A BASIC DISTRIBUTIVE LAW

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Abstract. We pursue distributive laws between monads, particularly in the context of KZ-doctrines, and show that a very basic distributive law has (constructively) completely distributive lattices for its algebras. Moreover, the resulting monad is shown to be also the double dualization monad (with respect to the subobject classifier) on ordered sets.

1. Introduction

1.1. In February of 1984, at a meeting in San Juan, two of us heard Fred Linton describe the category of frames as the category of algebras for a distributive law between monads on the category of *ordered* sets (while at the same time he pointed out that no such result holds over the category of sets). A moment's reflection on this suggests that an analogous result must hold for the category of completely distributive lattices. It does. The distributive law in question is particularly interesting though and warrants both description and study.

1.2. Distributive laws between monads in a bicategory can lead to rather large diagrams, especially by way of the 'pentagon' conditions. In [RW4] it was shown that for idempotent monads (and comonads) there is a major simplification — one triangle suffices. In this paper the distributive law on which we focus involves a 'KZ' monad and a 'co-KZ' monad. Such monads (or 'doctrines' as they are often called) are generalizations of idempotent monads, requiring one further categorical dimension to define them, so it is not too surprising that we are able to simplify the study of distributive laws between them. This we do in Section 4. We express our results for such monads on an object in an **ord**-**cat**-category, where **ord** denotes the 2-category of antisymmetric ordered sets.

1.3. A brief word on the level of generality may be helpful. In [STR] Street defined and studied monads on objects in an arbitrary 2-category. His results are easily extended to monads on objects in bicategories — either directly or by using the coherence theorem which states that each bicategory is biequivalent to a 2-category. It has become clear that KZ-doctrines should be studied in the context of pseudomonads on objects in a tricategory. Given the coherence result of [GPS], it suffices to study them in **Gray**-categories and this development has begun in [MO1], [MO2] and [MO3]. There is no doubt that substantial results of the kind we present can be proved in general **Gray**-categories, however, their pursuit here would take us too far afield from the main applications we have in mind.

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Our modifications (3-cells) are mere inequalities and we assume that any instance of $\leq$ is antisymmetric.

1.4. Section 3 is in the spirit of [STR] and the results are valid in any bicategory. We add a new formulation of distributive laws that is useful when the ambient bicategory does not necessarily admit the 'construction of algebras' in the terminology of [STR] (or, said otherwise, Eilenberg-Moore objects are not known to exist). This formulation is frequently useful and also clarifies the characterization of distributive laws in terms of Kleisli objects. (In this last connection we have in mind recent work of Pisani [PIS] and Johnstone [JOH].)

1.5. In Section 5 we introduce our 'basic distributive law' $r : UD \longrightarrow DU$, where D and U underlie the 'down-set' KZ-doctrine $\mathbf{D}$, respectively the 'up-set' co-KZ-doctrine $\mathbf{U}$, on the 2-category of ordered sets. We show that this law restricts to a number of important submonads of $\mathbf{D}$ and $\mathbf{U}$. From our results in Section 4 it follows that the basic law gives rise to the only possible distributive law in each case. In particular, leaving $\mathbf{D}$ unaltered and replacing $\mathbf{U}$ by the 'finitely-generated up-set' monad we obtain Linton's distributive law which captures frames in terms of sup-lattices and meet-semi-lattices, over the 2-category of (antisymmetric) ordered sets.

1.6. In Section 6 we show that the algebras for the monad arising from our basic law are precisely the constructively completely distributive — CCD — lattices introduced in [F&W] and further studied in [RW1], [RW2], [RW3] and [P&W]. The distributive law $r : UD \longrightarrow DU$ has a left adjoint which is also a distributive law and its algebras are those lattices whose dual is CCD. The distinction between such lattices and CCD lattices is not apparent with respect to boolean set theory but as with other papers that deal with CCD lattices our results are intuitionistically valid. Using our techniques of Section 5 we are also able to answer an interesting question of Paul Taylor: the algebras over ordered sets for double dualization with respect to the subobject classifier are also the CCD lattices.

2. KZ-Doctrines

2.1. Let $\mathbf{K}$ be an **ord-cat**-category. Thus, for each pair of objects $\mathcal{A}, \mathcal{B}$ in $\mathbf{K}$, we have an **ord**-category $\mathbf{K}(\mathcal{A}, \mathcal{B})$, meaning that for each pair of arrows (1-cells) A, B from $\mathcal{A}$ to $\mathcal{B}$ we have an ordered set $\mathbf{K}(\mathcal{A}, \mathcal{B})(A, B)$ of transformations (2-cells) from A to B. To fix notation, write

$$a \leq b : A \longrightarrow B : \mathcal{A} \longrightarrow \mathcal{B}$$

for an inequality providing a typical modification of $\mathbf{K}$. Composition of arrows in the underlying ordinary category of $\mathbf{K}$ is denoted by juxtaposition; we use $-\cdot-$ for composition in the $\mathbf{K}(\mathcal{A}, \mathcal{B})$. Since we assume the $\leq$ to be antisymmetric, $\mathbf{K}$ has an underlying 2-category. So given $x : X \longrightarrow Y : \mathcal{X} \longrightarrow \mathcal{A}$ and $a : A \longrightarrow B : \mathcal{A} \longrightarrow \mathcal{B}$, we have $Bx \cdot aX = aY \cdot Ax$ in the ordered set $\mathbf{K}(\mathcal{X}, \mathcal{B})(AX, BY)$ providing a well-defined $ax : AX \longrightarrow BY : \mathcal{X} \longrightarrow \mathcal{B}$. When we speak of a monad on an object $\mathcal{K}$ in $\mathbf{K}$ we are in the first instance considering the underlying 2-category of $\mathbf{K}$ and we have immediate access to the formal theory of monads as described in [STR]. (It should be stressed that replacing **K** by a general **Gray**-category considerably complicates this discussion.)

