(x,r) = P_R(G_L(x))$, but PutGet gives $G_R P_R(G_L(x)) = G_L(x)$, so $x\,C\,\mathrm{f}_a(x,r)$. C3 is just transitivity of equality. □

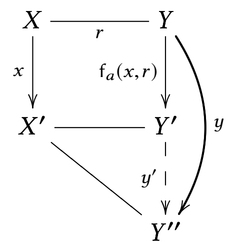
There is an important special case of d-lens (as shown in [6]) called the *c-lens*. We refer the reader to any of [6], [7], [10] or [11] for more detail including motivation and the notation used in the following:

Definition 2.9. A *c-lens* from $\mathbf{S}$ to $\mathbf{X}$ is a pair (G, P) where $\mathbf{S} \xrightarrow{G} \mathbf{X}$ and $(G, 1_{\mathbf{X}}) \xrightarrow{P} \mathbf{S}$ are functors satisfying

i) c-PutGet: $GP = RG$
ii) c-GetPut: $P\eta_G = 1_{\mathbf{S}}$
iii) c-PutPut: $P\mu_G = P(P, 1_{\mathbf{X}})$

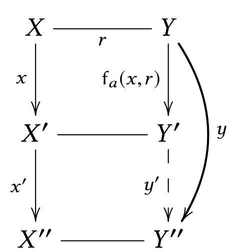
In [10] we defined a *least change* property for an fb-lens with a compatibility relation. The idea is that the propagations will satisfy a universal property.

Definition 2.10. An fb-lens L equipped with a compatibility relation C is called *least-change* if for any $x : X \longrightarrow X'$ and corr $r : X \leftrightarrow Y$ it is the case that $\mathrm{f}_a(x,r)$ satisfies the following universal property: For any $y : Y \longrightarrow Y''$ compatible with x there is a unique $y' : Y' \longrightarrow Y''$ with $y = y'\mathrm{f}_a(x,r)$ and $\mathrm{id}_{X'}\,C\,y'$: and similarly for the back propagation b.



We showed in [10] that if a cospan of c-lenses represents L, then L is least change. More generally, such a cospan satisfies the following more general condition that accounts for composites with x.

Definition 2.11. An fb-lens L equipped with a compatibility relation C is called *cartesian* if for any $x : X \longrightarrow X'$ and corr $r : X \leftrightarrow Y$ it is the case that $\mathrm{f}_a(x,r)$ satisfies the following universal property: For any $y : Y \longrightarrow Y''$ compatible with $x'x$ there is a unique $y' : Y' \longrightarrow Y''$ with $y = y'\mathrm{f}_a(x,r)$ and $x'\,C\,y'$: and similarly for the back propagation b.



3 COMPATIBILITY AND COSPANS

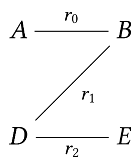
We will show that certain fb-lenses with compatibility give rise to cospans of d-lenses. Our first objective is the construction of a cospan of categories $G_L : \mathbf{X} \longrightarrow \mathbf{C} \longleftarrow \mathbf{Y} : G_R$ from an fb-lens $L = (\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathrm{f}, \mathrm{b})$ with compatibility relation C and we fix L and C for the rest of this section. We will need additional conditions to ensure that the cospan represents L. We note in passing that the results in Lemmas 3.1, 3.3 and 3.5 do not require property C2 of a compatibility relation (Definition 2.7), nor is it needed for the construction of objects and arrows of the base $\mathbf{C}$ of the cospan. However, to define the composition in $\mathbf{C}$ we do need C2.

We will call condition C3 of Definition 2.7 the *ZX property* – short for Z implies X.

Lemma 3.1. *The ZX property for a compatibility relation implies the ZX property for corrs. That is, for corrs r_0, r_1, r_2 with δ's as suggested by the left figure below, there is a corr r_3*

as in the right figure:

Proof. Given the figure of corrs:



consider the identities at all four corner objects and then C1 and C3 imply $\mathrm{id}_A\, C\, \mathrm{id}_E$, so by C0 there is a corr from A to E as required. □

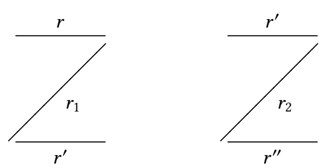
We remark that the "reversal of Z" implies X also holds, which we call *backwards ZX*. (Start from the diagram using the *other* diagonal.)

Definition 3.2. Define the relation N on corrs by rNr' for corrs r, r' if there is a corr r'' such that $\delta_{\mathrm{X}}(r'') = \delta_{\mathrm{X}}(r)$ and $\delta_{\mathrm{Y}}(r'') = \delta_{\mathrm{Y}}(r')$

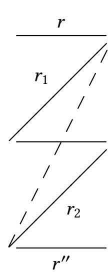
In other words, in a diagram of corrs, r and r' can be represented as the top and bottom, respectively, of a Z. By the ZX property, equivalently r and r' are the top and bottom, respectively, of a backwards Z of corrs.

Lemma 3.3. *N is an equivalence relation*

Proof. First $r\,N\,r$ is obvious. If $r\,N\,r'$ then $r'\,N\,r$ by ZX (and a vertical flip of the diagram). Now suppose $r\,N\,r'$ and $r'\,N\,r''$ via:



We can stack the Z's:



so there is a backwards Z formed by r_1, r', and r_2, and we apply ZX to get the top-right to bottom left corr witnessing $r\,N\,r''$. □

As we did for corrs, we now define a relation on C-related pairs of arrows.

Definition 3.4. Let C be a compatibility relation. Define a relation E on C-related pairs by $(x, y)\,E\,(x', y')$ if and only if $x'\,C\,y$ (or, equivalently by ZX, iff $x\,C\,y'$).

