Constructive Complete Distributivity III*

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Dedicated to the memory of our colleague and friend
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Abstract

A complete lattice L is *constructively completely distributive*, $(CCD)(L)$, if the *sup* map defined on down closed subobjects has a left adjoint. We characterize preservation of this property by left exact functors between toposes using a "logical comparison transformation". The characterization is applied to (direct images of) geometric morphisms to show that local homeomorphisms (in particular, product functors) preserve (CCD) objects, while preserving (CCD) objects implies openness.

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Introduction

A complete ordered set L is *constructively completely distributive*, abbreviated to $(CCD)(L)$, if there is a left adjoint to the *sup* map $\bigvee : \mathcal{D}L \longrightarrow L$, where $\mathcal{D}L$ is the set of down-closed subobjects of L ordered by inclusion. The condition is equivalent to

$$(\forall \mathcal{S} \subseteq \mathcal{D}L) \left(\bigwedge \left\{ \bigvee S \mid S \in \mathcal{S} \right\} = \bigvee \left\{ \bigwedge \{T(S) \mid S \in \mathcal{S}\} \mid T \in \Pi\mathcal{S} \right\} \right)$$

which in turn is the condition $(CD)(L)$ for complete distributivity of L when the power object $\mathcal{P}L$ replaces $\mathcal{D}L$. That is, (CCD) is the restriction of (CD) to down-closed subobjects. The name is motivated by the facts [3] that the axiom of choice (ac) implies the equivalence of (CD) with (CCD) for any L, while the equivalence of the two distributivity concepts implies (ac). In short, $(CD) \iff (CCD)$ is equivalent to choice. In any topos, $\mathcal{P}X$ is (CCD) for all X whereas the statement that $\mathcal{P}X$ is (CD) for all X is equivalent to (ac). Hence (CCD) is clearly a relevant notion in most toposes — those where (ac) fails.

The power objects $\mathcal{P}X$ are not generally boolean algebras in a topos $\mathbf{E}$ and the condition that they are such, denoted $(boo)(\mathbf{E})$, makes a difference in the theory of constructive complete distributivity. We write $(CCD^{op})(L)$ for $(CCD)(L^{op})$. In [6] we showed that a topos is boolean precisely when the conditions (CCD) and (CCD^{op}) are equivalent for all L.

This article is concerned with studying the preservation of the (CCD) property by left exact functors and geometric morphisms between toposes. Let $\mathbf{E}$ be a topos. We denote the 2-category of ordered objects in $\mathbf{E}$ by $\mathbf{ord}(\mathbf{E})$, where an order is assumed to be reflexive and transitive, but not necessarily antisymmetric, as in [3]. We denote by $\mathbf{idl}(\mathbf{E})$ the 2-category with the objects of $\mathbf{ord}(\mathbf{E})$, arrows the ideals, and transformations just containments.

Let $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ be a left exact functor between toposes. We define a "logical comparison transformation" $\gamma_E : \Gamma\mathcal{D}_{\mathbf{E}}E \longrightarrow \mathcal{D}_{\mathbf{S}}\Gamma E$ in $\mathbf{ord}(\mathbf{S})$ using ideals and their calculus. This γ_E is 2-natural and we show that Γ preserves cocomplete objects in $\mathbf{ord}(\mathbf{E})$ precisely if γ_E always has a left adjoint. It turns out that Γ preserves (CCD) objects precisely if γ_E has a left adjoint which has a left adjoint.

Turning to geometric morphisms, we use the results above to show that direct images of local homeomorphisms preserve (CCD) objects. Geometric morphisms which preserve (CCD) objects are open. Counterexamples show that neither of these implications is reversible, and that essential geometric morphisms need not preserve (CCD) objects. Since local homeomorphisms preserve (CCD) objects, so do arbitrary product functors. Thus a power of a (CCD) objects is a (CCD) object. Furthermore, the cotensor $\mathbf{ord}(\mathbf{E})(X, L)$, for L a (CCD) object and X in $\mathbf{ord}(\mathbf{E})$, is a (CCD) object [6].

1 Left Exact Functors

Let $\mathbf{E}$ and $\mathbf{S}$ be elementary toposes and $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ a left exact functor. It induces a 2-functor $\mathbf{ord}(\Gamma) : \mathbf{ord}(\mathbf{E}) \longrightarrow \mathbf{ord}(\mathbf{S})$ which we abbreviate to Γ. It is also convenient to define the 2-category $\mathbf{idl}(\mathbf{E})$ whose objects are those of $\mathbf{ord}(\mathbf{E})$, whose arrows are (order) ideals and whose transformations are "containments". For ordered objects X and Y, an ideal from X to Y is a subobject $r : R \rightarrowtail Y \times X$ which is "down-closed in Y" and "up-closed in X". In terms of "elements", we mean that if $y' \leq y$ and yRx and $x \leq x'$, then $y' \leq x'$. We write $r \leq s$ to indicate that r factors through $s : S \rightarrowtail Y \times X$. Horizontal composition is given by composition of relations.