2.2. Definition. *A* KZ-doctrine *on an object* $\mathcal{K}$ *in* **K** *consists of an arrow* $D : \mathcal{K} \longrightarrow \mathcal{K}$ *and a transformation* $d : 1_{\mathcal{K}} \longrightarrow D$ *which admit a fully-faithful adjoint string, in the sense of [RW4],* $Dd \dashv m \dashv dD : D \longrightarrow DD$ *(in* $\mathbf{K}(\mathcal{K}, \mathcal{K})$*).*

Precisely because our **Gray**-category **K** is merely ordered at the level of modifications, we are able to dispense with the coherence equation in the general definition of KZ-doctrine given in [MO1]. We also avoid having to consider the adjunctions $Dd \dashv m$ and $m \dashv dD$ as data. Note that if **K** is **ord-cat** then our definition of KZ-doctrine coincides with that in [KOK].

2.3. It is convenient to say that a *co-KZ-doctrine* on an object $\mathcal{K}$ in **K** consists of an arrow $U : \mathcal{K} \longrightarrow \mathcal{K}$ and a transformation $u : 1_{\mathcal{K}} \longrightarrow U$ which admit a fully-faithful adjoint string $uU \dashv n \dashv Uu : U \longrightarrow UU$. A KZ-doctrine (D, d), [co-KZ-doctrine (U, u)] gives rise to a monad, (D, d, m) [(U, u, n)]. We note that fully-faithfullness in the definition gives a modification $Dd \leq dD$ [$uU \leq Uu$]. A **T**-algebra for a monad $\mathbf{T} = (T, t, c)$ on $\mathcal{K}$ in **K** with domain $\mathcal{X}$ is an arrow $X : \mathcal{X} \longrightarrow \mathcal{K}$, together with a transformation $x : TX \longrightarrow X$ satisfying the usual axioms, which for convenience we have stated in 3.3. It is now classical, in fact the starting point of KZ study by Kock, that if (T, t, c) is KZ then (X, x) is a **T**-algebra if and only if $x \dashv tX$ (in $\mathbf{K}(\mathcal{X}, \mathcal{K})$), with tX fully-faithful. Similarly, if (T, t, c) is co-KZ then (X, x) is a **T**-algebra if and only if $tX \dashv x$, with tX fully-faithful. Of course, if t is fully-faithful then the fully-faithful requirement is satisfied automatically and a similar remark applies to Definition 2.2 if d itself is known to be fully-faithful.

3. Distributive Laws

3.1. For monads $\mathbf{D} = (D, d, m)$ and $\mathbf{U} = (U, u, n)$ on $\mathcal{K}$ a transformation $r : UD \longrightarrow DU$ is a *distributive law of* **U** *over* **D** if it satisfies the following four axioms of Beck as found in [BEK]:

$$
\begin{tikzcd}
 & U \arrow[dl, "Ud"'] \arrow[dr, "dU"] & \\
UD \arrow[rr, "r"'] & & DU \\
 & D \arrow[ul, "uD"] \arrow[ur, "Du"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
UDD \arrow[r, "rD"] \arrow[d, "Um"'] & DUD \arrow[r, "Dr"] & DDU \arrow[d, "mU"] \\
UD \arrow[rr, "r"'] & & DU \\
UUD \arrow[u, "nD"] \arrow[r, "Ur"'] & UDU \arrow[r, "rU"'] & DUU \arrow[u, "Dn"']
\end{tikzcd}
$$

Observe that each axiom involves precisely one of the structural transformations d, m, u, n of the monads in question. We will write $r[s]$, for s in $\{d, m, u, n\}$ to indicate that a mere transformation $r : UD \longrightarrow DU$ does at least satisfy the axiom involving s. (So, for example, $r[d]$ is the equation expressed by the top triangle above.)

3.2. In [BEK] it is shown that distributive laws $r : UD \longrightarrow DU$ involving monads on a *category* $\mathcal{K}$ are in bijective correspondence with multiplications $p : DUDU \longrightarrow DU$ for which: (DU, du, p) is a monad; $dU : U \longrightarrow DU \longleftarrow D : Du$ provide monad transformations; and the middle unitary law

$$\begin{tikzcd} DU \arrow[r, "DudU"] \arrow[dr, "1_{DU}"'] & DUDU \arrow[d, "p"] \\ & DU \end{tikzcd}$$

holds. It is further shown that distributive laws $r : UD \longrightarrow DU$ are in bijective correspondence with 'liftings' $\tilde{\mathbf{D}}$ of the monad $\mathbf{D}$ to $\mathcal{K}^{\mathbf{U}}$, the category of $\mathbf{U}$-algebras. 'Laws' and 'multiplications' are in a sense quite different from 'liftings'. The first two involve only a (natural) transformation satisfying equations while the last requires construction of an endo-functor on a category which is best viewed as a (lax) limit. It is obvious that the formal theory of monads, [STR], applies to the first two without reservation but we can only hope to speak of the last in a bicategory in which the requisite lax limit exists. It is fair to say that laws and multiplications are syntactic entities while liftings live at the level of semantics. There are results about distributive laws that are immediately apparent when translated as liftings but which require some opaque diagram chasing when considered directly. A good example is provided by: 'If $\mathbf{U}$ is an idempotent monad on $\mathcal{K}$ then, for any monad $\mathbf{D}$ on $\mathcal{K}$ there is at most one distributive law $UD \longrightarrow DU$'. Proof: The forgetful functor from the algebras for an idempotent monad is fully-faithful, so a lifting is a restriction and a functor either restricts or it doesn't. (In [RW4] a less memorable, but syntactic, proof is given.)

3.3. It transpires however that there is a syntactic formulation of 'lifting' that enables reasonably memorable proofs of results such as that at the end of 3.2 above. For a $\mathbf{U}$-algebra with domain $\mathcal{X}$, say $x : UX \longrightarrow X : \mathcal{X} \longrightarrow \mathcal{K}$, the requisite equations are:

$$\begin{tikzcd} X \arrow[r, "uX"] \arrow[dr, "1_X"'] & UX \arrow[d, "x"] & UUX \arrow[l, "nX"'] \arrow[d, "Ux"] \\ & X & UX \arrow[l, "x"] \end{tikzcd}$$

