Minimization and Minimal Realization in **Span**(**Graph**)

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Abstract

The context of this article is the program to study the bicategory of spans of graphs as an algebra of processes, with applications to concurrency theory. The objective here is to study functorial aspects of reachability, minimization and minimal realization. The compositionality of minimization has application to model-checking.

1 Introduction

In this article we extend the consideration of the bicategory of spans of graphs as an algebra of transition systems found in the work of Katis, Sabadini and Walters. They have shown that, with the operations also discussed in this article and certain other minimal constants, a discrete cartesian bicategory results. Moreover, this algebra is expressive of many examples in concurrency theory (see [KSW97b], [KSW97c], [KSW98], [KSW00]).

The main results here are a functorial minimization and minimal realization for labelled graphs. These are compositional and described by idempotent lax monads. We also find a functorial and compositional description of reachability by a colax comonad. The minimization is obtained using the concept of bisimulation from concurrency theory. We consider a behaviour functor that takes its values in (equivalence classes of) labelled trees (and more generally in forests). Thus, our objective here is the study of functorial relations among minimization, behaviour and minimal realization in variants of **Span**(**Graph**).

As an application, we provide a simple model checking algorithm that in many interesting cases avoids the state explosion problem for verification of distributed systems with parallel components [GSL96]. We apply it here to the case of the Dining Philosopher Problem, and note that there is an infinite class of examples with similar properties [KSW01].

By functorial minimization and minimal realization we understand the following general setup. For a pair of input and output objects A and B, which we may also view as interfaces, we require a category of machines from A to B. Classically these are some transducer model. Here they are spans of graphs (with a restriction on morphisms). These should be viewed as non-deterministic transition systems with left and right labellings specified by the span, that is the head graph defines states and transitions while the left and right legs of the span specify left and right labellings. Furthermore, the machines should have a composition providing a machine from A to C given one from A to B and one from B to C. Notice that this is distinct from the local composition in the category of machines from A to B. In fact we expect these two compositions to interact according to the *interchange law* so that we obtain a bicategory of machines. For a classical version, see [RSW98] in which the authors have also considered minimization and minimal realization in several bicategories of automata.

For minimization, a local minimization functor M is defined on each local machine category (from A to B say). It is an idempotent monad and the algebras (the minimized machines) determine a reflective subcategory. We expect this category to be essentially discrete, and it is so classically and below. Minimization usually requires a reachability restriction. The minimization process M will be an identity-on-objects lax functor (bicategory morphism) on the machine bicategory; that is, there will be a comparison from the composite of minimized machines to the minimization of their composite. M is locally an idempotent monad with discrete algebras and we get a lax monad in the sense of [CR91]. Thus, the minimized machines determine a locally discrete sub-bicategory of the machines.

There should be a behaviour functor E defined on each local machine category and taking values in a discrete behaviour category. This is classically a category of functions; below we take observational equivalence classes of trees. By minimal realization we understand the functorial construction of a universal machine realizing any behaviour, that is we expect the minimal realization process to be locally right adjoint to behaviour [Gog72, RSW98]. The minimal realization is denoted N (for Nerode). As with minimization, behaviour carries the structure of a lax functor, and minimized machines and behaviours are (bi-)equivalent. The situation we have been describing can be summed up in:

$$\begin{tikzcd}
& \mathbf{Mach} \arrow[dl, shift right, "M"'] \arrow[dr, shift left, "E"] & \\
\mathbf{MinMach} \arrow[ur, shift right, "I"'] \arrow[rr, shift left, "E'"] & & \mathbf{Beh} \arrow[ul, shift left, "N"] \arrow[ll, shift left, "N'"]
\end{tikzcd}$$

where the bicategory **MinMach** is locally discrete, **Beh** is locally discrete, M and E are locally left adjoint, I is locally an inclusion and E' and N' are an equivalence.

We wish to re-emphasize that the minimization process described here is locally functorial as well as compositional, and is expressible as a local adjunction. This gives a mathematically precise expression of the meaning of compositional minimal realization which has been a goal of other workers in model checking [AE98], [CLM89], [GSL96]. As was the case in the earlier [BSW96], the minimization process is essentially one which "kills the 2-cells", that is the morphisms between machines, so that the bicategory of minimized machines is locally discrete.

We begin in Section 2 with a study of a minimization for labelled graphs by generalizing the notion of bisimulation for transition systems [Arn92]. To make the minimization functorial we consider 'path lifting' morphisms of labelled graphs and then consider points and reachability. The resulting minimization process is an idempotent monad. In Section

3 we make the minimization compositional by defining a bicategory of spans of graphs on which our minimization is an idempotent lax monad [CR91]. In Section 4 we consider the possibility of null edges, labels and branching bisimulation [GW96] using spans of reflexive graphs and again obtain compositional minimization results.

In Section 5 we describe the setting for our minimal realization theory in considerable generality. We include a generalized version of our behaviour functor and show its relations with some toposes. We also show how reachability for spans of (multi-)pointed graphs is described by an idempotent comonad on each hom category. This provides an identity-on-objects comorphism of bicategories which is a colax comonad. We obtain a bicategory of coalgebras which are the spans of reachable graphs. This is similar to results found in [RSW98].

In Section 6 we consider a definition of behaviour for pointed labelled graphs. The labelled trees which arise in defining the local behaviour functors are those of interest in concurrency theory. The actual values of the local behaviour are bisimilarity classes of trees. We construct a minimal reachable pointed graph realizing such a class of trees. Behaviour followed by minimal realization defines an idempotent monad on each of the hom categories and we again have a lax monad. Our construction can be modified by adding final states to recover the classical minimal realization theory for finite automata.