Lemma 3.5. *E is an equivalence relation*

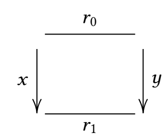
Proof. The proof uses arguments parallel to Lemma 3.3.

First, clearly $(x, y)\,E\,(x, y)$. If $(x, y)\,E\,(x', y')$ then by ZX $x\,C\,y'$ so $(x', y')\,E\,(x, y)$. Now suppose $(x, y)\,E\,(x', y')$ and $(x', y')\,E\,(x'', y'')$ so $x'\,C\,y$, $x'\,C\,y'$ and $x''\,C\,y'$. Now by ZX we get $x''\,C\,y$ whence $(x, y)\,E\,(x'', y'')$ as required for transitivity. □

Construction. We specify the data for a category we will call **C** and which will be the base of a cospan from **X** to **Y**.

Objects: $|\mathbf{C}|$, the objects of **C**, is the set R/N. So objects of **C** are N-equivalence classes of corrs. For a corr r, we denote its N equivalence class $[r]_N$, and occasionally omit the subscript.

Arrows: For objects A, B in $|\mathbf{C}|$, define $\mathbf{C}_{A,B} = \{(x, y) \in C \mid \exists r_0 \in A, r_1 \in B \text{ with } \delta_{\mathrm{X}}(r_0) = d_0(x), \delta_{\mathrm{Y}}(r_0) = d_0(y), \delta_{\mathrm{X}}(r_1) = d_1(x), \delta_{\mathrm{Y}}(r_1) = d_1(y)\}$. Graphically, we require:

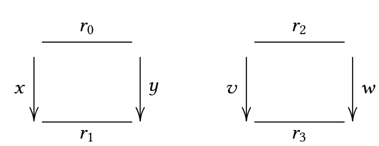


We denote by $E_{A,B}$ the restriction of E to $\mathbf{C}_{A,B}$, and remark that it is still an equivalence relation. For a compatible pair of arrows (x, y) we denote its E equivalence class $[(x, y)]_E$, and usually omit the subscript.

Define the hom-set $\mathbf{C}(A, B)$ to be $\mathbf{C}_{A,B}/E_{A,B}$.

Composition: To define the composite of equivalence classes $g = [(x, y)]_E \in \mathbf{C}(A, B)$ and $h = [(v, w)]_E \in \mathbf{C}(B, D)$ we will construct a representative of the second equivalence class which is directly composable with (x, y).

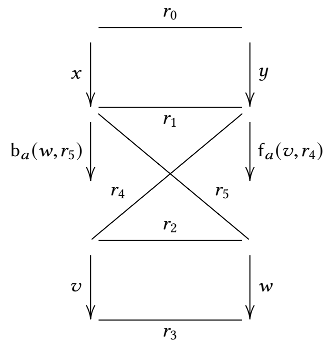
For the representatives (x, y) and (v, w) there are corrs with δs as indicated in the squares below:



Since r_1, r_2 are in B, there are corrs r_4 from $d_1(y)$ to $d_0(v)$ and r_5 from $d_1(x)$ to $d_0(w)$ which we use to define the composite

$$hg = [(\mathrm{b}_a(w, r_5)x, \mathrm{f}_a(v, r_4)y)] \tag{1}$$

as shown in the following diagram:



Notice that the definition of composition does not involve any of r_0, r_1, r_2, r_3. Furthermore, in the definition of hg we need to know that $\mathsf{b}_a(w, r_5)x \, C \, \mathsf{f}_a(v, r_4)y$. Since $x \, C \, y$, that follows by C1 if $\mathsf{b}_a(w, r_5) \, C \, \mathsf{f}_a(v, r_4)$. However, the latter follows from $v \, C \, \mathsf{f}_a(v, r_4)$, $v \, C \, w$, $\mathsf{b}_a(w, r_5) \, C \, w$ and ZX. Notice that here we finally use the condition C2.

Identities: To define the identity on an object A of $\mathbf{C}$, choose a corr r in A. The identity will be $[(\mathrm{id}_{\delta_{\mathrm{X}}(r)}, \mathrm{id}_{\delta_{\mathrm{Y}}(r)})]$.

The idea behind the definition of the composite (1) is to replace (v, w) with an E-equivalent pair which is directly composable with the pair (x, y). We need to show that the definitions of composition and identity are well-defined, that is independent of the choice of representatives.

Proposition 3.6. *The definition of identities for* $\mathbf{C}$ *is well-defined.*

Proof. The proof only depends on the fact that identities propagate to identities.

Suppose rNr'. Then there is an r'' with $\delta_{\mathrm{X}}(r'') = \delta_{\mathrm{X}}(r')$ and $\delta_{\mathrm{Y}}(r'') = \delta_{\mathrm{Y}}(r)$. Since $\mathsf{f}_a(\mathrm{id}_{\delta_{\mathrm{X}}(r')}, r'') = \mathrm{id}_{\delta_{\mathrm{Y}} r}$, we have $\mathrm{id}_{\delta_{\mathrm{X}}(r')} \, C \, \mathrm{id}_{\delta_{\mathrm{Y}} r}$ showing $(\mathrm{id}_{\delta_{\mathrm{X}}(r)}, \mathrm{id}_{\delta_{\mathrm{Y}}(r)}) \, E \, (\mathrm{id}_{\delta_{\mathrm{X}}(r')}, \mathrm{id}_{\delta_{\mathrm{Y}}(r')})$. □

We note the following useful lemma:

Lemma 3.7. *Suppose* $(x, y) \, E \, (x', y')$ *and* $(v, w) \, E \, (v', w')$, *and* $d_1(x) = d_0(v)$, $d_1(x') = d_0(v')$, $d_1(y) = d_0(w)$ *and* $d_1(y') = d_0(w')$ *(so that* (x, y) *directly composes with* (v, w) *and* (x', y') *directly composes with* (v', w')*), then* $(vx, wy) \, E \, (v'x', w'y')$.