While $\mathbf{ord}(\mathbf{E})$ is locally ordered, $\mathbf{idl}(\mathbf{E})$ is locally antisymmetrically ordered as a result of our having defined ideals in terms of subobjects rather than just monomorphisms. A word about our apparent switch in generality may be helpful. Given an ordered object X in $\mathbf{E}$ and x, y in X we write $x \cong y$ if and only if both $x \leq y$ and $y \leq x$. The associated antisymmetric ordered object is $X/\cong$ and we have $q : X \longrightarrow X/\cong$ in $\mathbf{ord}(\mathbf{E})$. However, we cannot conclude that q is an equivalence of ordered objects because there may be no splitting of q in $\mathbf{E}$. Now, for any object E in $\mathbf{E}$ write $\mathbf{mono}(E)$ for the ordered set of monomorphisms with codomain E and $\mathbf{sub}(E)$ for the antisymmetric ordered set of subobjects of E. Then $\mathbf{mono}(E) \longrightarrow \mathbf{sub}(E)$ is an instance of q in a suitably large category of sets. We do not assume choice for the latter, but if we regard the pullbacks of "true", $\tau : 1 \longrightarrow \Omega$, in $\mathbf{E}$ as being canonically specified (which we regard as a reasonable working convention), then $\mathbf{mono}(E) \longrightarrow \mathbf{sub}(E)$ has a canonical splitting for each E and it seems sensible to use it. Were we to carry out this discussion in an arbitrary regular category we would adopt the approach of [2].

Ideals $X \longrightarrow Y$ are in order-isomorphic correspondence with $\mathbf{ord}(\mathbf{E})$ arrows $Y^{op} \times X \longrightarrow \Omega$ which in turn are in order isomorphic correspondence with $\mathbf{ord}(\mathbf{E})$ arrows $X \longrightarrow \mathcal{D}Y$. We write

$$(\ )^{\wedge} : \mathbf{idl}(\mathbf{E})(X, Y) \overset{\sim}{\rightleftarrows} \mathbf{ord}(\mathbf{E})(X, \mathcal{D}Y) : (\ )^{\vee}$$

and note that $y \in \varphi^{\wedge} x$ if and only if $y \varphi x$; $y h^{\vee} x$ iff $y \in hx$. If $f : X \longrightarrow Y$ in $\mathbf{ord}(\mathbf{E})$ we define $f_+ : X \longrightarrow Y$ in $\mathbf{idl}(\mathbf{E})$ by $y f_+ x$ if and only if $y \leq fx$. Similarly, $f^+ : Y \longrightarrow X$ is defined by $x f^+ y$ if and only if $fx \leq y$. We have $f_+ \dashv f^+$ in $\mathbf{idl}(\mathbf{E})$ and $(\ )_+ : \mathbf{ord}(\mathbf{E}) \longrightarrow \mathbf{idl}(\mathbf{E})$ is proarrow equipment. (See [7, 8].) Sometimes we suppress $(\ )_+$ when we feel that it does not cause confusion. Note that applying Γ to subobjects does not give a 2-functor $\mathbf{idl}(\Gamma) : \mathbf{idl}(\mathbf{E}) \longrightarrow \mathbf{idl}(\mathbf{S})$ unless Γ happens to preserve images. However, (together with $\mathbf{ord}(\Gamma)$) it does give an arrow of "$\mathcal{F}$" as described in great detail in [1]. Abbreviating $\mathbf{idl}(\Gamma)$ by Γ, we record those facts about $\mathbf{idl}(\Gamma)$ which we need for this paper:

(i) $\Gamma(1_X) = 1_{\Gamma X}$ for all X in $\mathbf{idl}(\mathbf{E})$

(ii) $\Gamma(f_+) = (\Gamma f)_+$ for all f in $\mathbf{ord}(\mathbf{E})$

(iii) $\Gamma(f^+) = (\Gamma f)^+$ for all f in $\mathbf{ord}(\mathbf{E})$

(iv) $\Gamma(\varphi f) = \Gamma\varphi\Gamma f$ for all f in $\mathbf{ord}(\mathbf{E})$ and φ in $\mathbf{idl}(\mathbf{E})$

(v) $\Gamma(f^+\varphi) = \Gamma f^+\Gamma\varphi$ for all f in $\mathbf{ord}(\mathbf{E})$ and φ in $\mathbf{idl}(\mathbf{E})$

Note too that $\mathbf{ord}(\Gamma)$ preserves full faithfulness of an arrow $f : X \longrightarrow Y$ in $\mathbf{ord}(\mathbf{E})$. For f is fully faithful if and only if $f^+f = 1_X$.

As usual, we write $\downarrow_E : E \longrightarrow \mathcal{D}_{\mathbf{E}}E$ in $\mathbf{ord}(\mathbf{E})$ for the down-segment (yoneda) arrow given by $\downarrow_E(x) = \downarrow x = \{y \in E \mid y \leq x\}$. We define $\gamma_E : \Gamma\mathcal{D}_{\mathbf{E}}E \longrightarrow \mathcal{D}_{\mathbf{S}}\Gamma E$ in $\mathbf{ord}(\mathbf{S})$ by $\gamma_E = (\Gamma\downarrow_E)^{+\wedge}$.