In section 7 we consider the tensor structure and feedback on spans of graphs and their relationship to our minimization. We describe the resulting compositional model checking algorithm and apply it to the Dining Philosopher Problem. Finally, we extend the behaviour-minimal realization theory from trees to forests.

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2 Bisimulation and minimization

In this section we will use the concept of bisimulation, and in particular greatest self bisimulation to describe a minimization functor on a suitable category of labelled graphs viewed as a slight generalization of labelled transition systems. We find that minimization is adjoint to the inclusion of the minimized labelled graphs.

Transition systems are often used in modelling parallel processes.

Definition 1 *[Arn92] Let A be an alphabet, i.e. a finite set. A* transition system labelled *by A is $\mathcal{A} = (S, T, \alpha, \beta, \lambda)$ where S and T are sets of* states *and* transitions, *$\alpha : T \longrightarrow S$ and $\beta : T \longrightarrow S$ define the* source *and* target *of transitions, and the* labelling *$\lambda : T \longrightarrow A$*

makes the mapping $\langle \alpha, \lambda, \beta \rangle : T \longrightarrow S \times A \times S$ injective. A transition t with source s and target s' labelled by a is denoted $t : s \xrightarrow{a} s'$.

Arnold is ambiguous about whether the condition on λ is required. It means that there is only one transition with a given label between two states, and is sometimes expressed by saying that T is a subset of $S \times A \times S$ (e.g. [JNW96]).

In the sequel we will be mainly concerned with variants of the category of *(directed) graphs*, **Graph**. This category has as objects the parallel pairs of mappings $G : G_1 \underset{t_G}{\overset{s_G}{\rightrightarrows}} G_0$ which specify the set of edges G_1 and the set of vertices (or nodes) G_0, and the source and target mappings, s_G and t_G. A morphism $f : G \longrightarrow G'$ of graphs is a pair of mappings $(f_1 : G_1 \longrightarrow G'_1, f_0 : G_0 \longrightarrow G'_0)$ that is compatible with source and target, i.e. $f_0 s_G = s_{G'} f_1$ and $f_0 t_G = t_{G'} f_1$. To determine notation, we recall that for any object G of **Graph**, the *slice category* **Graph**$/G$ has as objects pairs (G', f) with G' a graph and $f : G' \longrightarrow G$ a morphism of graphs with codomain G. A morphism in **Graph**$/G$ from (G', f) to (G'', g) is a graph morphism $h : G' \longrightarrow G''$ satisfying $gh = f$.

An alphabet A may be viewed as a graph with a single node and an edge for each member of the alphabet. With this point of view, a *graph G labelled by A* is a pair (G, l) where G is a graph and l is a graph morphism from G to A (all nodes of G map to A's single node, edges of G map to members of the alphabet A). That is, we have an object of **Graph**$/A$. A label-preserving morphism of graphs is the same thing as a morphism of **Graph**$/A$. It is useful to note for the sequel that *a transition system labelled by A is exactly the special case of an object of* **Graph**$/A$ *satisfying the condition on λ in Definition 1.*

A *homomorphism* of transition systems labelled by an alphabet A is a morphism of **Graph**$/A$ between labelled transition systems. Indeed, the full subcategory of transition systems is reflective in **Graph**$/A$.

More generally, if A is *any* graph then an object of **Graph**$/A$ is a *graph labelled by A*. Here the label of an edge e of G must be an edge of A whose source and target are compatible under the labelling with the source and target of e. Thus, any object of **Graph**$/A$ may be viewed as a labelled transition system with a varying set of labels which does not necessarily satisfy the condition on λ, and a morphism of **Graph**$/A$ is a label-preserving homomorphism of labelled graphs.

A notion which has received considerable attention in the study of concurrency is *(strong) bisimulation.*

Definition 2 *[Arn92] Let $\mathcal{A}_1 = (S_1, T_1, \alpha_1, \beta_1, \lambda_1)$ and $\mathcal{A}_2 = (S_2, T_2, \alpha_2, \beta_2, \lambda_2)$ be two transition systems labelled by the same alphabet A. A* bisimulation*[Arn92] between $\mathcal{A}_1$ and $\mathcal{A}_2$ is a relation R from S_1 to S_2 such that*

(i a) The projection of R on S_1 is onto
(i b) The projection of R on S_2 is onto
(ii a) For every transition $t_1 : s_1 \xrightarrow{a} s_2$ in $\mathcal{A}_1$ and state s'_1 such that $s_1 R s'_1$ there is a transition $t_2 : s'_1 \xrightarrow{a} s'_2$ with $s_2 R s'_2$ in $\mathcal{A}_2$
(ii b) Similarly for transitions in $\mathcal{A}_2$.

It should be noted that there are several related concepts in the literature and that just defined is usually called 'strong bisimulation'. The idea is that any letter which may act on a state of the first transition system can also act on *any* related state in the second and yield a state of the second system related to the result state when the letter acts on the first system, and vice versa.

Bisimulations have been studied extensively, but our interest is primarily in the *greatest bisimulation* between a transition system and itself. It is easy to see that the union of a family of bisimulations is also a bisimulation. Further, for a labelled transition system $\mathcal{A}$, the identity relation on its states is clearly a bisimulation. Thus, the union of all bisimulations which contain the identity relation on S is the *greatest self bisimulation* $\sim_{\mathcal{A}}$ on the transition system $\mathcal{A}$. Moreover, $\sim_{\mathcal{A}}$ is an equivalence relation on S. For details we refer to [Arn92].