Proof. Since $(x, y) \, E \, (x', y')$ we have $x' \, C \, y$. Similarly, since $(v, w) \, E \, (v', w')$ we have $v' \, C \, w$. So by C1, $v'x' \, C \, wy$ showing that $(vx, wy) \, E \, (v'x', w'y')$. □

Proposition 3.8. *The composite specified by* (1) *is well-defined, that is it is independent of the choice of* (x, y), (v, w), r_4 *and* r_5.

Proof. We begin with independence of the choice of r_4: Suppose that s_4 is a corr "parallel" to r_4, that is $\delta_{\mathrm{X}}(r_4) = \delta_{\mathrm{X}}(s_4)$ and $\delta_{\mathrm{Y}}(r_4) = \delta_{\mathrm{Y}}(s_4)$. As noted above, $\mathsf{b}_a(w, r_5) \, C \, \mathsf{f}_a(v, r_4)$ and by the same argument $\mathsf{b}_a(w, r_5) \, C \, \mathsf{f}_a(v, s_4)$. Thus we have $(\mathsf{b}_a(w, r_5), \mathsf{f}_a(v, r_4)) \, E \, (\mathsf{b}_a(w, r_5), \mathsf{f}_a(v, s_4))$. Then, since the domains of $\mathsf{f}_a(v, r_4)$ and $\mathsf{f}_a(v, s_4)$ are the same, by Lemma 3.7 we have that $(\mathsf{b}_a(w, r_5)x, \mathsf{f}_a(v, r_4)y) \, E \, (\mathsf{b}_a(w, r_5)x, \mathsf{f}_a(v, s_4)y)$, so $[(\mathsf{b}_a(w, r_5)x, \mathsf{f}_a(v, r_4)y)] = [(\mathsf{b}_a(w, r_5)x, \mathsf{f}_a(v, s_4)y)]$ as required.

Independence of the choice of r_5 is similar.

Now consider (v, w): Suppose that $(v, w) \, E \, (v', w')$. We will show that the composites according to (1) of (x, y) with each of (v, w) and (v', w') are equivalent. Let $f = \mathsf{f}_a(v, r_4)$ and $b = \mathsf{b}_a(w, r_5)$ as in (1). Similarly, let $f' = \mathsf{f}_a(v', r_4')$ and $b' = \mathsf{b}_a(w', r_5')$ as in (1) for (v', w'). Now $(v, w) \, E \, (b, f)$ and $(v', w') \, E \, (b', f')$. Since $(v, w) \, E \, (v', w')$, transitivity now gives $(b, f) \, E \, (b', f')$ whence by Lemma 3.7 the composites (bx, fy) and $(b'x, f'y)$ are E-equivalent as required.

Finally, we consider independence of the choice of (x, y): Suppose that $(x, y) \, E \, (x', y')$. As above, let $f = \mathsf{f}_a(v, r_4)$ and $b = \mathsf{b}_a(w, r_5)$ as in (1). This time, let $f'' = \mathsf{f}_a(v, r_4'')$ and $b'' = \mathsf{b}_a(w, r_5'')$ as in (1) for (x', y'). Now $(v, w) \, E \, (b, f)$ and $(v, w) \, E \, (b'', f'')$, so by transitivity $(b, f) \, E \, (b'', f'')$, and applying Lemma 3.7, we have $(bx, fy) \, E \, (b''x', f''y')$ as required. □

Proposition 3.9. $\mathbf{C}$ *is a category.*

Proof. We have seen above that there is a well-defined composition for $\mathbf{C}$. Clearly the identities defined above act as units for the composition.

Associativity of composition is the only further requirement. For this, refer to Figure 1 and suppose that the arrows $[(x, y)]$, $[(v, w)]$, and $[(u, z)]$ are composable. From composability of $[(x, y)]$, $[(v, w)]$ there are corrs t, t', From composability of $[(v, w)]$, and $[(u, z)]$ there are corrs s, s'. Where possible, in what follows we will elide brackets, subscripts and commas, and write, for example $\mathsf{f}xr$ instead of $\mathsf{f}_a(x, r)$. Now let $v' = \mathsf{b}wt'$, $w' = \mathsf{f}vt$, $u' = \mathsf{b}zs'$, $z' = \mathsf{f}us$, $b = \mathsf{b}_a(z'w, t')$ and $f = \mathsf{f}_a(u'v, t)$. Since $v' \, C \, w$ and $v \, C \, w'$, there are corrs r, r'. Define $u'' = \mathsf{b}z'r'$ and $z'' = \mathsf{f}u'r$.

It is the case that the right bracketed composite, namely $[(x, y)]([(v, w)][(u, z)])$, is $[(\mathsf{b}_a(z'w, t')x, \mathsf{f}_a(u'v, t)y))]$ and by compositionality of f and b, we get $[(x, y)]([(v, w)][(u, z)]) = [(\mathsf{b}z'r' \cdot \mathsf{b}wt' \cdot x, \mathsf{f}u'r \cdot \mathsf{f}vt \cdot y)]$. Since $(v, w) \, E \, (v', w')$ there are corrs $c_0 \, N \, c_1$ linking the codomains as shown. Since $[(v, w)]$ and $[(u, z)]$ are composable, there is a corr c_2 at the domains of the pair (u, z) and we have $c_2 \, N \, c_1$. So by transitivity $c_0 \, N \, c_2$ and there are corrs q, q' as shown. It is also the case that $([(x, y)][(v, w)])[(u, z)] = [(\mathsf{b}zq' \cdot \mathsf{b}wt' \cdot x, \mathsf{f}uq \cdot \mathsf{f}vt \cdot y)]$.