Lemma 1 *$(\ )^\vee : \mathbf{ord}(\mathbf{E})(X, \mathcal{D}Y) \longrightarrow \mathbf{idl}(\mathbf{E})(X, Y)$ is given by composition with the arrow $\downarrow_Y^+ : \mathcal{D}Y \longrightarrow Y$. That is*

$$(X \xrightarrow{h} \mathcal{D}Y)^\vee = X \xrightarrow{h_+} \mathcal{D}Y \xrightarrow{\downarrow_Y^+} Y$$

Proof. We have

$$\begin{array}{rll} y(\downarrow_Y^+ h_+)x & \text{iff} & (\exists T \in \mathcal{D}Y)(y\downarrow_Y^+ T h_+ x) \\ & \text{iff} & (\exists T \in \mathcal{D}Y)(\downarrow_Y(y) \subseteq T \subseteq hx) \\ & \text{iff} & \quad (\downarrow_Y(y) \subseteq hx) \\ & \text{iff} & \quad (y \in hx) \end{array}$$

∎

Corollary 2 *For $X \xrightarrow{f} Y$ and $Y \xrightarrow{h} \mathcal{D}Z$ in $\mathbf{ord}(\mathbf{E})$, $(hf)^\vee = h^\vee f_+$ in $\mathbf{idl}(\mathbf{E})$.*

Proof.

$$(hf)^\vee = \downarrow_Z^+(hf)_+ = \downarrow_Z^+ h_+ f_+ = h^\vee f_+$$

∎

Lemma 3 *For $\varphi : F \longrightarrow E$ in* $\mathbf{idl}(\mathbf{E})$

$$\begin{tikzcd}
 & \Gamma\mathcal{D}_{\mathbf{E}}E \arrow[dr, "\gamma_E"] & \\
\Gamma F \arrow[ur, "\Gamma\varphi^\wedge"] \arrow[rr, "(\Gamma\varphi)^\wedge"'] & \arrow[u, phantom, "\|" near start] & \mathcal{D}_{\mathbf{S}}\Gamma E
\end{tikzcd}$$

Proof. It suffices to show $\Gamma\varphi = (\gamma_E \cdot \Gamma\varphi^\wedge)^\vee$

$$\begin{array}{rcll}
(\gamma_E \cdot \Gamma\varphi^\wedge)^\vee & = & ((\Gamma{\downarrow_E}^+)^\wedge \cdot \Gamma\varphi^\wedge)^\vee & \\
 & = & \Gamma{\downarrow_E}^+ \cdot \Gamma\varphi^\wedge & \text{(by Corollary 2)} \\
 & = & \Gamma({\downarrow_E}^+ \cdot \varphi^\wedge) & \\
 & = & \Gamma(\varphi^{\wedge\vee}) & \text{(by Lemma 1)} \\
 & = & \Gamma\varphi &
\end{array}$$

∎

Corollary 4 *For E in* $\mathbf{E}$

$$\begin{tikzcd}
 & \Gamma\mathcal{D}_{\mathbf{E}}E \arrow[dr, "\gamma_E"] & \\
\Gamma E \arrow[ur, "\Gamma{\downarrow_E}"] \arrow[rr, "{\downarrow_{\Gamma E}}"'] & \arrow[u, phantom, "\|" near start] & \mathcal{D}_{\mathbf{S}}\Gamma E
\end{tikzcd}$$

in $\mathbf{ord}(\mathbf{S})$.

Proof. $(E \xrightarrow{1_E} E)^\wedge = {\downarrow_E} : E \longrightarrow \mathcal{D}_{\mathbf{E}}E.$ ∎

Lemma 5 *For $h : X \longrightarrow \mathcal{D}Y$ and $g : Z \longrightarrow Y$ in* $\mathbf{ord}(\mathbf{E})$, *$(\mathcal{D}g \cdot h)^\vee = g^+h^\vee$ in* $\mathbf{idl}(\mathbf{E})$.

Proof. Consider

$$
\begin{tikzcd}
X \arrow[r, "h"] & \mathcal{D}Y & \mathcal{D}Z \arrow[l, "g_!"'] \\
 & Y \arrow[u, "\downarrow_Y"] & Z \arrow[l, "g"] \arrow[u, "\downarrow_Z"']
\end{tikzcd}
$$

where $g_!$ denotes the left adjoint to $\mathcal{D}g : \mathcal{D}Y \longrightarrow \mathcal{D}Z$. ($\mathcal{D}$ is given on arrows by inverse image.) The square commutes because $\downarrow$ is natural from the identity to $(\ )_!$. Taking right adjoints of the arrows in the square, the result follows immediately from Lemma 1. ∎

Corollary 6 *For E in $\mathbf{E}$*

$$
\begin{tikzcd}
\Gamma\mathcal{D}_{\mathbf{E}}E \arrow[rr, "\gamma_E"] \arrow[dr, "\downarrow_{\Gamma\mathcal{D}_{\mathbf{E}}E}"'] & \arrow[d, phantom, "||"] & \mathcal{D}_{\mathbf{S}}\Gamma E \\
 & \mathcal{D}_{\mathbf{S}}\Gamma\mathcal{D}_{\mathbf{E}}E \arrow[ur, "\mathcal{D}_{\mathbf{S}}\Gamma\downarrow_E"'] &
\end{tikzcd}
$$

Proof.

$$
\begin{aligned}
(\mathcal{D}_{\mathbf{S}}\Gamma\downarrow_E \cdot \downarrow_{\Gamma\mathcal{D}_{\mathbf{E}}E})^{\vee} &= \Gamma{\downarrow_E}^{+} \cdot {\downarrow_{\Gamma\mathcal{D}_{\mathbf{E}}E}}^{\vee} \quad \text{by Lemma 5} \\
&= \Gamma{\downarrow_E}^{+} \cdot 1_{\Gamma\mathcal{D}_{\mathbf{E}}E} \\
&= {\gamma_E}^{\vee}
\end{aligned}
$$