In particular, in the context of monads (D, d, m) and (U, u, n), we can examine $\mathbf{U}$-algebras on $DU : \mathcal{K} \longrightarrow \mathcal{K}$. Let us write $a : UDU \longrightarrow DU$ for such an algebra. We will consider such algebras satisfying, in addition to the two basic equations above, the following three equations:

$$
\begin{tikzcd}
UDUU \arrow[r, "UDn"] \arrow[d, "aU"'] & UDU \arrow[d, "a"] \\
DUU \arrow[r, "Dn"'] & DU
\end{tikzcd}
\qquad
\begin{tikzcd}
UU \arrow[r, "UdU"] \arrow[d, "n"'] & UDU \arrow[d, "a"] \\
U \arrow[r, "dU"'] & DU
\end{tikzcd}
\qquad
\begin{tikzcd}
UDDU \arrow[r, "UmU"] \arrow[d, "UDuDU"'] & UDU \arrow[ddd, "a"] \\
UDUDU \arrow[d, "aDU"'] & \\
DUDU \arrow[d, "Da"'] & \\
DDU \arrow[r, "mU"'] & DU
\end{tikzcd}
$$

Replacing the generic (X, x) in the basic **U**-algebra equations by (DU, a), we will label the resulting five equations for (DU, a) that have appeared above as $a[u]$, $a[n]$, $a[Dn]$, $a[dU]$ and $a[mU]$ respectively. Each of $a[Dn]$, $a[dU]$ and $a[mU]$ asserts that the arrow on the bottom side of the square in question is a homomorphism of **U**-algebras. That the domain of the first is in fact a **U**-algebra by way of structure aU follows from the general fact that if $x : UX \longrightarrow X : \mathcal{X} \longrightarrow \mathcal{K}$ is an algebra, then for any $Y : \mathcal{Y} \longrightarrow \mathcal{X}$, $xY : UXY \longrightarrow XY : \mathcal{Y} \longrightarrow \mathcal{K}$ is a **U**-algebra. (We may see such as the Y'th *instance* of (X, x).) Of course (UU, n) is a **U**-algebra and for the third we note:

3.4. LEMMA. *If (U, u, n) is a monad on an object $\mathcal{K}$ in a 2-category, $D : \mathcal{K} \longrightarrow \mathcal{K}$ is any arrow, and $a : UDU \longrightarrow DU$ is a* **U***-algebra structure satisfying $a[Dn]$ then*

$$UDDU \xrightarrow{UDuDU} UDUDU \xrightarrow{aDU} DUDU \xrightarrow{Da} DDU$$

is a **U***-algebra.*

Proof. The proof is a large diagram chase that is nevertheless easily found using the DU'th instance of $a[Dn]$. ■

3.5. PROPOSITION. *Given monads (D, d, m) and (U, u, n) on an object in a 2-category, there is a bijective correspondence between distributive laws $r : UD \longrightarrow DU$ and* **U***-algebras $a : UDU \longrightarrow DU$ satisfying $a[Dn]$, $a[dU]$, and $a[mU]$; given by:*

$$r \longmapsto \quad \alpha(r) = (UDU \xrightarrow{rU} DUU \xrightarrow{Dn} DU)$$

with inverse given by:

$$a \longmapsto \quad \rho(a) = (UD \xrightarrow{UDu} UDU \xrightarrow{a} DU)$$

Proof. We will just give the equations (other than monad equations and 'interchange') that are relevant at each step.

i) To show that $\alpha(\rho(a)) = a$, use $a[Dn]$;

ii) For $\rho(\alpha(r)) = r$, no 'r' equations are needed;

iii) To show that $\alpha(r)$ satisfies $\alpha(r)[Dn]$, no 'r' equations are needed;

iv) For $\alpha(r)[u]$, use $r[u]$;

v) For $\alpha(r)[n]$, use $r[n]$;

vi) For $\alpha(r)[dU]$, use $r[d]$;

vii) For $\alpha(r)[mU]$, use $r[m]$;

viii) For $\rho(a)[u]$, use $a[u]$;

ix) For $\rho(a)[n]$, use $a[n]$ and $a[Dn]$;

x) For $\rho(a)[d]$, use $a[dU]$;

xi) For $\rho(a)[m]$, use $a[mU]$.

∎

The following result is sometimes helpful.

3.6. Lemma. *For $a = \alpha(r)$, the* **U**-*algebra structure on DDU given in Lemma 3.4 is*

$$UDDU \xrightarrow{rDU} DUDU \xrightarrow{DrU} DDUU \xrightarrow{DDn} DDU.$$

∎

3.7. The considerations of 3.3, in which we have isolated special *left* **U**-algebra structures on DU, suggest that it is reasonable to examine *right* **D**-algebra structures on DU. Quite generally, given a monad (D, d, m) on an object $\mathcal{K}$ in a 2-category and an arrow $X : \mathcal{K} \longrightarrow \mathcal{X}$, a right **D**-algebra structure on X — which we would prefer to call a *D-opalgebra* structure following the terminology of [STR] — is a transformation $x : XD \longrightarrow X$ which is unitary and associative. In the case of $X = DU$ and a transformation $b : DUD \longrightarrow DU$ we will write

$b[d]$ for $b \cdot DUd = 1_{DU}$

$b[m]$ for $b \cdot DUm = b \cdot bD$

(the unitary and assciative conditions respectively) and to these add

$b[mU]$ for $mU.Db = b \cdot mUD$

$b[Du]$ for $Du \cdot m = b \cdot DuD$

$b[Dn]$ for $Dn \cdot bU \cdot DUb \cdot DUdUD = b \cdot DnD$

quite analogously to the extra equations in 3.3. With these at hand a dualization of Proposition 3.5 gives:

3.8. Proposition. *Given monads (D,d,m) and (U,u,n) on an object in a 2-category, there is a bijective correspondence between distributive laws $r : UD \longrightarrow DU$ and* **D**-*opalgebras $b : DUD \longrightarrow DU$ satisfying $b[mU]$, $b[Du]$, and $b[Dn]$; given by:*

$$r \longmapsto \quad \beta(r) = (DUD \xrightarrow{Dr} DDU \xrightarrow{mU} DU)$$

with inverse given by:

$$b \longmapsto \quad \sigma(b) = (UD \xrightarrow{dUD} DUD \xrightarrow{b} DU)$$

■

We will sketch below how the **U**-algebra structures of 3.3 are in bijective correspondence with liftings of **D** to the Eilenbeg-Moore object $\mathcal{K}^{\mathbf{U}}$. From this point of view it is clear that the **D**-opalgebra structures of 3.7 are in bijective correspondence with extensions of **U** to the Kleisli object $\mathcal{K}_{\mathbf{D}}$. This last observation seems to have been largely overlooked but bears on recent work of others, for example [PIS] and [JOH]. Accordingly, we summarize formally:

3.9. Theorem. *For monads $\mathbf{D} = (D,d,m)$ and $\mathbf{U} = (U,u,n)$ on an object $\mathcal{K}$ in a 2-category, the following structures are in bijective correspondence:*

- *i) Distributive laws $UD \longrightarrow DU$;*
- *ii) Monad structures as in* 3.2 *on DU;*
- *iii)* **U**-*algebra structures as in* 3.3 *on DU;*
- *iv)* **D**-*opalgebra structures as in* 3.7 *on DU;*

and, if the 2-category admits Eilenberg-Moore algebras then i) through iv) are in bijective correspondence with

- *v) Liftings of* **D** *through $\mathcal{K}^{\mathbf{U}} \longrightarrow \mathcal{K}$;*

and, if the 2-category admits Klesili opalgebras then i) through iv) are in bijective correspondence with

- *vi) Extensions of* **U** *along $\mathcal{K} \longrightarrow \mathcal{K}_{\mathbf{D}}$.*

Proof. In view of the discussion above and the account in [BEK], it suffices to sketch the correspondence between iii) and v); that of iv) and vi) being a dual. To give a lifting of **D** through $\mathcal{K}^{\mathbf{U}} \longrightarrow \mathcal{K}$ is to prescribe a monad on $\mathcal{K}^{\mathbf{U}}$ whose structure commutes with that of **D** via $\mathcal{K}^{\mathbf{U}} \longrightarrow \mathcal{K}$, where $\mathcal{K}^{\mathbf{U}} \longrightarrow \mathcal{K}$ (the forgetful functor when the 2-category is **CAT**) is the arrow part of the universal **U**-algebra

$$\begin{tikzcd} \mathcal{K}^{\mathbf{U}} \arrow[r] \arrow[dr] & \mathcal{K} \arrow[d, "U"] \arrow[dl, phantom, "\downarrow"] \\ & \mathcal{K} \end{tikzcd}$$

Thus to give even an arrow $\tilde{D} : \mathcal{K}^{\mathbf{U}} \longrightarrow \mathcal{K}^{\mathbf{U}}$ with

$$\begin{tikzcd}
\mathcal{K}^{\mathbf{U}} \arrow[r, "\tilde{D}"] \arrow[d] & \mathcal{K}^{\mathbf{U}} \arrow[d] \\
\mathcal{K} \arrow[r, "D"'] & \mathcal{K}
\end{tikzcd}$$

is to give a $\mathbf{U}$-algebra structure on $\mathcal{K}^{\mathbf{U}} \longrightarrow \mathcal{K} \xrightarrow{D} \mathcal{K}$. Such an algebra structure when preceded by the left adjoint to $\mathcal{K}^{\mathbf{U}} \longrightarrow \mathcal{K}$ (whose existence follows from the universal property of $\mathcal{K}^{\mathbf{U}}$, see [STR]) gives a $\mathbf{U}$-algebra structure on DU, since the composite $\mathcal{K} \longrightarrow \mathcal{K}^{\mathbf{U}} \longrightarrow \mathcal{K}$ is U. It should be clear now how the required correspondence is constructed. ■

By way of illustration of the use of iii) of 3.9 let us return to the fact stated at the end of 3.2: 'If $\mathbf{U}$ is an idempotent monad on an object $\mathcal{K}$ in any 2-category then, for any monad $\mathbf{D}$ on $\mathcal{K}$, there is at most one distributive law $UD \longrightarrow DU$'. Proof: For *any* $X : \mathcal{X} \longrightarrow \mathcal{K}$ there is a $\mathbf{U}$-algebra structure on X if and only if $uX : X \longrightarrow UX$ is invertible, in which case it is given by $(uX)^{-1}$. In particular this holds for $X = DU$. To illustrate the use of vi) we note that it would sometimes seem to be desirable to extend a monad $\mathbf{T}$ on **set**, the category of sets and functions, along **set** $\longrightarrow$ **rel**, where **rel** is the category of sets and relations and the functor interprets a function as the relation given by its graph. However **set** $\longrightarrow$ **rel** is the Kleisli opalgebra **set** $\longrightarrow$ $\mathbf{set}_{\mathbf{P}}$, where $\mathbf{P}$ is the power-set monad, so it follows that the desired extensions correspond to distributive laws $TP \longrightarrow PT$.

4. KZ-Doctrines and Distributive Laws

4.1. Proposition. *For an object $\mathcal{K}$ in any* **ord-cat**-*category* $\mathbf{K}$*,*

i) If $\mathbf{U}$ is either a KZ-doctrine or a co-KZ-doctrine on $\mathcal{K}$ and $\mathbf{D}$ is any monad on $\mathcal{K}$ then there is at most one distributive law $UD \longrightarrow DU$;

ii) If $\mathbf{D}$ is either a KZ-doctrine or a co-KZ-doctrine on $\mathcal{K}$ and $\mathbf{U}$ is any monad on $\mathcal{K}$ then there is at most one distributive law $UD \longrightarrow DU$.

Proof. For i) and the case in which $\mathbf{U}$ is co-KZ apply iii) of Theorem 3.9 and recall 2.3. The arrow DU supports a $\mathbf{U}$-algebra structure (not a priori satisfying all the requirements of iii) of Theorem 3.9) if and only if uDU has a right adjoint, which in this case is the structure arrow. The other case of i) appeals to existence of left adjoints and ii) is similar except that it uses iv) of Theorem 3.9. ■

4.2. It is natural to conjecture that if a distributive law $r : UD \longrightarrow DU$ involves KZ-doctrines or co-KZ-doctrines then the conditions $r[m]$ and $r[n]$ can be derived from $r[d]$ and $r[u]$. In the diagrams which follow it is convenient to display instances of modifications with unlabelled arrows $\longrightarrow$ rather than inequality symbols $\leq$.