The definition of bisimulation and the considerations of the preceding paragraph can immediately be extended to **Graph**$/A$ for any labelling graph A.

Recall that we denote objects of **Graph**$/A$ by $G = (G, l)$ where $G \xrightarrow{l} A$ is the labelling.

Definition 3 *Let $G = (G, l)$ and $G' = (G', l')$ be objects of* **Graph**$/A$. *A* bisimulation *from G to G' is a relation R from G_0 to G'_0 which satisfies:*
(i a) The projection of R on G_0 is onto
(i b) The projection of R on G'_0 is onto
(ii a) For every edge $e : s_1 \xrightarrow{a} s_2$ in G and node s'_1 such that $s_1 R s'_1$ there is an edge $e' : s'_1 \xrightarrow{a} s'_2$ in G' with $s_2 R s'_2$
(ii b) Similarly for edges in G'.

This definition clearly generalizes Definition 2, and in the sequel when we write 'bisimulation' we mean it in the sense just given.

Lemma 4 *(i) The union of bisimulations is a bisimulation, as is the identity relation.*
(ii) There is a largest self bisimulation $\sim_G$ on any G in **Graph**$/A$. *$\sim_G$ is an equivalence relation on G.* ∎

The proofs are easy and follow [Arn92].

The quotient set of the states S of a labelled transition system $\mathcal{A}$ by the largest self bisimulation $\sim_{\mathcal{A}}$ is the state set for a quotient transition system with the same labels [Arn92]. Though objects of **Graph**$/A$ allow multiple edges with the same label between two states, it is still easy to define a suitable quotient. We first note that $\sim_G$ induces an equivalence relation $\sim_G^e$ on the edges G_1 of G defined by $e_1 \sim_G^e e_2$ iff e_1 and e_2 have the same label and their sources and targets are $\sim_G$ related (and note that this does not require that the sources are different).

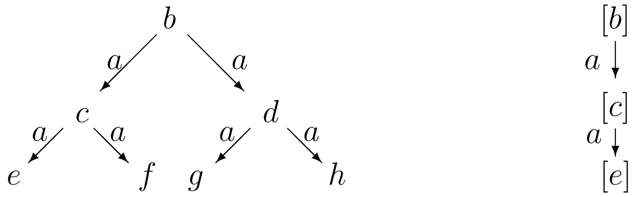
Definition 5 *Let $G = (G, l)$ be an object of* **Graph**$/A$. *The* minimization *of G is the labelled graph $MG = (MG, l_M)$ in* **Graph**$/A$ *where MG is the graph whose:*
- nodes MG_0 are the quotient set of the nodes G_0 of G by $\sim_G$;
- edges MG_1 are the quotient of the edges G_1 of G by $\sim_G^e$;
- $(l_M)_0([g]) = l_0(g)$ where $[g]$ is the $\sim_G$ class of the node g in G.
- $(l_M)_1([e]) = l_1(e)$ where $[e]$ is the $\sim_G^e$ class of the edge e in G_1.
The projection morphism *$\pi_G : G \longrightarrow MG$ in* **Graph**$/A$ *is defined on nodes g and edges e of G by $(\pi_G)_0(g) = [g]$, $(\pi_G)_1(e) = [e]$.*

Notice that the definition of $\sim_G^e$ implies that there is at most one edge labelled a from $[g]$ to $[g']$ in MG, so MG is actually a transition system.

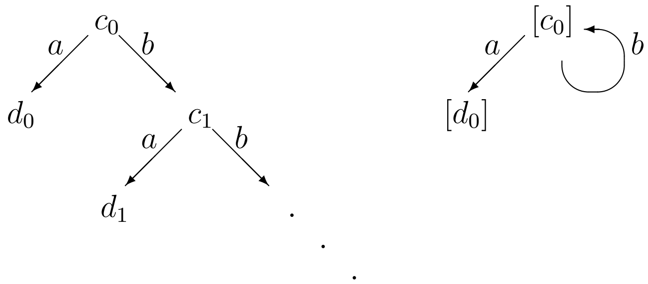
Proposition 6 *MG and the projection morphism $\pi_G : G \longrightarrow MG$ are well-defined.*

Proof. We need to show that the labelling l_M is independent of representatives and that π_G preserves labelling. The latter is obvious when the former is proved, so suppose that $g \sim_G g'$. If there is an edge labelled a leaving g say, then there is also one leaving g', hence both g and g' are labelled by the same node of the labelling graph A, that is $l_0(g) = l_0(g')$. If neither g nor g' has any edges leaving, then they must be identified by $\sim_G$ since (iia) and (iib) are trivially satisfied. Thus l_M is well-defined on nodes. By the definition of $\sim_G^e$ it is clear that if $e_1 \sim_G^e e_2$ then they have the same label, so l_M is well-defined on edges. ∎

Example 7 Consider the depth two binary tree with all edges labelled a (the left illustration below). Its minimization is the depth two unary tree (on the right below):



Consider the infinite ‘comb’ at left below. Its minimization is the labelled graph to the right.