Thus we need to show that $[(\mathsf{b}z'r' \cdot \mathsf{b}wt' \cdot x, \mathsf{f}u'r \cdot \mathsf{f}vt \cdot y)] = [(\mathsf{b}zq' \cdot \mathsf{b}wt' \cdot x, \mathsf{f}uq \cdot \mathsf{f}vt \cdot y)]$. By Lemma 3.7 it is enough to show $[(\mathsf{b}z'r', \mathsf{f}u'r)] = [(\mathsf{b}zq', \mathsf{f}uq)]$, which would follow from $\mathsf{b}z'r' \, C \, \mathsf{f}uq$. But $\mathsf{b}z'r' \, C \, z'$ $(z' = \mathsf{f}us)$, $u \, C \, \mathsf{f}us$ and $u \, C \, \mathsf{f}uq$, so by ZX $\mathsf{b}z'r' \, C \, \mathsf{f}uq$. □

The next step is to define a cospan from $\mathbf{X}$ to $\mathbf{Y}$ with base $\mathbf{C}$.

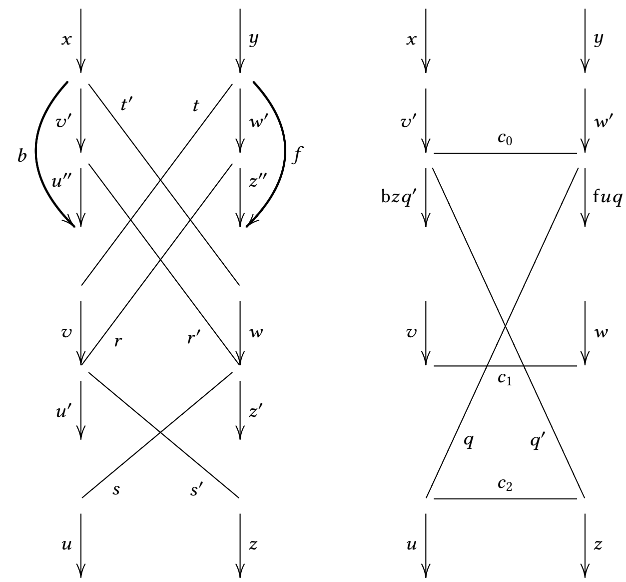


Figure 1: Associativity

Assumption. *For the rest of this section we assume that the fb-lens L with compatibility relation C is δ-surjective.*

We now define the left Get functor $G_L : \mathbf{X} \longrightarrow \mathbf{C}$ as follows:

For an object X of $\mathbf{X}$ there is a corr r with $X = \delta_\mathbf{X}(r)$ and let $G_L(X) = [r]_N$. To see that $G_L(X)$ is well-defined, notice that if also $X = \delta_\mathbf{X}(r')$ then r' itself proves that $[r']_N = [r]_N$.

For an arrow $x : X \longrightarrow X'$ of $\mathbf{X}$ there is a corr r with $X = \delta_\mathbf{X}(r)$ and we have $x\, C\, \mathsf{f}_a(x, r)$. Define $G_L(x) = [(x, \mathsf{f}_a(x, r))]$. To see that $G_L(x)$ is well-defined, we suppose that also $X = \delta_\mathbf{X}(r')$. We need to know that $(x, \mathsf{f}_a(x, r))\, E\, (x, \mathsf{f}_a(x, r'))$, but this is proved by $x\, C\, \mathsf{f}_a(x, r)$.

Proposition 3.10. *G_L is a functor.*

Proof. We note first that $G_L(\mathrm{id}_X) = [(\mathrm{id}_X, \mathsf{f}_a(\mathrm{id}_X, r)] = [(\mathrm{id}_X, \mathrm{id}_{\delta_\mathbf{Y}(r)})]$. Next suppose that $x : X \longrightarrow X'$ and $x' : X' \longrightarrow X''$ are composable in $\mathbf{X}$. There is a corr r with $X = \delta_\mathbf{X}(r)$ and $G_L(x) = [(x, \mathsf{f}_a(x, r))]$. Let $r' = \mathsf{f}_c(x, r)$, so $G_L(x') = [(x', \mathsf{f}_a(x', r'))]$ while

$$\begin{aligned} G_L(x'x) &= [(x'x, \mathsf{f}_a(xx', r))] \\ &= [(x', \mathsf{f}_a(x', r'))][(x, \mathsf{f}_a(x, r))] \\ &= G_L(x')G_L(x) \end{aligned}$$

The first and third equalities are by definition. The second may seem obvious by compositionality of f, but requires a short comment. By definition $[(x', \mathsf{f}_a(x', r'))][(x, \mathsf{f}_a(x, r))] = [(\mathsf{b}_a(\mathsf{f}_a(x', r'), r')x, \mathsf{f}_a(x', r')\mathsf{f}_a(x, r))]$. But $x'\, C\, \mathsf{f}_a(x', r')$, so we have that $(\mathsf{b}_a(\mathsf{f}_a(x', r'), r'), \mathsf{f}_a(x', r'))\, E\, (x', \mathsf{f}_a(x', r'))$. And finally

$$\begin{aligned} &[(\mathsf{b}_a(\mathsf{f}_a(x', r'), r')x, \mathsf{f}_a(x', r')\mathsf{f}_a(x, r))] \\ &\quad = \quad [(x'x, \mathsf{f}_a(x', r')\mathsf{f}_a(x, r))] \end{aligned}$$

so we are done. □

The right Get functor $G_R : \mathbf{Y} \longrightarrow \mathbf{C}$ is defined similarly: on an arrow y of $\mathbf{Y}$, define $G_R(y) = [(y, \mathsf{b}_a(y, r))]$ where r satisfies $d_0(y) = \delta_\mathbf{Y}(r)$