4.3. Lemma. *For monads* $\mathbf{D}$ *and* $\mathbf{U}$ *and a transformation* $r : UD \longrightarrow DU$*,*

i) If (D, d, m) *is either KZ or co-KZ then* $r[d]$ *implies* $r[m]$*;*

ii) If (U, u, n) *is either KZ or co-KZ then* $r[u]$ *imples* $r[n]$*.*

Proof. For i) assume that (D, d, m) is KZ and consider the *two* modifications below whose conjunction is $r[m]$.

$$\begin{tikzcd}
UDD \arrow[r, "rD"] \arrow[d, "Um"'] & DUD \arrow[r, "Dr"] \arrow[dl, phantom, "\longrightarrow"] \arrow[dr, phantom, "\longleftarrow"] & DDU \arrow[d, "mU"] \\
UD \arrow[rr, "r"'] & & DU
\end{tikzcd}$$

Using simple instances of mates as in [K&S], we see that existence of the first of the modifications above follows from the first below and similarly for the second.

$$\begin{tikzcd}
UDD \arrow[r, "rD"] & DUD \arrow[r, "Dr"] & DDU \\
UD \arrow[u, "UDd"] \arrow[rr, "r"'] & & DU \arrow[ul, "DUd"'] \arrow[u, "DdU"'] \arrow[u, phantom, shift left=4ex, "\longrightarrow"]
\end{tikzcd}
\qquad
\begin{tikzcd}
UDD \arrow[r, "rD"] & DUD \arrow[r, "Dr"] & DDU \\
UD \arrow[u, "UdD"] \arrow[u, phantom, shift right=4ex, "\longleftarrow"] \arrow[ur, "dUD"'] \arrow[rr, "r"'] & & DU \arrow[u, "dDU"']
\end{tikzcd}$$

Now to assume $r[d]$ is to assume $r \cdot Ud \leq dU$ and $dU \leq r \cdot Ud$. The first of these inequalities upon application of D gives $Dr \cdot DUd \leq DdU$ — the first triangle above. The second inequality applied to D gives $dUD \leq rD \cdot UdD$ — the second triangle. This completes the proof of *i)* in the case (D, d, m) is KZ. The proof when (D, d, m) is co-KZ is entirely similar (in fact dual), as is that of *ii)*. ■

In fact, we can do slightly better in reducing the requirements for a distributive law $r : UD \longrightarrow DU$ in the present context. Of the original two triangles and two pentagons in 3.1, 'one and half triangles suffice'.

4.4. Proposition. *For monads* $\mathbf{D}$ *and* $\mathbf{U}$ *and a transformation* $r : UD \longrightarrow DU$*:*

i) If (D, d, m) *is KZ and* (U, u, n) *is either KZ or co-KZ then* $r : UD \longrightarrow DU$ *is a distributive law if it satisfies* $r[d]$ *and* $r \cdot uD \leq Du$*;*

ii) If (U, u, n) *is co-KZ and* (D, d, m) *is either KZ or co-KZ then* $r : UD \longrightarrow DU$ *is a distributive law if it satisfies* $r[u]$ *and* $r \cdot Ud \leq dU$

Proof. For i) consider the following diagram, in which the triangle surmounting the square is D applied to $r[d]$.

$$\begin{tikzcd}
& & DU \arrow[d, "DUd"'] \arrow[dr, "DdU"] \arrow[drr, "1_{DU}"] & & \\
D \arrow[urr, "Du"] \arrow[r, shift left=2, "Dd"] \arrow[r, shift right=2, "dD"'] \arrow[r, phantom, "\downarrow"] \arrow[drr, "uD"'] & DD \arrow[r, "DuD"] & DUD \arrow[r, "Dr"] & DDU \arrow[r, "mU"] & DU \\
& & UD \arrow[u, "dUD"] \arrow[r, "r"'] & DU \arrow[u, "dDU"'] \arrow[ur, "1_{DU}"'] &
\end{tikzcd}$$

All regions commute except for that given by $Dd \leq dD$, which we have since (D, d, m) is KZ. So the diagram gives $Du \leq r \cdot uD$. This inequality and the given inequality then provide $r[u]$, so that invoking Lemma 4.3 we have a distributive law. The second statement is dual. ∎

In case we have an adjunction $l \dashv r : UD \longrightarrow DU$, the task of checking that either l or r is a distributive law is facilitated somewhat by the following:

4.5. Lemma. *For monads (D, d, m) and (U, u, n) on $\mathcal{K}$ and $l \dashv r : UD \longrightarrow DU$,*

i) If (D, d, m) is either KZ or co-KZ then $l[d]$ implies $r[m]$;

i) If (U, u, n) is either KZ or co-KZ then $l[u]$ implies $r[n]$;

i) If (D, d, m) is either KZ or co-KZ then $r[d]$ implies $l[m]$;

i) If (U, u, n) is either KZ or co-KZ then $r[u]$ implies $l[n]$.

Proof. We prove just the first half of the first implication. Again, the other calculations are similar. The equality $r[m]$, see the first diagram in the proof of Lemma 4.3, is equivalently given as an equality between left adjoints and in the case that (D, d, m) is KZ and $l[d]$ holds we have

$$\begin{tikzcd}
UDD & DUD \arrow[l, "lD"'] & DDU \arrow[l, "Dl"'] \\
UD \arrow[u, "UDd"] & & DU \arrow[ll, "l"] \arrow[ul, "DUd"] \arrow[u, "DdU"']
\end{tikzcd}$$

∎

5. Ordered Sets

5.1. For the rest of the paper $D : \mathbf{ord} \longrightarrow \mathbf{ord}$ will denote the 2-functor which sends an ordered set X to the set of down-sets of X ordered by inclusion and which is defined on arrows by down-closure of direct image. Of course DX is naturally isomorphic to $[X^{op}, \Omega]$, the ordered set of functors from X^{op} to to the subobject classifier. To help clarify notation, let $f : X \longrightarrow A$ be an arrow in $\mathbf{ord}$. Here, for $S \in DX$ we have

$$Df(S) = \{fx|x \in S\}^{\overline{\downarrow}} = \{a \in A|(\exists x \in S)(a \leq fx)\}.$$

We will write $\mathcal{D} = [(-)^{op}, \Omega] : \mathbf{ord}^{coop} \longrightarrow \mathbf{ord}$. Then, modulo identification of $[X^{op}, \Omega]$ with DX, $\mathcal{D}f$ is given by inverse image. For all $f : X \longrightarrow A$ in $\mathbf{ord}$, we have $Df \dashv \mathcal{D}f$. In the context of D as above we will understand $dX : X \longrightarrow DX$ to be the yoneda functor that sends x to $\{y|y \leq x\}$. We may write $\downarrow x$ for $dX(x)$. It is well known that $\mathbf{D} = (D, d)$ is a KZ-doctrine and that the 2-category of $\mathbf{D}$-algebras is $\mathbf{sup}$, the 2-category of complete lattices, sup-preserving functors and inequalities. See, for example, Chapter III.3 of [J&T].