We also note that:

Proposition 8 *The identity relation Δ_{MG_0} on MG_0 is the largest self bisimulation $\sim_{MG}$ on MG, and $\sim^e_{MG}$ is the identity relation on MG_1. Hence the projection $MG \stackrel{\pi_{MG}}{\longrightarrow} MMG$ is an identity arrow, and minimization is an idempotent process.*

Proof. It is clear that the largest bisimulation on MG is the identity relation: this relation is a bisimulation and unequal nodes of MG cannot be related by $\sim_{MG}$. Indeed if unequal nodes could be made bisimilar, then $\sim_G$ would have already related their members. It follows that if two edges in MG_1 are related by $\sim^e_{MG}$, they must have the same source, label and target (since $\sim_M G$ is the identity), and any two such edges were already related by $\sim^e_G$, so the two edges are identical. Thus the quotient arrow $\pi_{MG} : MG \longrightarrow MMG = MG/\sim_{M(G)}= MG$ is the identity on the nodes and edges of MG. ∎

Remark 9 In the case of a *finite* labelled graph there is an algorithm to compute the largest self bisimulation on a labelled graph G. It begins with the total relation on the nodes of G and successively separates nodes which cannot be related by any bisimulation. The iterative step is as follows:

> for any edge $g \xrightarrow{a} g'$, if every edge labelled a emanating from a node h has target not equivalent to g', then g and h are not equivalent.

The algorithm terminates when the relation stabilizes at the largest self bisimulation. It relies on finiteness for termination, is detailed in [KRSW00] and is similar to the algorithm for maximal self bisimulation in [Arn92].

To extend minimization to arrows we will need a suitable restriction:

Definition 10 *Let G and G' be labelled graphs in* **Graph**/A. *A* **Graph**/A *morphism $\varphi : G \longrightarrow G'$ is called* path lifting *if whenever $e' : \varphi(g) \xrightarrow{a} g'$ is an an a-labelled edge in G' with source $\varphi(g)$ and target g', then there is an (a-labelled) edge $e : g \xrightarrow{a} g_1$ in G which satisfies $\varphi(e) = e'$ and hence $\varphi(g_1) = g'$.*

Path lifting morphisms have been considered by several authors (see [JNW96] or [BF00] and references there). Clearly, a graph isomorphism is path lifting and path lifting morphisms compose, so there is a category of labelled graphs and path-lifting morphisms. It is an easy exercise to show that for labelled graphs G, G' a relation $r : R \hookrightarrow G_0 \times G'_0$ on their node sets is a bisimulation if and only if

$$p_0 r : G \longleftarrow \hat{R} \longrightarrow G' : p_1 r$$

is a span of path lifting morphisms, where $\hat{R}$ is the graph with nodes R and an edge $(g_1, g'_1) \xrightarrow{a} (g_2, g'_2)$ exactly when $g_1 \xrightarrow{a} g_2$ and $g'_1 \xrightarrow{a} g'_2$. Moreover, a path lifting morphism $\varphi : G \longrightarrow G'$ provides the path lifting span $1_G : G \longleftarrow G \longrightarrow G' : \varphi$ which is isomorphic to one of the sort just described, and hence provides a *functional bisimulation* from G to G'.

Important for us is:

Proposition 11 *The projection morphism $\pi_G : G \longrightarrow MG$ is a path lifting arrow in* **Graph**/A.

Proof. This is immediate: any edge $[e]$ in MG is the image under π_G of an edge in G with the same label. ∎

For the study of minimization and minimal realization we are interested in *reachable pointed graphs*. Notice that the construction of MG above does not require either an (initial) point or reachability. The definition of transition system in [JNW96], for example, includes an 'initial state' as part of the data, and morphisms preserve initial states. Indeed, pointed graphs and their morphisms are appropriate for our considerations. Moreover, as is usual in automata theory, we require reachability for our compositional minimization. Note that this was not needed above, and there is a similar situation in [RSW98].

A *pointed graph* is a pair (G, g_0) where G is an object of **Graph** and g_0 is an element of G_0. A *morphism of pointed graphs* is a **Graph** morphism that preserves the specified element, and we call the resulting category Pt**Graph**. This category may also be described as $N/$**Graph** where N is the graph with one node and no edges (see Section 5). A pointed graph (G, g_0) is *reachable* if there is a path of edges in G_1 from g_0 to any node in G_0. We will have more to say about these concepts in Section 5. For now let us denote the full subcategory of Pt**Graph** whose objects are reachable by RchPt**Graph**. For any pointed graph G its *reachable part* is the subgraph consisting of reachable nodes (and edges) that we denote by G_R. We write $\rho_G : G_R \longrightarrow G$ for the inclusion.

Notation The (non-full) subcategory of RchPt**Graph** with the same objects as RchPt**Graph** and morphisms which are path lifting graph morphisms in RchPt**Graph** is denoted by PL**Graph**.

Let $(A, *)$ be a pointed graph. We will write $\mathsf{PL}\mathbf{Graph}_A$ for the category in which an object $((G, g_0), l)$ is a pointed graph morphism $l : (G, g_0) \longrightarrow (A, *)$ from a reachable pointed graph (G, g_0) to $(A, *)$ and an arrow $\varphi : ((G, g_0), l) \longrightarrow ((G', g_0'), l')$ is a path lifting morphism $\varphi : (G, g_0) \longrightarrow (G', g_0')$ of reachable pointed graphs such that $l'\varphi = l$. Thus:

Notation $\mathsf{PL}\mathbf{Graph}_A$ has reachable pointed graphs (G, l) labelled by $(A, *)$ as objects. Its morphisms are path lifting and label-preserving.

Since the domains of objects of $\mathsf{PL}\mathbf{Graph}_A$ are pointed, there is no harm in making the same assumption for our labelling graph A. The results of this section do not depend on this assumption, but our compositional minimization theory will do so. Note that we do not require $(A, *)$ to be reachable nor l to be path lifting.