Even before defining Puts, note that the constructed cospan defines a relation on pairs (x, y) of arrows from $\mathbf{X}$ and $\mathbf{Y}$ that we denote $C' = \{(x, y) \mid G_L(x) = G_R(y)\}$ which (except for C2) has the properties of a compatibility relation. We show immediately:

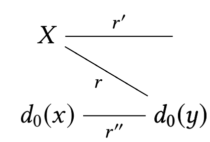
Proposition 3.11. *$C' = C$*

Proof. For $C' \subseteq C$, suppose that $x\, C'\, y$ so $[(x, \mathsf{f}_a(x, r_x))] = [(\mathsf{b}_a(y, r_y), y)]$ for suitable corrs r_x, r_y. Now $x\, C\, \mathsf{f}_a(x, r_x)$ and $\mathsf{b}_a(y, r_y)\, C\, y$, and by hypothesis $\mathsf{b}_a(y, r_y)\, C\, \mathsf{f}_a(x, r_x)$, so by ZX we have $x\, C\, y$ as required.

For the reverse inclusion, if $x\, C\, y$ we have $x\, C\, \mathsf{f}_a(x, r_x)$ and $\mathsf{b}_a(y, r_y)\, C\, y$ so by ZX, $\mathsf{b}_a(y, r_y)\, C\, \mathsf{f}_a(x, r_x)$. Hence we have $(x, \mathsf{f}_a(x, r_x))\, E\, (\mathsf{b}_a(y, r_y), y)$, which means that $[(x, \mathsf{f}_a(x, r_x))] = [(\mathsf{b}_a(y, r_y), y)]$ and $x\, C'\, y$ as required. □

We want to show that the original fb-lens with compatibility C is represented by the cospan $G_L : \mathbf{X} \longrightarrow \mathbf{C} \longleftarrow \mathbf{Y} : G_R$ of d-lenses with Puts P_L and P_R to be defined now.

First we define P_L. Suppose we are given X in $\mathbf{X}$ and an arrow $[(x, y)]$ in $\mathbf{C}$ with $d_0([(x, y)]) = G_L(X)$ where $G_L(X) = [r']_N$ and $\delta_\mathbf{X}(r') = X$. Since $x\, C\, y$, there is a corr r'' with $\delta_\mathbf{Y}(r'') = d_0(y)$ and $\delta_\mathbf{X}(r'') = d_0(x)$ and $r'\, N\, r''$. Thus there is $r : X \leftrightarrow d_0(y)$ as in



Now define $P_L(X, [(x, y)]) = \mathsf{b}_a(y, r)$. P_R is defined similarly.

We need to know that the definition of P_L is independent of the choice of (x, y) and r. We assume the following condition to ensure the former, and later that r is unique for the latter.

Definition 3.12. For arrows x, x' we say xKx' if there is a y such that $x\, C\, y$ and $x'\, C\, y$. Similarly for yKy'. An fb-lens L *satisfies condition κ* iff whenever yKy' and there are corrs $r : X \leftrightarrow d_0(y)$, $r' : X \leftrightarrow d_o(y')$, we have $\mathsf{b}_a(y, r) = \mathsf{b}_a(y', r')$, and the similar condition for f.

The following shows that condition κ is necessary for our construction.

Proposition 3.13. *Let L be the fb-lens constructed from the cospan $(G_L, P_L) : \mathbf{X} \longrightarrow \mathbf{S} \longleftarrow \mathbf{Y} : (G_R, P_R)$ of d-lenses.*

Then L satisfies condition κ. Moreover, there is at most one corr between any pair of objects X, Y.

Proof. Note that yKy' means there is x with $G_L(x) = G_R(y) = G_R(y')$, so the X has $G_L(X) = d_0(x) = d_0(y)$ and $\mathrm{b}_a(y,r) = P_L(X, G_R(y)) = P_L(X, G_R(y')) = \mathrm{b}_a(y', r')$. Uniqueness of corrs is by their definition. □

To ensure that the definition of P_L does not depend on the choice of r we have the following.

Definition 3.14. An fb-lens L *has unique corrs* (or say L is *u-corr*) iff for any pair of objects X, Y there is at most one corr r with $X = \delta_{\mathbf{X}}(r)$ and $Y = \delta_{\mathbf{Y}}(r)$.

We remark that the fb-lens constructed from a cospan of d-lenses clearly has unique corrs.

Proposition 3.15. *Let L be a δ-surjective fb-lens with compatibility relation C, satisfying condition κ and having unique corrs, then P_L and P_R are well-defined.*

Proof. As noted, we require that $P_L(X, [(x,y)])$ as defined above does not depend on the choice of (x,y) or r. Suppose that $[[x,y)] = [(x',y')]$ so that $(x,y)\,E\,(x',y')$ and hence $x\,C\,y$ and $x\,C\,y'$ so that yKy'. Now suppose further that $G_L(X) = d_0([(x,y)]) = d_0([(x',y')])$. As above there are corrs $r : X \leftrightarrow d_0(y)$, $r' : X \leftrightarrow d_0(y')$. By κ, we have $P_L(X, [(x,y)]) = \mathrm{b}_a(y,r) = \mathrm{b}_a(y',r') = P_L(X, [(x',y')])$ and of course r and r' are unique.