5.2. It is convenient to write

$$U = (D(-)^{op})^{op} : \mathbf{ord} \longrightarrow \mathbf{ord} \quad \text{and} \quad \mathcal{U} = (\mathcal{D}(-)^{op})^{op} : \mathbf{ord} \longrightarrow \mathbf{ord}^{coop}.$$

So UX is the set of up-sets of X ordered by reverse inclusion and is naturally isomorphic to $[X, \Omega]^{op}$. From this last observation it follows easily that $\mathcal{U}$ is the left 2-adjoint of $\mathcal{D}$. For reference later, note that $\mathcal{U}\mathcal{D} : \mathbf{ord} \longrightarrow \mathbf{ord}$ is simply double dualization with respect to Ω, that is $[[-, \Omega], \Omega]$, which we regard as a 2-monad on $\mathbf{ord}$ via the structure of the 2-adjunction. For all $f : X \longrightarrow A$ in $\mathbf{ord}$, we have $\mathcal{U}f \dashv Uf$ but note that it is $\mathcal{U}f$ which is given by inverse image while Uf is up-closure of direct image. In the context of U we will understand $uX : X \longrightarrow UX$ to be the coyoneda functor that sends x to $\{y|x \leq y\}$. We may write $\uparrow x$ for $uX(x)$. Now $\mathbf{U} = (U, u)$ is a co-KZ-doctrine, the 2-category of algebras for which is $\mathbf{inf}$, the 2-category of complete lattices, inf-preserving functors and inequalities.

5.3. The elegant notion of *yoneda structure* on a 2-category as defined in [S&W] has $\mathbf{ord}$ together with the yoneda functors $dX : X \longrightarrow DX$ of 5.1 as an important example. Following [S&W] we recall that for any $f : X \longrightarrow A$ in $\mathbf{ord}$ we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "dX"'] \arrow[r, phantom, shift right=4, "\longrightarrow"] & A \arrow[d, "{A(f,1)}"] \\ & DX \end{tikzcd}$$

which is both a left (kan) extension and an absolute left lifting. (Extensions and liftings are carefully explained in [S&W].) Here it is straightforward to show that $A(f, 1)$ sends

a to the down-set $\{x \in X | fx \le a\}$. Now taking $UdX : UX \longrightarrow UDX$ for $f : X \longrightarrow A$ above and writing rX for $UDX(UdX, 1)$ we have, for each X in **ord**

$$\begin{tikzcd} UX \arrow[r, "UdX"] \arrow[dr, "dUX"'] & UDX \arrow[d, "rX"] \\ & DUX \end{tikzcd}$$

The left extension triangle commutes because UdX is fully-faithful, which we have since dX is so. It is clear that the construction defines $r : UD \longrightarrow DU$, 2-naturally.

5.4. Using the description of $A(f, 1)$ in 5.3 we can calculate rX explicitly. Observe first that, for $T \in UX$,

$$\begin{array}{rcl} UdX(T) & = & \{S \in DX | (\exists x \in T)(\downarrow x \subseteq S\} \\ & = & \{S \in DX | (\exists x \in T)(x \in S)\} \\ & `=' & \{S \in DX | T \cap S \text{ is non-empty}\}, \end{array}$$

where the last 'equation' is intuitively helpful but intuitionistically unhelpful. For $\mathcal{T} \in UDX$, we have

$$\begin{array}{rcl} rX(\mathcal{T}) & = & \{T \in UX | UdX(T) \supseteq \mathcal{T}\} \\ & = & \{T \in UX | (\forall S \in \mathcal{T})(S \in UdX(T))\} \\ & = & \{T \in UX | (\forall S \in \mathcal{T})(\exists x \in T)(x \in S)\} \\ & `=' & \{T \in UX | (\forall S \in \mathcal{T})(T \cap S \text{ is non-empty})\}. \end{array}$$

From the last 'equation' above it is clear that $r : UD \longrightarrow DU$ has a left adjoint $l : DU \longrightarrow UD$ which, for $\mathcal{S} \in DUX$, is given by

$$\begin{array}{rcl} lX(\mathcal{S}) & = & \{S \in DX | (\forall T \in \mathcal{S})(\exists x \in S)(x \in T)\} \\ & `=' & \{S \in DX | (\forall T \in \mathcal{S})(T \cap S \text{ is non-empty})\}. \end{array}$$

It is the case that the lX also arise by consideration of the coyoneda structure on **ord** given by the $uX : X \longrightarrow UX$ and observations dual to those in 5.3.

5.5. Proposition. *The transformation $r : UD \longrightarrow DU :$ **ord** $\longrightarrow$ **ord** is a distributive law of* **U** *over* **D** *and the transformation $l : DU \longrightarrow UD :$ **ord** $\longrightarrow$ **ord** is a distributive law of* **D** *over* **U**.

Proof. By construction of r we have $r[d]$, so by Proposition 4.4 it suffices, for the first claim, to show that $r \cdot uD \le Du$. In other words, we must show that for all S in DX, $rX(\uparrow S) \subseteq \{\uparrow x | x \in S\}^{\mapsdown}$. But if T is in $rX(\uparrow S)$ then for all S' which contain S there is an x in $T \cap S'$. In particular, there is an x in $T \cap S$ and now $T \supseteq \uparrow x$ and $x \in S$ shows that $T \in \{\uparrow x | x \in S\}^{\mapsdown}$. The calculations for l are similar but can be shown to follow from those above by duality. ∎

5.6. Remark. If we write **idl** for the bicategory of ordered sets, order ideals and inclusions then a down-set S of X can be regarded as an arrow $S : 1 \longrightarrow\!\!\!\!\!\!\!+\;\; X$ in **idl**. Similarly, an up-set T of X can be regarded as an arrow $T : X \longrightarrow\!\!\!\!\!\!\!+\;\; 1$ in **idl**. A composite $TS : 1 \longrightarrow\!\!\!\!\!\!\!+\;\; 1$ of such admits a comparison $TS \subseteq 1_1$ (because $1_1 : 1 \longrightarrow\!\!\!\!\!\!\!+\;\; 1$ is terminal in $\mathbf{idl}(1,1)$). To say that $(\exists x)(x \in T \text{ and } x \in S)$, the condition which arises in the definitions of both r and l in 5.4 is to say that $TS \subseteq 1_1$ is an equality.