∎

Lemma 7 *For $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ left exact, γ is a 2-natural transformation:*

$$
\begin{tikzcd}
\mathbf{ord}(\mathbf{E})^{coop} \arrow[r, "\mathcal{D}_{\mathbf{E}}"] \arrow[d, "\mathbf{ord}(\Gamma)^{coop}"'] & \mathbf{ord}(\mathbf{E}) \arrow[d, "\mathbf{ord}(\Gamma)"] \arrow[dl, phantom, "\swarrow\gamma"] \\
\mathbf{ord}(\mathbf{S})^{coop} \arrow[r, "\mathcal{D}_{\mathbf{S}}"'] & \mathbf{ord}(\mathbf{S})
\end{tikzcd}
$$

Proof. Because $\mathbf{ord}(\mathbf{S})$ is a locally ordered 2-category, it suffices to prove ordinary naturality of γ. Let $f : X \longrightarrow Y$ be an arrow of $\mathbf{ord}(\mathbf{E})$ and consider:

$$\begin{tikzcd}
\Gamma\mathcal{D}_{\mathbf{E}}Y \arrow[r, "\Gamma\mathcal{D}_{\mathbf{E}}f"] \arrow[d, "\gamma_Y"'] & \Gamma\mathcal{D}_{\mathbf{E}}X \arrow[d, "\gamma_X"] \\
\mathcal{D}_{\mathbf{S}}\Gamma Y \arrow[r, "\mathcal{D}_{\mathbf{S}}\Gamma f"'] & \mathcal{D}_{\mathbf{S}}\Gamma X
\end{tikzcd}$$

$$\begin{array}{rcll}
(\gamma_X \cdot \Gamma\mathcal{D}_{\mathbf{E}}f)^{\vee} & = & ((\Gamma{\downarrow}_X)^{+\wedge} \cdot \Gamma\mathcal{D}_{\mathbf{E}}f)^{\vee} & \\
& = & ((\Gamma{\downarrow}_X)^{+}(\Gamma\mathcal{D}_{\mathbf{E}}f)_{+} & \text{(Lemma 2)} \\
& = & \Gamma({\downarrow}_X^{+}\, \mathcal{D}_{\mathbf{E}}f) & \\
& = & \Gamma(f^{+}\cdot {\downarrow}_Y^{+}) & \\
& = & \Gamma f^{+} \cdot (\Gamma\, {\downarrow}_Y)^{+} & \\
& = & ((\mathcal{D}_{\mathbf{S}}\Gamma f)(\Gamma\, {\downarrow}_Y)^{+\wedge})^{\vee} & \text{(Lemma 5)} \\
& = & ((\mathcal{D}_{\mathbf{S}}\Gamma f)\gamma_Y)^{\vee}. &
\end{array}$$

∎

Theorem 8 *For $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ a left exact functor between toposes, the following statements for all E in* $\mathbf{ord}(\mathbf{E})$ *are equivalent:*

(i) γ_E has a left adjoint

(ii) E cocomplete $\Rightarrow \gamma_E$ has a left adjoint

(iii) E cocomplete $\Rightarrow \Gamma E$ cocomplete

(iv) $\Gamma\mathcal{D}_{\mathbf{E}}E$ is cocomplete.

Proof. (i) $\Rightarrow$ (ii) Trivial. (ii) $\Rightarrow$ (iii) E is cocomplete iff ${\downarrow}_E : E \longrightarrow \mathcal{D}_{\mathbf{E}}E$ has a left adjoint ($\bigvee_E$). The implication follows immediately from Lemma 4 since $\mathbf{ord}(\Gamma)$ preserves adjunctions. (iii) $\Rightarrow$ (iv) Trivial, since $\mathcal{D}_{\mathbf{E}}E$ is cocomplete for all E in $\mathbf{ord}(\mathbf{E})$. (iv) $\Rightarrow$ (i) Consider the factorization of γ_E given in Corollary 6. The left adjoint of $\mathcal{D}_{\mathbf{S}}\Gamma{\downarrow}_E$ is, in any event, $(\Gamma{\downarrow}_E)_!$. Cocompleteness of $\Gamma\mathcal{D}_{\mathbf{E}}E$ means, precisely, the existence of a left adjoint for ${\downarrow}_{\Gamma\mathcal{D}_{\mathbf{E}}E}$. ∎

Our goal is change of base criteria for (CCD) objects. Write **pamord**$(\mathbf{E})$ for the locally full sub-2-category of **ord**$(\mathbf{E})$ determined by the complete objects of **ord**$(\mathbf{E})$ and the inf-preserving arrows between them and define the cocomplete objects therein to be those E for which $\downarrow_E : E \longrightarrow \mathcal{D}_{\mathbf{E}}E$ has a left adjoint in **pamord**$(\mathbf{E})$. Since an arrow is inf-preserving precisely if it has a left adjoint we see that (CCD) objects of **ord**$(\mathbf{E})$ are the cocomplete objects of **pamord**$(\mathbf{E})$ and this suggests that we structure our preservation of (CCD) criteria parallel to those of Theorem 8.