We note the following.

Lemma 12 *The arrows of* $\mathsf{PL}\mathbf{Graph}_A$ *are onto on nodes and edges, and so all arrows of* $\mathsf{PL}\mathbf{Graph}_A$ *are epi.*

Proof. Let $\varphi : (G, l) \longrightarrow (G', l')$ be a morphism of $\mathsf{PL}\mathbf{Graph}_A$ and g' a node of G'. Since G' is reachable, there is a path of edges in G', $e_1', \ldots, e_n'$ labelled $a_1, \ldots, a_n$, say, from the point of G' to g'. Since the point of G' is the image of the point of G, the path lifting property provides a path of edges $e_1, \ldots, e_n$ (and labelled $a_1, \ldots, a_n$) which φ maps to the given path.

Thus g' is in the image of φ. Now if e' is any edge from g', the path lifting property shows that it is in the image of φ.

The epimorphism cancellation property for all arrows of $\mathsf{PL}\mathbf{Graph}_A$ follows immediately. ∎

Indeed, as the proof shows, any path lifting morphism whose codomain is a reachable graph is onto on nodes and edges.

As the next two lemmas show, morphisms of $\mathsf{PL}\mathbf{Graph}_A$ behave well with respect to bisimulation.

Lemma 13 *If $G \xrightarrow{\varphi} H$ is a morphism of $\mathsf{PL}\mathbf{Graph}_A$ and R_G is a bisimulation on G, then the relation R_H on H defined by hR_Hh' iff $\exists g, g'$ such that gR_Gg' & $h = \varphi(g)$ & $h' = \varphi(g')$ is a bisimulation on H.*

Proof. First, for any node h of H, there is a g such that $h = \varphi(g)$ and a g' such that gR_Gg'. Hence $hR_H\varphi(g')$, so (ia) is satisfied for R_H; similarly for (ib).

Now suppose that $e' : h \xrightarrow{a} h_1$ and hR_Hh'. Since φ is onto we have g with $\varphi(g) = h$, g_1 and $e : g \xrightarrow{a} g_1$ with $\varphi(e) = e'$ and g' with gR_Gg' and $\varphi(g') = h'$. By the path lifting property and (*iia*) for R_G, there is then an edge $e_1 : g' \xrightarrow{a} g'_1$ with $g_1R_Gg'_1$. Thus $\varphi(g'_1)R_H\varphi(g_1)$ and $\varphi(e_1) : h' \xrightarrow{a} \varphi(g'_1)$. So (iia) is satisfied for R_H; similarly for (iib). ∎

A more conceptual proof of the preceding lemma is provided by the observations that a self-bisimulation R on G is the same thing as a jointly monic span of path lifting arrows $G \longleftarrow R \longrightarrow G$ and that path lifting arrows compose.

Lemma 14 *If $G \xrightarrow{\varphi} H$ is a morphism of $\mathsf{PL}\mathbf{Graph}_A$ and R_H is a bisimulation on H, then the relation R_G defined by gR_Gg' iff $\varphi(g)R_H\varphi(g')$ is a bisimulation on G.*

Proof. First, for any node g of G, there is an h such that $\varphi(g)R_Hh$, but $h = \varphi(g')$ for some node g' of G, hence gR_Gg', so (ia) is satisfied for R_G; similarly for (ib).

Now suppose that $g \xrightarrow{a} g_1$ and gR_Gg' in G. Thus $\varphi(g) \xrightarrow{a} \varphi(g_1)$ and $\varphi(g)R_H\varphi(g')$ so there is $e' : \varphi(g') \xrightarrow{a} h$ in H with $\varphi(g_1)R_Hh$. But $h = \varphi(g'_1)$ for some g'_1 in G so $g_1R_Gg'_1$ and since φ is path lifting there is an edge $e : g' \xrightarrow{a} g'_1$ in G with $\varphi(e) = e'$. Hence (iia) is satisfied; similarly for (iib). ∎

We are now in a position to define M on arrows of $\mathsf{PL}\mathbf{Graph}_A$.

Definition 15 *Let* $G \xrightarrow{\varphi} H$ *be a morphism of* $\mathsf{PL}\mathbf{Graph}_A$*. We define* $MG \xrightarrow{M\varphi} MH$ *on nodes of* G *by* $(M\varphi)_0([g]) = [\varphi(g)]$ *and on edges by* $(M\varphi)_1([e]) = [\varphi(e)]$.

To see that $M\varphi$ is well-defined, suppose that $g \sim_G g'$. By Lemma 13 there is a bisimulation R_H on H with $\varphi(g)R_H\varphi(g')$, so it follows that $\varphi(g) \sim_H \varphi(g')$. Similarly, φ respects edges which are identified by $\sim^e_G$. Since φ is path lifting, so is $M\varphi$.

Proposition 16 *With the definitions above,* $M : \mathsf{PL}\mathbf{Graph}_A \longrightarrow \mathsf{PL}\mathbf{Graph}_A$ *is a functor. The arrows* π_G *are the components of a natural transformation* $\pi : 1_{\mathsf{PL}\mathbf{Graph}_A} \longrightarrow M$.