P_R is similar. □

Assumption. *For the rest of this section we assume that the fb-lens L with compatibility relation C which is under consideration and which defines G_L, P_L, G_R and P_R both satisfies condition κ and has unique corrs.*

Proposition 3.16. *Let L be a δ-surjective fb-lens with compatibility relation C, satisfying condition κ and with unique corrs, then (G_L, P_L) and (G_R, P_R) are d-lenses.*

Proof. By construction, the domain of $P_L(X, [(x,y)]) = \delta_{\mathbf{X}}(r)$ is X, so P_L satisfies d-PutInc.

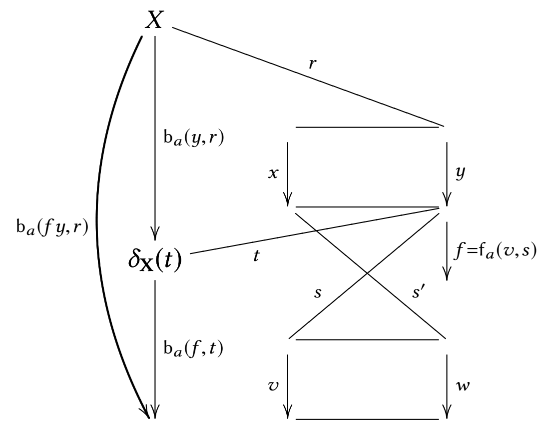
Next observe that the identity on $G_L(X)$ is $[(\mathrm{id}_X, \mathrm{id}_{\delta_{\mathbf{Y}}(r)})]$ for a corr r, so we have $P_L(X, [(\mathrm{id}_X, \mathrm{id}_{\delta_{\mathbf{Y}}(r)})]) = \mathrm{b}_a(\mathrm{id}_{\delta_{\mathbf{Y}(r)}}, r) = \mathrm{id}_X$. Thus P_L satisfies d-PutId.

Now consider d-PutGet. Suppose that $d_0([(x,y)]) = G_L(X)$. We need to show that $G_L P_L(X, [(x,y)]) = [(x,y)]$. Let r', r, r'' be as in the definition of P_L above, so $G_L P_L(X, [(x,y)]) = G_L(\mathrm{b}_a(y,r)) = [(\mathrm{b}_a(y,r)), \mathrm{f}_a(\mathrm{b}_a(y,r),r)]$ and $\delta_{\mathbf{X}}(r) = X$. But now $\mathrm{b}_a(y,r)\,C\,y$ so $(x,y)\,E\,(\mathrm{b}_a(y,r), \mathrm{f}_a(\mathrm{b}_a(y,r),r))$ and so $[(\mathrm{b}_a(y,r)), \mathrm{f}_a(\mathrm{b}_a(y,r),r)] = [(x,y)]$ as required.

Finally, we consider d-PutPut.

Suppose $[(x,y)]$ and $[(v,w)]$ are composable and $G_L(X) = d_0([(x,y)])$. We need to show that $P_L(X, [(v,w)][(x,y)]) = P_L(X', [(v,w)])P_L(X, [(x,y)])$ where $X' = d_1(P_L(X, [(x,y)]))$.

Consider the following:



Where we shorten $\mathrm{f}_a(v,s)$ to f. The right hand squares are the relevant part of the definition of $[(v,w)][(x,y)]$. Since $P_L(X, [(x,y)]) = \mathrm{b}_a(y,r)$ is C related to y, there is a corr we denote t with $\delta_{\mathbf{X}}(t) = d_1(P_L(X, [(x,y)])) = X'$. Since $[(v,w)] = [(v, \mathrm{f}_a(v,s))] = [(v,f)]$, we have $P_L(X', [(v,w)]) = P_L(\delta_{\mathbf{X}}(t), [(v,f)]) = \mathrm{b}_a(f,t)$. Thus the composite of Puts is

$$P_L(X', [(v,w)])P_L(X, [(x,y)]) = \mathrm{b}_a(y,r)\mathrm{b}_a(f,t)$$

On the other hand, the Put of the composite is

$$P_L(X, [(v,w)][(x,y)]) = \mathrm{b}_a(fy,r) = \mathrm{b}_a(f,t)\mathrm{b}_a(y,r)$$

by compositionality of b_a, so we are done. The proof for (G_R, P_R) is the same. □

We have completed the construction of a cospan of d-lenses from a suitable fb-lens L. There is an fb-lens L' constructed from this cospan of d-lenses. It is, of course, not the same as L, but it has closely related behaviour and indeed:

Proposition 3.17. *Let L be a δ-surjective fb-lens with compatibility relation C, satisfying condition κ and with unique corrs. The fb-lens L' constructed from the cospan*

$$(G_L, P_L) : \mathbf{X} \longrightarrow \mathbf{C} \longleftarrow \mathbf{Y} : (G_R, P_R)$$

of d-lenses defined above satisfies $L \equiv_{\mathrm{fb}} L'$.

Proof. In the construction of L', we define its set of corrs to be $R' = \{(X,Y) \mid G_L(X) = G_R(Y)\}$. To show that $L \equiv_{\mathrm{fb}} L'$ we need a relation σ from R to R'. For r in R and (X,Y) in R', define $r\,\sigma\,(X,Y)$ iff $\delta_{\mathbf{X}}(r) = X$ and $\delta_{\mathbf{Y}}(r) = Y$. We need to show that σ satisfies the properties of Definition 2.5.