Such an approach also suggests the question of a left exact version. For if we write **lexord**$(\mathbf{E})$ for the locally full sub-2-category of **ord**$(\mathbf{E})$ determined by the finitely complete objects of **ord**$(\mathbf{E})$ and the left exact arrows between them and define cocompleteness of E in **lexord**$(\mathbf{E})$ to mean the existence of a left adjoint to $\downarrow_E$ in **lexord**$(\mathbf{E})$, then the cocomplete objects of **lexord**$(\mathbf{E})$ are the locales of **ord**$(\mathbf{E})$. We write $(LOC)(E)$ for the statement "E is a locale" and record without proof

Theorem 9 *For* $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ *a left exact functor between toposes the following steatements for all* E *in* **lexord**$(\mathbf{E})$ *are equivalent:*

(i) γ_E *has a left exact left adjoint*

(ii) $(LOC)(E) \Rightarrow \gamma_E$ *has a left exact left adjoint*

(iii) $(LOC)(E) \Rightarrow (LOC)(\Gamma E)$

(iv) $(LOC)(\Gamma \mathcal{D}_{\mathbf{E}} E)$.

∎

In fact we can say somewhat more.

Theorem 10 *For* $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ *a left exact functor between toposes, each of the equivalent statements of Theorem 8 is equivalent to each statement of Theorem 9.*

Proof. ((iii) of Theorem 8 $\Rightarrow$ (iii) of Theorem 9) Let E be finitely complete. If E is a locale then certainly by (iii) of Theorem 8, ΓE is cocomplete. However, Γ, being left exact, preserves Heyting algebras; so if E is a locale then ΓE is a locale. ((iv) of Theorem 9 $\Rightarrow$ (iv) of Theorem 8). Let E be an object of **ord**$(\mathbf{E})$. We have $\downarrow_E : E \rightarrowtail \mathcal{D}_{\mathbf{E}}E$, fully faithful, and hence, via $\Gamma(\downarrow_E)_! \dashv \Gamma\mathcal{D}_{\mathbf{E}}\downarrow_E$, $\Gamma\mathcal{D}_{\mathbf{E}}E$ is coreflective in $\Gamma\mathcal{D}_{\mathbf{E}}\mathcal{D}_{\mathbf{E}}E$. For any E, $\mathcal{D}_{\mathbf{E}}E$ is in **lexord**($\mathbf{E}$); by assumption $\Gamma\mathcal{D}_{\mathbf{E}}\mathcal{D}_{\mathbf{E}}E$ is a locale and hence cocomplete; finally $\Gamma\mathcal{D}_{\mathbf{E}}E$ is complete and hence cocomplete. ∎

Theorem 11 *For $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ a left exact functor between toposes the following statements for all E in* **pamord**($\mathbf{E}$) *are equivalent:*

(i) γ_E has a left adjoint which has a left adjoint.

(ii) $(CCD)(E) \Rightarrow \gamma_E$ has a left adjoint which has a left adjoint.

(iii) $(CCD)(E) \Rightarrow (CCD)(\Gamma E)$

(iv) $(CCD)(\Gamma\mathcal{D}_{\mathbf{E}}E)$.

Proof. (i) $\Rightarrow$ (ii) Trivial. (ii) $\Rightarrow$ (iii) $(CCD)(E)$ iff ${\downarrow_E}$ has a left adjoint which has a left adjoint. The implication follows immediately from Corollary 4 (iii) $\Rightarrow$ (iv) Trivial, since $(CCD)(\mathcal{D}_{\mathbf{E}}E)$ for all E in **ord**($\mathbf{E}$). (iv) $\Rightarrow$ (i) Consider the factorization of γ_E given in Corollary 6. Here we have

$$(\Gamma{\textstyle\bigvee_E})_! \dashv (\Gamma{\downarrow_E})_! \dashv \mathcal{D}_{\mathbf{E}}\Gamma{\downarrow_E} \quad \text{and} \quad \Downarrow_{\Gamma\mathcal{D}_{\mathbf{E}}E} \dashv \bigvee_{\Gamma\mathcal{D}_{\mathbf{E}}E} \dashv {\downarrow_{\Gamma\mathcal{D}_{\mathbf{E}}E}}$$

where, as in [3], we write $\Downarrow \dashv \bigvee$ for (CCD) objects. ∎

Remark One might enquire whether condition (i) in each of Theorems 8, 9 and 11 is expressible in terms of the power object functor $\mathcal{P}$. For Theorem 8 this is easy. For any X in $\mathbf{E}$ we have $\mathcal{P}X = \mathcal{D}DX$, where $D : \mathbf{E} \longrightarrow$ **ord**($\mathbf{E}$) is the discrete order functor, left adjoint to the forgetful functor, $|\ |$. Certainly, then (i) of Theorem 8 implies

$$(\forall X \text{ in } E)(\gamma_X : \Gamma\mathcal{P}_{\mathbf{E}}X \longrightarrow \mathcal{P}_{\mathbf{S}}\Gamma X \;\; \text{has a left adjoint})$$

(where we have abbreviated γ_{DX} by γ_X and noted that **ord**(Γ) preserves discreteness.) This condition implies (i) of Theorem 8. For any E in **ord**($\mathbf{E}$) we have the counit for the adjunction, $D \dashv |\ |, D|E| \longrightarrow E$. Invoking the left adjoint to $\mathcal{D}$ of this arrow, which is down closure, $(\ )^{\blacktriangledown}$, and naturality of γ, Lemma 7, it is easy to describe a left adjoint for γ_E in terms of a left adjoint for $\gamma_{D|E|}$.