Proof. The first point is clear from the definition of M on arrows, for if $G \xrightarrow{\varphi} G' \xrightarrow{\psi} G''$ in $\mathsf{PL}\mathbf{Graph}_A$ then

$$M\psi M\varphi([g]) = M\psi([\varphi(g)]) = [\psi\varphi(g)] = M(\psi\varphi)([g])$$

Naturality of π is equally immediate. ∎

Moreover,

Proposition 17 *If* $G \xrightarrow{\varphi} H$ *is a morphism of* $\mathsf{PL}\mathbf{Graph}_A$ *then* $MG \xrightarrow[\cong]{M\varphi} MH$.

Proof. By Lemma 12, since $M\varphi$ path lifting, it is onto. To see that $M\varphi$ is an isomorphism we need only observe that it is injective. So suppose that $M\varphi([g]) = M\varphi([g'])$, i.e. $[\varphi(g)] = [\varphi(g')]$ or $\varphi(g) \sim_H \varphi(g')$. By Lemma 14 there is a bisimulation R_G on G such that gR_Gg', hence $g \sim_G g'$. Thus $[g] = [g']$. ∎

The following are crucial for the sequel:

Lemma 18 *If* $G \xrightarrow{\alpha} G$ *is a morphism in* $\mathsf{PL}\mathbf{Graph}_A$*, then the relation* R_α *defined on* G_0 *by* $gR_\alpha g'$ *iff* $\alpha(g) = g'$ *is a bisimulation on* G.

Proof. Since R_α is the graph of an onto function (ia) and (ib) of Definition 3 are immediate. Since a morphism is label-preserving, whenever $e : g \xrightarrow{a} g_1$ is an edge labelled a and $gR_\alpha g'$ we have $g' = \alpha(g)$ and $\alpha(e) : g' \xrightarrow{a} \alpha(g_1)$, so $g_1R_\alpha\alpha(g_1)$ and (iia) is satisfied. For (iib), suppose $e' : g' \xrightarrow{a} g'_1$ and $gR_\alpha g'$, so $g' = \alpha(g)$. Since α is onto on edges, there is $e : g \xrightarrow{a} g_1$ with $\alpha(e) = e'$. Thus $g'_1 = \alpha(g_1)$ and $g_1R_\alpha g'_1$ as required for (iib). ∎

Corollary 19 *The only automorphism of MG in $\mathsf{PL}\mathbf{Graph}_A$ is the identity.*

Proof. Suppose $MG \xrightarrow{\alpha} MG$ is an automorphism. By the previous lemma its graph is a bisimulation R_α and is contained in the largest bisimulation on MG, namely the identity relation by Proposition 8. The only graph of a function contained in the identity relation is the identity relation. ∎

Lemma 20 *If $G \underset{\beta}{\overset{\alpha}{\rightrightarrows}} H$ are in $\mathsf{PL}\mathbf{Graph}_A$, then $M\alpha = M\beta$.*

Proof. By Proposition 17 $M\alpha$ is invertible. Thus $(M\alpha)^{-1}M\beta$ is an automorphism of MG, hence it is the identity by Corollary 19 and the result follows. ∎

Lemma 21 *If $H \underset{\beta}{\overset{\alpha}{\rightrightarrows}} MG$ are in $\mathsf{PL}\mathbf{Graph}_A$, then $\alpha = \beta$.*

Proof. Consider

$$\begin{tikzcd} H \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\pi_H"'] & MG \arrow[d, "\pi_{MG}"] \\ MH \arrow[r, "M\alpha = M\beta"'] & MMG \end{tikzcd}$$

Since π_{MG} is the identity by Proposition 8, naturality of π and Lemma 20 give $\alpha = M\alpha\pi_H = M\beta\pi_H = \beta$. ∎

Corollary 22 *The projection $G \xrightarrow{\pi_G} MG$ is the unique arrow from G to MG in $\mathsf{PL}\mathbf{Graph}_A$, and hence $\pi_{MG} = M\pi_G$.* ∎

Lemma 23 *If $G \xrightarrow{\alpha} H$ is in $\mathsf{PL}\mathbf{Graph}_A$, there is a unique $H \xrightarrow{\beta} MG$ such that $\beta\alpha = \pi_G$.*

$$\begin{tikzcd} G \arrow[rr, "\alpha"] \arrow[dr, "\pi_G"'] & & H \arrow[dl, "\beta"] \\ & MG & \end{tikzcd}$$

Proof. Define $\beta = (M\alpha)^{-1}\pi_H$, then $\beta\alpha = \pi_G$ by Corollary 22. If also $\beta'\alpha = \pi_G$, then $\beta = \beta'$ since α is epi by Lemma 12. ∎

By Corollary 22, the pair (M, π) is the data for an idempotent monad $M : \mathsf{PL}\mathbf{Graph}_A \longrightarrow \mathsf{PL}\mathbf{Graph}_A$.

Notation The full subcategory of $\mathsf{PL}\mathbf{Graph}_A$ whose objects are the image of the idempotent monad M is denoted $\mathsf{MPL}\mathbf{Graph}_A$. We write M_A for the minimization functor $\mathsf{M}_A : \mathsf{PL}\mathbf{Graph}_A \longrightarrow \mathsf{MPL}\mathbf{Graph}_A$ and $\mathsf{I}_A : \mathsf{MPL}\mathbf{Graph}_A \longrightarrow \mathsf{PL}\mathbf{Graph}_A$ for the inclusion, so $M = \mathsf{I}_A\mathsf{M}_A$.