5.7. Suppose now that $d : 1 \longrightarrow D$ $[u : 1 \longrightarrow U]$ factorizes as $1 \xrightarrow{d'} D' \xrightarrow{i} D$ $[1 \xrightarrow{u'} U' \xrightarrow{j} U]$, with i $[j]$ fully-faithful. From the proof of Theorem 3.8 in [RW5] it follows that if D'- $[U'$-$]$ unions (in both cases) of D'- $[U'$-$]$ sets are D'- $[U'$-$]$ sets then $\mathbf{D}' = (D', d')$ $[\mathbf{U}' = (U', u')]$ is also a KZ [co-KZ] doctrine and i $[j]$ is a monad arrow. In this situation, we can *attempt* to define $r' : U'D' \longrightarrow D'U'$ by modifying the description of r in 5.4 so as to have

$$r'X(\mathcal{T}) = \{T \in U'X | (\forall S \in \mathcal{T})(\exists x \in T)(x \in S)\}$$

and similarly for an $l' : D'U' \longrightarrow U'D'$. This definition of r' certainly gives an arrow $U'D' \longrightarrow DU'$ and it factorizes through $D'U' \longrightarrow DU'$ if the set displayed above is a D'-set. Similarly for l', the obvious putative definition makes sense if the defining set is a U'-set. If r does restrict to give r' then the general considerations of Section 4 show that we have a distributive law $r' : U'D' \longrightarrow D'U'$ of $\mathbf{U}'$ over $\mathbf{D}'$ (and that this is the only possibility for such a law). It should be noted that r may restrict to give such an r' without l restricting to give such an l'. Observe that if $D' = D$ then the condition is automatically satisfied for $r' : U'D \longrightarrow DU'$ to be a distributive law. In particular take U' to be given by finitely-generated up-sets. These are closed with respect to finite unions and the resulting co-KZ-doctrine is well known to be that for which the algebras are meet-semi-lattices. The algebras for the composite monad $\mathbf{DU}'$ are frames. This is the law that we attributed to Linton in 1.1. In this case, l does not restrict to give an l'.

5.8. In [RW5] there is an extended discussion of the case where D' is given by bounded down-sets and U' by non-empty up-sets. There it is shown that $r' : U'D' \longrightarrow D'U'$ is well-defined. We note here that in this case $l' : D'U' \longrightarrow U'D'$ is also well-defined.

If D' is given by finitely-generated down-sets and U' by finitely-generated up-sets then both r' and l' are well-defined.

If D' is given by up-directed down-sets and U' by finitely-generated up-sets then r' is well-defined.

We should point out here that the basic law r is sensitive to the base 2-category **ord**. In particular, r does not preserve all finite joins so that it is not possible to consider the restrictions of such monads $\mathbf{D}'$ and $\mathbf{U}'$ as are under under consideration to, say, the 2-category of distributive lattices and obtain a distributive law whose components are the rL, where L is a distributive lattice.

6. Complete Distributivity

6.1. By **DU** we will understand the monad on **ord** constructed on DU with the help of r. Similarly, **UD** is the composite monad obtained with the help of l. From [F&W] we recall that a (constructively) completely distributive lattice is an ordered set L for which dL has a left adjoint which has a left adjoint. We often call such L CCD lattices and a number of characterizations of these are given in [RW3]. Now to say that L^{op} is CCD is to say that uL has a right adjoint which has a right adjoint. Here we will call such an L an ^{op}CCD lattice. Classically the notions CCD and ^{op}CCD coincide but it was shown in [RW1] that relative to a general elementary topos, coincidence of CCD and ^{op}CCD is equivalent to booleaness of the topos in question. We write **ccd** [^{op}**ccd**] for the 2-category of CCD [^{op}CCD] lattices, functors that preserve both sups and infs, and inequalities.

6.2. THEOREM.

i) $\mathbf{ord}^{\mathbf{DU}} \cong \mathbf{ccd}$

ii) $\mathbf{ord}^{\mathbf{UD}} \cong {}^{op}\mathbf{ccd}$

Proof. It suffices to give a proof of *i)*; that of *ii)* is dual to it. From Section 2. of [BEK] we know that a **DU**-algebra is a **U**-algebra that carries a **D**-algebra structure for which the **D**-structure arrow is a **U**-homomorphism. If L is a **U**-algebra, that is an object of **inf**, it is necessarily an object of **sup** but its **D**-algebra structure, that is $\bigvee : DL \longrightarrow L$, is a **U**-homomorphism iff $\bigvee$ preserves infima iff $\bigvee$ has a left adjoint iff L is CCD. Of course **DU**-homomorphisms are just arrows that are both **U**-homomorphisms (inf-preserving) and **D**-homomorphisms (sup-preserving). ∎

6.3. Consider the adjunction $[-,\Omega] \dashv [-,\Omega] : \mathbf{ord}^{op} \longrightarrow \mathbf{ord}$ which gives rise to the monad on **ord** known as double dualization with respect to Ω. We are grateful to Paul Taylor for asking us to consider the algebras for this monad. As pointed out in 5.2 this monad admits the apparently more complicated description $\mathcal{DU}$, which is obtained by composing the adjunction above with the isomorphism $(-)^{op} \dashv (-)^{op} : \mathbf{ord}^{coop} \longrightarrow \mathbf{ord}^{op}$.

6.4. LEMMA. $\mathcal{DU} = DU : \mathbf{ord} \longrightarrow \mathbf{ord}$

Proof. Consider an arrow $f : X \longrightarrow A$ in **ord**. As noted in 5.2, $\mathcal{U}f \dashv Uf$ and since $\mathcal{D}$ is a 2-functor, $\mathcal{DU}f \dashv \mathcal{D}Uf$. On the other hand, for the arrow Uf we have $DUf \dashv \mathcal{D}Uf$, as noted in 5.1. Since $\mathcal{DU}f$ and DUf have the same right adjoint they are equal. ∎

Lemma 6.4 is at first surprising since both D and U are defined in terms of direct image while both $\mathcal{D}$ and $\mathcal{U}$ are given by inverse image. Certainly, for a function $f : X \longrightarrow A$,

$$\mathcal{P}(\mathcal{P}f) \neq \exists(\exists f) : \mathcal{PP}X \longrightarrow \mathcal{PP}A,$$

where $\mathcal{P}$ is the inverse-image power set functor and $\exists f$ is direct image (the left adjoint of $\mathcal{P}f$).