The situation in Theorems 9 and 11 is not so simple with respect to γ, however the following characterization is also useful. ∎

Theorem 12 *For* $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ *a left exact functor between toposes,*

$$(\forall E \textit{ in } \mathbf{ord}(\mathbf{E}))((CCD)(E) \Longrightarrow (CCD)(\Gamma E))$$

if and only if

$$(\forall X \textit{ in } \mathbf{E})((CCD)(\Gamma\mathcal{P}X)).$$

Proof. Power objects are always (CCD) . For the converse, invoke the Raney-Buchi presentation of (CCD) objects:

$$\begin{tikzcd} E \arrow[r, shift left=4] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4] & \mathcal{D}E \arrow[l] \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{P}(D|E|) \arrow[l, shift right=4] \arrow[l, shift left=4] \end{tikzcd}$$

(see [3]) and note that $\mathbf{ord}(\Gamma)$ preserves adjunction and full faithfulness. Then $(CCD)(\Gamma E)$ follows from Propositions 10 and 11 of [3]. ∎

It is clear that Γ preserves $\mathcal{D}$ (in the sense that γ is an isomorphism) if and only if Γ preserves $\mathcal{P}$. Hence:

Theorem 13 *Logical functors between toposes preserve* (CCD) *objects (and, of course, locales and complete objects.)* ∎

2 Geometric Morphisms

We turn now to a brief discussion of geometric morphisms.

Let $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ be geometric with inverse image $\Delta : \mathbf{S} \longrightarrow \mathbf{E}$, unit $\eta : 1_{\mathbf{S}} \longrightarrow \Gamma\Delta$ and counit $\varepsilon : \Delta\Gamma \longrightarrow 1_{\mathbf{E}}$. The previous considerations for Γ apply to Δ too and we write $\delta_S : \Delta\mathcal{D}_{\mathbf{S}}S \longrightarrow \mathcal{D}_{\mathbf{E}}\Delta S$ for $(\Delta{\downarrow_S})^{+\wedge}$. We refer to γ, respectively δ, as the logical comparison transformation associated to Γ, respectively Δ.

Lemma 14 *For* $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ *geometric,*

$$\begin{tikzcd} \Gamma\mathcal{D}_{\mathbf{E}}E \arrow[r, "\Gamma\mathcal{D}_{\mathbf{E}}\varepsilon_E"] \arrow[rrrd, "{\gamma_E}^{\vee}"'] & \Gamma\mathcal{D}_{\mathbf{E}}\Delta\Gamma E \arrow[r, "\Gamma(\delta^{+}_{\Gamma E})"] & \Gamma\Delta\mathcal{D}_{\mathbf{S}}\Gamma E \arrow[r, "(\eta_{\mathcal{D}_{\mathbf{S}}\Gamma E})^{+}"] & \mathcal{D}_{\mathbf{S}}\Gamma E \arrow[d, "\downarrow^{+}_{\Gamma E}"] \\ & & & \Gamma E \end{tikzcd}$$

Proof. Consider the following diagram:

$$
\begin{tikzcd}
\Gamma\mathcal{D}_{\mathbf{E}}E \arrow[r, "\Gamma\mathcal{D}_{\mathbf{E}}\varepsilon_E"] \arrow[d, "\Gamma\downarrow^+_E"'] & \Gamma\mathcal{D}_{\mathbf{E}}\Delta\Gamma E \arrow[r, "\Gamma(\delta^+_{\Gamma E})"] \arrow[dr, "\Gamma\downarrow^+_{\Delta\Gamma E}"'] & \Gamma\Delta\mathcal{D}_{\mathbf{S}}\Gamma E \arrow[r, "(\eta_{\mathcal{D}_{\mathbf{S}}\Gamma E})^+"] \arrow[d, "\Gamma\Delta\downarrow^+_{\Gamma E}"] & \mathcal{D}_{\mathbf{S}}\Gamma E \arrow[d, "\downarrow^+_{\Gamma E}"] \\
\Gamma E \arrow[rr, "\Gamma\varepsilon^+_E"'] \arrow[rrr, bend right, "1_{\Gamma E}"'] & & \Gamma\Delta\Gamma E \arrow[r, "(\eta_{\Gamma E})^+"'] & \Gamma E
\end{tikzcd}
$$

The right hand square commutes by naturality of η and application of $(\ )^+$. The middle triangle is an instance of Lemma 4 (followed by application of $(\ )^+$ and Γ). For the left hand trapezoid recall that $\downarrow$ is a natural transformation from the identity to $(\ )_!$ and consider the ε_E'th instance of the naturality square; take right adjoints, noting that $(\varepsilon_E)_!^+ = \mathcal{D}_{\mathbf{E}}\varepsilon_E$; apply Γ. The bottom region of the diagram commutes by application of one of the triangular identities for the $\Delta \dashv \Gamma$ adjunction. Finally,

$$\gamma_E{}^{\vee} = \Gamma{\downarrow_E}^{+\wedge\vee} = \Gamma{\downarrow_E}^{+}.$$