By Lemma 21, $\mathsf{MPL}\mathbf{Graph}_A$ is essentially discrete, and we sum up with:

Proposition 24 $\mathsf{MPL}\mathbf{Graph}_A$ *is an essentially discrete full subcategory of* $\mathsf{PL}\mathbf{Graph}_A$. *There is an adjunction:*

$$\mathsf{M}_A \dashv \mathsf{I}_A : \mathsf{MPL}\mathbf{Graph}_A \longrightarrow \mathsf{PL}\mathbf{Graph}_A$$

∎

This is the local part of the left hand adjunction in the diagram of the Introduction.

3 Compositional minimization

In this section we consider the relationship of our minimization theory with composition of transition systems. The composition we use here is based on that for the bicategory of spans of graphs studied as an algebra of transition systems in [KSW97b]. We find that there is a *comparison* from the composite of minimizations to the minimization of a composite. Indeed, we find that minimization is an idempotent *lax monad* [CR91] on our bicategory.

To begin, note that if A and B are pointed graphs (and usually we will not mention the point), then an object $(G, \langle l, r\rangle)$ of $\mathsf{PL}\mathbf{Graph}_{A\times B}$ has an underlying span of graphs:

$$A \xleftarrow{l} G \xrightarrow{r} B.$$

G is called the 'head' graph of the span, and l and r are its left and right legs.

Our first objective is to define a bicategory $\mathsf{SpPL}\mathbf{Graph}$ whose objects are those of $\mathsf{Pt}\mathbf{Graph}$, and whose hom category for objects A and B is $\mathsf{PL}\mathbf{Graph}_{A\times B}$. That is, the 1-cells of $\mathsf{SpPL}\mathbf{Graph}$ from A to B are to be objects of $\mathsf{PL}\mathbf{Graph}_{A\times B}$ which it is convenient to think of as spans, and the 2-cells are morphisms of $\mathsf{PL}\mathbf{Graph}_{A\times B}$.

The identity arrow from A to A in $\mathsf{SpPL}\mathbf{Graph}$ is $(A_R, \langle \rho_A, \rho_A \rangle)$ where A_R is the reachable nodes of A and ρ_A is the inclusion $A_R \longrightarrow A$. Notice that this is the reason for the requirement mentioned in the last section that a labelling graph (an object of $\mathsf{SpPL}\mathbf{Graph}$) be pointed, for without this requirement there would be no candidate for an identity arrow. Furthermore, for an object (G, l) of $\mathsf{PL}\mathbf{Graph}_A$ we did not require the labelling graph A to be reachable. The reason for this is the following: Even if A and B are reachable it is not usually the case that $A \times B$ is reachable, but we have defined our hom categories are of the form $\mathsf{PL}\mathbf{Graph}_{A\times B}$ so that we can apply the minimization theory of the previous section directly to them. It is true that we could have taken only reachable pointed graphs as the objects of our bicategory, and then defined the hom categories to be spans of these. However doing so would require some reworking of the results above (which we consider to be of independent interest) before application of them in this section and would not avoid the main considerations in the next paragraph. The interested reader can provide the details.

Next we consider the composition of 1-cells. Suppose that $(G, \langle l, r \rangle)$ and $(G', \langle l', r' \rangle)$ are 1-cells in $\mathsf{SpPL}\mathbf{Graph}(A, B)$ and $\mathsf{SpPL}\mathbf{Graph}(B, C)$ respectively. Thus we have underlying spans $G = A \xleftarrow{l} G \xrightarrow{r} B$ and $G_1 = B \xleftarrow{l'} G' \xrightarrow{r'} C$. Their composite $G'' = G'G$ must be in $\mathsf{SpPL}\mathbf{Graph}(A, C)$, so we need a reachable pointed graph G'' and a span $G'' = A \xleftarrow{l''} G'' \xrightarrow{r''} C$ whose legs are morphisms of pointed graphs. Now pullbacks exist in the category $\mathsf{Pt}\mathbf{Graph}$ of pointed graphs – just take the $\mathbf{Graph}$ pullback with the point given by the pair of original points. However, the $\mathsf{Pt}\mathbf{Graph}$ pullback of reachable pointed graphs need not be reachable. Thus, we define $G'' = (G \times_B G')_R$ to be the reachable part of the pointed graph pullback $G \times_B G'$ of r along l'. Denoting the inclusion $i : G'' \longrightarrow G \times_B G'$, we define $l'' = lpi$ and $r'' = r'p'i$ where p and p' are projections. This composition respects the identity 1-cells defined above. That requires the observation that

$$(A_R \times_A G)_R \cong (A \times_A G)_R \cong G_R = G$$

for reachable pointed graphs G, whose easy proof we leave to the reader. It is also easy to see directly that this composition is associative up to isomorphism. These remarks also follow from the fact that spans of pointed graphs with reachable head are the coalgebra bicategory for the reachability colax comonad on spans of pointed graphs (see Section 5).

To complete the definition of $\mathsf{SpPL}\mathbf{Graph}$ we need to consider horizontal composition of 2-cells. Fortunately, the fact that they are path lifting means that this operation is essentially inherited from that for spans of graphs:

Lemma 25 *Suppose that $\psi : G_0 \longrightarrow G_1$ is an arrow of spans of graphs from $G_0 = A \xleftarrow{l_0} G_0 \xrightarrow{r_0} B$ to $G_1 = A \xleftarrow{l_1} G_1 \xrightarrow{r_1} B$ and $\phi : G_2 \longrightarrow G_3$ is an arrow of spans of graphs from*

$G_2 = B \xleftarrow{l_2} G_2 \xrightarrow{r_2} C$ *to* $G_3 = B \xleftarrow{l_3} G_3 \xrightarrow{r_3} C$. *If* ψ *and* ϕ *are path lifting then so is their* **Span**(**Graph**) *horizontal composite* $\phi \circ \psi$.