6.5. THEOREM. *The 2-monad arising from the 2-adjunction* $\mathcal{U} \dashv \mathcal{D}$ *is* **DU**.

Proof. After 6.4 we have only to check that the units and multiplications coincide. The unit η for the monad on $\mathcal{DU}$ is the unit for the 2-adjunction $\mathcal{U} \dashv \mathcal{D}$. We have $\eta X : X \longrightarrow [[X, \Omega], \Omega] = DUX$ given by the familiar 'evaluation' formula $\eta X(x)(T) = T(x)$. In terms of subsets this translates as

$$\begin{aligned}
\eta X(x) &= \{T \in UX | x \in T\} \\
&= \{T \in UX | T \supseteq \uparrow x\} \\
&= \downarrow\uparrow x \\
&= dUX(uX(x)) \\
&= (dU \cdot u)X(x)
\end{aligned}$$

but $dU \cdot u = Du \cdot d$ is the unit for $\mathbf{DU}$.

The multiplication for the monad on $\mathcal{DU}$ is $\mathcal{D}\epsilon\mathcal{U}$, where $\epsilon : \mathcal{UD} \longrightarrow 1_{\mathbf{ord}^{coop}}$ is the counit for the adjunction $\mathcal{U} \dashv \mathcal{D}$. As an arrow in **ord**, $\epsilon X : X \longrightarrow UDX = [[X^{op}, \Omega], \Omega]^{op}$ is again given by 'evaluation' and arguing as we did for the units we have

$$\begin{tikzcd}
1 \arrow[r, "d"] \arrow[d, "u"'] \arrow[dr, "\epsilon"] & D \arrow[d, "uD"] \\
U \arrow[r, "Ud"'] & UD
\end{tikzcd}$$

Now for each X in **ord**, $\mathcal{D}\epsilon\mathcal{U}X$ is by 5.1 the right adjoint of $D\epsilon UX$. From the square above then we can write $DuDU.DdU = D\epsilon U \dashv \mathcal{D}\epsilon\mathcal{U}$.

From [BEK] we know that the multiplication for $\mathbf{DU}$ is $mn \cdot DrU = Dn \cdot mUU \cdot DrU$, where m is the multiplication for $\mathbf{D}$ and n is the multiplication for $\mathbf{U}$. Now $Dn \cdot mUU \cdot DrU$ has a left adjoint given by $DlU \cdot DdUU \cdot DuU$. So finally, to show that the multiplications coincide we can show $DuDU \cdot DdU = DlU \cdot DdUU \cdot DuU$. This follows from 2-naturality of u and $l[d]$ as in the diagram below.

$$\begin{tikzcd}
DU \arrow[r, "DuU"] \arrow[d, "DdU"'] & DUU \arrow[r, "DdUU"] \arrow[dr, "DUdU"'] & DDUU \arrow[d, "DlU"] \\
DDU \arrow[rr, "DuDU"'] & & DUDU
\end{tikzcd}$$

∎

6.6. Corollary. *The 2-category of algebras for the double dualization with respect to Ω monad on* **ord** *is* **ccd**.

∎

6.7. Remark. Taking the $(-)^{coop}$ duals of our $\mathcal{D}$ and $\mathcal{U}$ we can prove that the monad on **ord** arising from $\mathcal{D}^{coop} \dashv \mathcal{U}^{coop}$ is **UD**, whence the 2-category of algebras is ${}^{op}\mathbf{ccd}$.

6.8. Remark. In any topos we have an adjunction $s \dashv c : \mathcal{U} \longrightarrow \mathcal{D}$, where all components of both s and c are given by complementation, giving rise to a commutative square

$$\begin{tikzcd} \mathcal{U}\mathcal{D} \arrow[d, "c\mathcal{D}"'] & \mathcal{U}\mathcal{U} \arrow[l, "\mathcal{U}s"'] \arrow[d, "c\mathcal{U}"] \\ \mathcal{D}\mathcal{D} & \mathcal{D}\mathcal{U} \arrow[l, "\mathcal{D}s"] \end{tikzcd}$$

If the base topos is *boolean* then c and s are inverse isomorphisms and consequently $Ds = (\mathcal{D}s)^{-1}$ and $Us = (\mathcal{U}s)^{-1}$ and we have

$$\begin{tikzcd} \mathcal{U}\mathcal{D} \arrow[r, "Us"] \arrow[d, "c\mathcal{D}"'] & \mathcal{U}\mathcal{U} \arrow[d, "c\mathcal{U}"] \\ \mathcal{D}\mathcal{D} \arrow[r, "Ds"'] & \mathcal{D}\mathcal{U} \end{tikzcd}$$

Using boolean set theory it is then easy to show that for any ordered set X, $DsX \cdot c\mathcal{D}X \subseteq rX \subseteq c\mathcal{U}X \cdot UsX$. A similar result holds for $l : \mathcal{D}\mathcal{U} \longrightarrow \mathcal{U}\mathcal{D}$ and it then follows that r and l are inverse isomorphisms, as mentioned in 4.2 of [RW5].

Conversely, assume only that r and l are inverse isomorphisms. Then in particular $r\emptyset$ is an isomorphism. From our description of r in 5.3, it is easy to see that in any topos $r\emptyset = \neg : \Omega^{op} \longrightarrow \Omega$. Thus in this case the base topos *is* boolean.

This remark extends to give another proof, quite different from the one in [RW1], that CCD = ^{op}CCD characterizes boolean toposes.

6.9. Remark. Mindful of the celebrated theorem of Paré, saying that $\Omega^{(-)}$ is monadic over the base topos, see [PAR], one might ask if $[-,\Omega]; \mathbf{ord}^{op} \longrightarrow \mathbf{ord}$ is monadic. The answer is 'no' and can be deduced from the results in [RW3] about those special CCD lattices of the form DX, for X an ordered set.

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