∎

Corollary 15 *If*

$$\Gamma\mathcal{D}_{\mathbf{E}}\Delta\Gamma E \xrightarrow{\Gamma(\delta^+_{\Gamma E})} \Gamma\Delta\mathcal{D}_{\mathbf{S}}\Gamma E \xrightarrow{(\eta_{\mathcal{D}_{\mathbf{S}}\Gamma E})^+} \mathcal{D}_{\mathbf{S}}\Gamma E$$

is ($(\ )_+$ of) an arrow of $\mathbf{ord}(\mathbf{S})$ *then*

$$\gamma_E = (\eta_{\mathcal{D}_{\mathbf{S}}\Gamma E})^+ \cdot \Gamma(\delta_{\Gamma E})^+ \cdot \Gamma\mathcal{D}_{\mathbf{E}}\varepsilon_E.$$

Proof. Given the hypothesis, the composite of the three arrows shown horizontally in the statement of Lemma 14 is ($(\ )_+$ of) an arrow of $\mathbf{ord}(\mathbf{S})$. By Lemma 1, $(\ )^{\vee}$ applied to it is given by postcomposition with $\downarrow^+_{\Gamma E}$. The result follows since $(\ )^{\vee}$ is an order isomorphism. ∎

Theorem 16 *Direct images of local homeomorphisms preserve (CCD) objects.*

Proof. Let $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ be a local homeomorphism (that is, assume Δ is logical.) In this case δ is an isomorphism, so $(\delta_{\Gamma E})^+ = \delta_{\Gamma E}^{-1}$. Also in this case, by [5, Sublemma p. 88] both η and ε have both left and right adjoints at all complete arguments. $\mathcal{D}_{\mathbf{S}}\Gamma E$ is complete so $(\eta_{\mathcal{D}_{\mathbf{S}}\Gamma E})^+$ is an arrow of $\mathbf{ord}(\mathbf{S})$ and the hypothesis of Corollary 15 is satisfied so that $\gamma_E = (\eta_{\mathcal{D}_{\mathbf{S}}\Gamma E})^+ \cdot \Gamma(\delta_{\Gamma E})^+ \cdot \Gamma\mathcal{D}_{\mathbf{E}}\varepsilon_E$. It suffices to show that each of the above three factors have left adjoints which have left adjoints for cocomplete E. The middle factor, being an isomorphism, trivially does. By the result of [5] mentioned above, $(\eta_{\mathcal{D}_{\mathbf{S}}\Gamma E})^+$ satisfies the condition for all E. For all E we have $\Gamma(\varepsilon_E)_! \dashv \Gamma\mathcal{D}_{\mathbf{E}}\varepsilon_E$. For (co)complete E we have $\lambda_E \dashv \varepsilon_E$, again by [5] and, in this case, $\Gamma(\lambda_E)_! \dashv \Gamma(\varepsilon_E)_!$ too. ∎

Corollary 17 *For all $\alpha : I \longrightarrow J$ in $\mathbf{S}$, the functor $\Pi_\alpha : \mathbf{S}/I \longrightarrow \mathbf{S}/J$ preserves (CCD) objects. In particular, $\Pi_I : \mathbf{S}/I \longrightarrow \mathbf{S}$ and $- \times - : \mathbf{S} \times \mathbf{S} \longrightarrow \mathbf{S}$ preserve (CCD) objects.* ∎

Corollary 18 *If L is a (CCD) object of $\mathbf{ord}(\mathbf{S})$ and I is in $\mathbf{S}$ then L^I is a (CCD) object of $\mathbf{ord}(\mathbf{S})$.* ∎

Corollary 19 *If L is a (CCD) object of $\mathbf{ord}(\mathbf{S})$ and X is in $\mathbf{ord}(\mathbf{S})$ then $\mathbf{ord}(\mathbf{S})(X, L)$ is a (CCD) object of $\mathbf{ord}(\mathbf{S})$.*

Proof. The canonical arrow $D|X| \longrightarrow X$ in $\mathbf{ord}(\mathbf{S})$ allows us to define another arrow $\mathbf{ord}(\mathbf{S})(X, L) \longrightarrow \mathbf{ord}(\mathbf{S})(D|X|, L) \cong L^{|X|}$, the inclusion of order preserving arrows in all arrows from X to L. Since L is complete the latter has both left and right adjoints, given by Kan extensions. Since $L^{|X|}$ is (CCD) the adjoint string exhibits $\mathbf{ord}(\mathbf{S})(X, L)$ as a (CCD) object. See [3, Proposition 10]. ∎

It is not true that direct images of essential geometric morphisms necessarily preserve (CCD) objects.

Counterexample 1 Consider $\Phi : \mathbf{set} \longrightarrow \mathbf{set^2}$ given by $\Phi X = (X \longrightarrow \mathbf{1})$. We have $\Delta \dashv \Gamma \dashv \Phi$ where $\Gamma(X : X_0 \longrightarrow X_1) = X_0$ and $\Delta X = (X \xrightarrow{=} X)$. Abbreviate $\mathcal{D}_{\mathbf{set^2}}$ by $\mathcal{D}_{\mathbf{2}}$ and $\mathcal{D}_{\mathbf{set}}$ by $\mathcal{D}$. Note that $\mathcal{D}_{\mathbf{2}}(X : X_0 \longrightarrow X_1)$ is given by $(\mathcal{D}_{\mathbf{2}})_0 \longrightarrow \mathcal{D}X_1$, where $(\mathcal{D}_{\mathbf{2}})_0$ is the set of all $S_0 \longrightarrow S_1$ with S_0 in $\mathcal{D}X_0$, S_1 in $\mathcal{D}X_1$ satisfying