Condition 1. is immediate by the definition of σ. For condition 2., if r in R, with $\delta_{\mathbf{X}}(r) = X$ and $\delta_{\mathbf{Y}}(r) = Y$, we have $G_L(X) = [r]_N = G_R(Y)$, so (X,Y) in R' with $r\,\sigma\,(X,Y)$. Conversely, if (X,Y) in R', then for some r', r'' in R, we have $[r'] = G_L(X) = G_R(Y) = [r'']$ with $\delta_{\mathbf{X}}(r') = X$ and $\delta_{\mathbf{Y}}(r'') = Y$, so $r'\,N\,r''$ and there is r with $\delta_{\mathbf{X}}(r) = X$ and $\delta_{\mathbf{Y}}(r) = Y$. Thus $r\,\sigma\,(X,Y)$.

For condition 3., suppose $r\,\sigma\,(X,Y)$ and $\delta_{\mathbf{X}}(r)$ (which must be X!) is the domain of x. We need to show that $\mathsf{f}_a(x,r) = \mathsf{f}'_a(x,(X,Y))$ and $\mathsf{f}_c(x,r)\,\sigma\,\mathsf{f}'_c(x,(X,Y))$. Now by definition, since $r : X \leftrightarrow Y$, we have $\mathsf{f}'_a(x,(X,Y)) = P_R(Y,[(x,\mathsf{f}_a(x,r))])$, but $d_0(\mathsf{f}_a(x,r)) = Y$ and corrs (in L) are unique, so we have $P_R(Y,[(x,\mathsf{f}_a(x,r))]) = \mathsf{f}_a(x,r)$ as required. Moreover, that means that the second component $\mathsf{f}_c(x,r)$ is the unique corr from $d_1(x)$ to $d_1(\mathsf{f}_a(x,r))$. The corr $\mathsf{f}_c(x,r)$ proves that $(d_1(x), d_1(\mathsf{f}_a(x,r)))$ is in R' and is, of course, $\mathsf{f}'_c(x,(X,Y))$. Thus $\mathsf{f}_c(x,r)\,\sigma\,\mathsf{f}'_c(x,(X,Y))$. The argument for condition 4. is similar. □

Indeed, from the construction more is true:

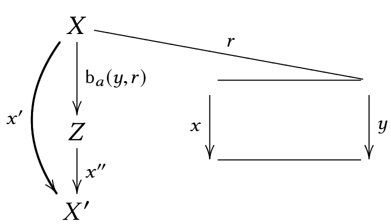
Corollary 3.18. *With the hypotheses of the Proposition, the cospan represents L.*

Proof. We just note that the mapping $r \mapsto (\delta_{\mathbf{X}}(r), \delta_{\mathbf{Y}}(r))$ from R to R' is actually a bijection since L is δ-surjective and has unique corrs. □

So far we do not have a definition for the equivalence of two cospans of asymmetric lenses and we will have not yet explored the effect of constructing a symmetric lens from a cospan and then using the construction above to form a (presumably equivalent) cospan from the resulting symmetric lens.

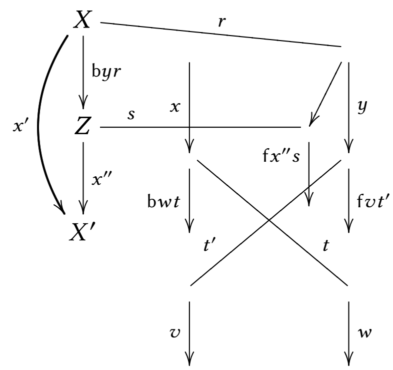
Proposition 3.19. *Let $L = (\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathsf{f}, \mathsf{b})$ be an fb-lens between $\mathbf{X}$ and $\mathbf{Y}$ with unique corrs and a compatibility relation C. If L is least change then the d-lenses in the cospan $G_L : \mathbf{X} \longrightarrow \mathbf{C} \longleftarrow \mathbf{Y} : G_R$ are pre-cartesian. If L is cartesian then the d-lenses are c-lenses.*

Proof. We show first that if L is least change then P_L is pre-cartesian. We need to show that P_L satisfies a universal property, so consider $\mathsf{b}_a(y,r) = P_L(X,[(x,y)])$ where $X = \delta_{\mathbf{X}}(r)$ and $d_0(y) = \delta_{\mathbf{Y}}(r)$. Suppose that $x' : X \longrightarrow X'$ and $G_L(x') = [(x,y)]$. We also have, by the definition, that $G_L(x') = [(x',\mathsf{f}_a(x',r))]$. Denote the codomain of $\mathsf{b}_a(y,r)$ by Z and consider:



Thus we have that $[(x,y] = [(x',\mathsf{f}_a(x',r))]$, so that $x'\,C\,y$ and by least change there is a unique $x'' : d_1(\mathsf{b}_a(y,r)) = Z \longrightarrow X'$ with $x''\,C\,\mathrm{id}_{d_1(y)}$. Now for a corr r' from Z to $d_1(y)$ we have $G_L(x'') = [(x'',\mathsf{f}_a(x'',r'))]$ and we also have $\mathrm{id}_{d_1(x)}\,C\,\mathrm{id}_{d_1(y)}$ so via $x''\,C\,\mathrm{id}_{d_1(y)}$ we have $[(x'',\mathsf{f}_a(x'',r'))] = [(\mathrm{id}_{d_1(x)}, \mathrm{id}_{d_1(y)}] = \mathrm{id}_{d_1([(x,y)])}$ as required. Similarly for P_R.

Now suppose that L is cartesian. W show that P_L delivers cartesian arrows. Consider $P_L(X,[x,y])$ which by definition is $\mathsf{b}yr$. Suppose that $G_L(x')$ factors as $[vw][xy]$. Using the definition of composition, referring the diagram below and again eliding brackets, write $[vw][xy] = [(\mathsf{b}wt\cdot x, \mathsf{f}vt'\cdot y)]$. By definition $G_L(x') = [(x', \mathsf{f}x'r)]$, so $x'\,C\,\mathsf{f}vt'\cdot y$ and we can apply the cartesian property for b to obtain a unique x'' with $x''\,C\,\mathsf{f}vt'$. Furthermore, $G_L(x'') = (x'',\mathsf{f}x''s)$ (where for example $s = \mathsf{f}_c((\mathsf{b}y,r),r))$ and since $x''\,C\,\mathsf{f}vt'$, $G_L(x'')$ lies in the same equivalence class as $(\mathsf{b}wt,\mathsf{f}vt')]$ which in turn is equivalent to (v,w). This completes the proof.