$$\begin{tikzcd}
S_0 \arrow[r, tail] \arrow[d] & X_0 \arrow[d] \\
S_1 \arrow[r, tail] & X_1
\end{tikzcd}$$

and $\mathcal{D}_{\mathbf{2}}X(S_0 \longrightarrow S_1) = S_1$. In particular, $(\mathcal{D}_{\mathbf{2}}\Phi X) = \mathcal{D}_{\mathbf{2}}(X \longrightarrow \mathbf{1})$ can be regarded as $\mathbf{1} + \mathcal{D}X \longrightarrow \mathbf{2} = \{0 < 1\}$ with $\perp \in \mathbf{1}$ and $\perp < S$ for all S in $\mathcal{D}X$; $\perp \mapsto 0$; $S \mapsto 1$. Writing φ for the logical comparison transformation, $\varphi_X : \Phi\mathcal{D}X \longrightarrow \mathcal{D}_{\mathbf{2}}\Phi X$ can be shown to be

$$\begin{tikzcd}
\mathcal{D}X \arrow[r, "i"] \arrow[d] & \mathbf{1} + \mathcal{D}X \arrow[d] \\
\mathbf{1} \arrow[r, "1"'] & \mathbf{2}
\end{tikzcd}$$

where i is the injection into the sum. This i has a left adjoint $(\perp \mapsto \emptyset, S \mapsto S)$ which has a left adjoint $(\emptyset \mapsto \perp, \emptyset \neq S \mapsto S)$. However, the latter does not commute with the left adjoint to the left adjoint of 1 (which is 0) for non-empty X. Thus φ_X does not have a left adjoint in $\mathbf{ord}(\mathbf{set}^{\mathbf{2}})$ for any complete X in $\mathbf{ord}(\mathbf{set})$. ∎

We also note that the condition "γ_1 has a left adjoint which has a left adjoint", for Γ geometric, is precisely openness as defined in [4, p. 56]. Thus geometric morphisms which preserve (CCD) objects are open. However, neither the converse of this remark nor the converse of Theorem 16 is valid.

Counterexample 2 Let L be a locale in **set** which is not (CCD) . For example, take L to be the lattice of open subsets of the reals. Then the global sections functor $\Gamma : \mathbf{sh}(L) \longrightarrow \mathbf{set}$ is open, as is any geometric morphism with codomain **set**, but $\Gamma(\Omega_{\mathbf{sh}(L)}) \cong L$ implies that Γ does not preserve (CCD) objects. ∎

Counterexample 3 Consider the global sections (or domain) functor $\Gamma : \mathbf{set}^{\mathbf{2}} \longrightarrow \mathbf{set}$. It is evidently not a local homeomorphism. We show that it does preserve (CCD) objects. Let $X : X_0 \longrightarrow X_1$ be any object of $\mathbf{set}^{\mathbf{2}}$. It is well known that $\mathcal{P}_{\mathbf{set}^{\mathbf{2}}}X$ is the projection shown in the following lax limit diagram in $\mathbf{ord}(\mathbf{set})$.

$$\begin{tikzcd} & \mathcal{P}_{\mathbf{set}}X_0 \arrow[dd, "\exists_X"] & = \mathcal{P}_{\mathbf{set}}\Gamma X \\ \Gamma\mathcal{P}_{\mathbf{set^2}}X \arrow[ur, "\gamma_{DX}"] \arrow[dr, "\mathcal{P}_{\mathbf{set^2}}X"'] \arrow[r, phantom, "\geq"] & & \\ & \mathcal{P}_{\mathbf{set}}X_1 & \end{tikzcd}$$

(Here $\exists_X$ is direct image, the left adjoint of $\mathcal{P}_{\mathbf{set}}X = X^{-1}$. A routine calculation shows that the other projection is a component of the logical comparison transformation as suggested by the above notation.)

Thus $\Gamma\mathcal{P}_{\mathbf{set^2}}X = \{(S_0, S_1) \in \mathcal{P}X_0 \times \mathcal{P}X_1 | X(S_0) \subseteq S_1\}$. It is easy to show that the projections paired give $\Gamma\mathcal{P}_{\mathbf{set^2}}X \rightarrowtail \mathcal{P}X_0 \times \mathcal{P}X_1$ with left adjoint given by the formula: $(T_0, T_1) \mapsto (T_0, X(T_0) \bigcup (T_1)$ and right adjoint given by: $(T_0, T_1) \mapsto (T_0 \bigcap X^{-1}(T_1), T_1)$. By Corollary 17, $\mathcal{P}X_0 \times \mathcal{P}X_1$ is (CCD) . By [3, Proposition 10], $\Gamma\mathcal{P}_{\mathbf{set^2}}X$ is (CCD) . By Theorem 11, Γ preserves (CCD) objects. ∎

This last Counterexample together with Counterexample 1 also show that a right adjoint of a (CCD) preserving functor, even if the latter is geometric, need not be (CCD) preserving. Indeed, *domain* : $\mathbf{set^2} \longrightarrow \mathbf{set}$ has right adjoint the Φ of Counterexample 1.

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