□

The following is a useful weakening of condition κ

Definition 3.20. Let L be an fb-lens with compatibility C. An fb-lens L *satisfies condition κ up to iso* iff whenever yKy' and there are corrs $r : X \leftrightarrow d_0(y)$, $r' : X \leftrightarrow d_o(xy')$, we have $d_1(\mathsf{b}_a(y,r)) \cong d_1(\mathsf{b}_a(y',r'))$ and the iso commutes with the bs, and the similar condition for f.

Proposition 3.21. *Let L be a least change fb-lens then L satisfies condition κ up to iso.*

Proof. Suppose that $x\,C\,y$, $x\,C\,y'$ and there are corrs $r : X \leftrightarrow d_0(y)$, $r' : X \leftrightarrow d_0(y')$. We want to show that $d_1(\mathsf{b}_a(y,r)) \cong d_1(\mathsf{b}_a(y',r'))$.

Now $\mathsf{b}_a(y',r')\,C\,y'$, so by ZX $\mathsf{b}_a(y',r')\,C\,y$. By least change there is a unique z with $z : d_1(\mathsf{b}_a(y,r)) \longrightarrow d_1(\mathsf{b}_a(y',r'))$ and $z\,C\,d_1(y)$. Similarly, there is a unique z' with $z'\,C\,d_1(y')$ and $z' : d_1(\mathsf{b}_a(y',r')) \longrightarrow d_1(\mathsf{b}_a(y',r'))$. As usual, since identities also satisfy the property for $z'z$ and zz', we have z' is the inverse of z which completes the proof. □

The point of this proposition is that if L is least change (or better cartesian) then we (almost) get condition κ for free which simplifies the requirements for Proposition 3.16 and what follows.

For an fb-lens, the following weak invertibility property is clearly desirable and resembles the $rlr = r$ property from [4].

Definition 3.22. [3] Let L be an fb-lens. L is *weakly invertible* iff for all x, r we have $\mathsf{f}_a(\mathsf{b}_a(\mathsf{f}_a(x,r),r),r) = \mathsf{f}_a(x,r)$ and the similar equation involving b.

Weak invertibility does not follow from the definition of an fb-lens, but we do have the following.

Proposition 3.23. *Suppose that the fb-lens L is constructed from the cospan $G_L : \mathbf{X} \longrightarrow \mathbf{C} \longleftarrow \mathbf{Y} : G_R$ of d-lenses, then L is weakly invertible.*

Proof. This is a straightforward calculation using the PutGet law. Suppose r is the corr (X, Y) from $G_L(X) = G_R(Y)$ and $d_0(x) = X$. Then

$$\begin{aligned} \mathsf{f}_a(\mathsf{b}_a(\mathsf{f}_a(x,r),r),r) &= \mathsf{f}_a(\mathsf{b}_a(P_R(Y,G_L(x)),r),r) \\ &= \mathsf{f}_a(P_L(X,G_R((P_R(Y,G_L(x)))),r) \\ &= \mathsf{f}_a(P_L(X,G_L(x)),r) \\ &= P_R(Y,G_L(P_L(X,G_L(x))) \\ &= P_R(Y,G_L(x)) \\ &= \mathsf{f}_a(x,r) \end{aligned}$$

The third and the fifth equalities are from PutGet and the rest are by definition. The equation for back propagation is similar. □

4 CONCLUSION

Suppose we begin with $(G_L, P_L) : \mathbf{X} \longrightarrow \mathbf{C} \longleftarrow \mathbf{Y} : (G_R, P_R)$, a cospan of d-lenses, and construct the fb-lens L as above. We have shown that L is δ-surjective, has a compatibility relation C, satisfies condition κ and has unique corrs. So these conditions are *necessary* for a lens to arise from a cospan. The bulk of the work in this paper has been to show that those conditions are also *sufficient*: Given a symmetric lens L satisfying those conditions we can construct a cospan of d-lenses which represents L.

Thus, we now know how to identify those symmetric lenses which can be represented by cospans of d-lenses and these have desirable properties for software engineering and cyber security.

5 FUTURE WORK

The question of why we have so frequently been able to find such cospans of lenses in practice remains, and we have some interesting hypotheses to explore. Meanwhile there are also important new mathematical questions that are opened up by the analysis presented here.

Recall that in Proposition 3.21 we demonstrated that least change lenses automatically satisfy condition κ up to isomorphism. Similarly, it appears that the functors of G_L and G_R, which are defined for a merely δ-surjective fb-lens with compatibility relation, also come equipped with Puts except that they need not satisfy d-PutPut. Yet again, when the fb-lens is least change the putative Puts do appear to satisfy d-PutPut up to isomorphism. The move from equational axioms to weaker systems that replace equalities with coherent isomorphisms is rarely straightforward, but usually very productive, and it seems likely that in such a theory of d-lenses, least change lenses will play a special role.

It may be the case that because our consultancy work involved least change lenses the remarks of the preceding paragraph, once the mathematics is completed, might further explain why we so regularly were able to find cospans of lenses: With minor adjustments for δ-surjectivity the remaining required conditions may be automatically satisfied, up to isomorphism, by least change lenses.

6 ACKNOWLEDGEMENTS

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