Proof. For simplicity we assume that A and C have only a single node. Suppose that $(e_3, e_1) : \phi \circ \psi(g_2, g_0) \xrightarrow{a} (g_3, g_1)$ is an edge labelled a, i.e. there are edges $e_3 : \phi g_2 \xrightarrow{a} g_3$ and $e_1 : \psi g_0 \xrightarrow{a} g_1$. But path lifting for ϕ and ψ then provides $e_2 : g_2 \xrightarrow{a} g_5$ and $e_1 : g_1 \xrightarrow{a} g_4$ in G' and G respectively which ϕ and ψ send to e_3 and e_1. Thus there is an edge $(e_2, e_0) : (g_2, g_0) \xrightarrow{a} (g_5, g_4)$ of $G'G$ which $\phi \circ \psi$ sends to the original (e_3, e_1). ∎

Now the composition of 1-cells in SpPL**Graph** differs from the composition of spans of pointed graphs by the restriction to the reachable part of the ordinary composite, but the important point is that the path lifting property is maintained when taking reachable parts and so we have:

Proposition 26 *For 2-cells* φ *in* SpPL**Graph**(A, B) *and* ψ *in* SpPL**Graph**(B, C)*, their* **Span**(**Graph**) *horizontal composite* $\varphi \circ \psi$ *restricted to reachable parts is in fact in the subcategory* SpPL**Graph**(A, C).

Proof. Follows from the preceding paragraph and Lemma 25. ∎

We note further that the interchange law inherited from **Span**(**Graph**) holds for our proposed vertical and horizontal composites, so this completes our definition.

Notation The bicategory SpPL**Graph** has pointed graphs as objects, and for objects A and B the hom category is $\mathsf{PL}\mathbf{Graph}_{A\times B}$.

The compositional minimization which we seek is defined locally by the minimization functors in the previous section. While the composite of minimizations need not equal the minimization of the composite, there is a comparison that makes the local minimization lax functorial:

Proposition 27 *The functors* $M_{(A,B)} : \mathsf{SpPL}\mathbf{Graph}(A, B) \longrightarrow \mathsf{SpPL}\mathbf{Graph}(A, B)$ *defined by* $M_{(A,B)} = M : \mathsf{PL}\mathbf{Graph}_{A\times B} \longrightarrow \mathsf{PL}\mathbf{Graph}_{A\times B}$ *determine an idempotent lax monad on* SpPL**Graph**.

Proof. A lax monad [CR91] is an identity on objects morphism of bicategories (lax functor) that is locally a monad and satisfies several equations. When the local monads are idempotent, Proposition 42 of [RSW98] simplifies the equations. In the case at hand, we need to define a lax functor

$$(M, \tau) : \mathsf{SpPL}\mathbf{Graph} \longrightarrow \mathsf{SpPL}\mathbf{Graph}$$

which is locally an idempotent monad, has identity components τ_A given by the local monad units and satisfies the equation at the end of this proposition making τ compatible with the monad units. By the results of the previous section, the minimization functors $M_{(A,B)}$ already provide local idempotent monads.

We next consider the lax functor structure. Let

$$G = A \xleftarrow{l} G \xrightarrow{r} B \quad \text{and} \quad G' = B \xleftarrow{l'} G' \xrightarrow{r'} C$$

be spans defining arrows of $\mathsf{SpPL}\mathbf{Graph}$. We require a comparison arrow $\tau_{G'G}$ from $MG'MG$ to $MG'G$. We define this on a pair of representative nodes $[g'], [g]$ from MG and MG' which appear in the composite $MG'MG$ by noting that then (g', g) is in the composite $G'G$, so we let $\tau_{G'G}([g'], [g]) = [(g', g)]$. A similar formula defines $\tau_{G'G}$ on edges. Checking that this $\tau_{G'G}$ is well-defined and path lifting is done easily by recalling Propositions 6 and 11. We also need, for each object A of $\mathsf{SpPL}\mathbf{Graph}$, an identity component arrow $\tau_A : 1_A \longrightarrow M_{(A,A)}(1_A)$ (remembering that M is identity on objects so that $1_{MA} = 1_A$). For these τ_A we take the (path lifting) projections π_{A_R} from Section 2, the units for the local monads. With these definitions, it is easy to see that (M, τ) satisfies the requirements to be a lax functor.

It remains to check only compatibility of τ with the monad units, that is that the following commutes:

$$\begin{tikzcd}
G'G \arrow[rr, "\pi_{G'G}"] \arrow[dr, "\pi_{G'} \circ \pi_G"'] & & M(G'G) \\
& MG'MG \arrow[ur, "\tau_{G'G}"'] &
\end{tikzcd}$$

but this is another exercise in equating equivalence classes. ∎

Notation We denote by $\mathsf{SpMPL}\mathbf{Graph}$ the algebras for the lax monad M on $\mathsf{SpPL}\mathbf{Graph}$.

The 1-cells in $\mathsf{SpMPL}\mathbf{Graph}(A, B)$ are the full subcategory of $\mathsf{SpPL}\mathbf{Graph}(A, B)$ consisting of the minimized graphs $\mathsf{MPL}\mathbf{Graph}_{A\times B}$. Note however that the composite of MG and MG' in $\mathsf{SpMPL}\mathbf{Graph}$ is *not* their $\mathsf{SpPL}\mathbf{Graph}$ composite, but rather its minimization (see [CR91] and [RSW98]).

The point of Proposition 27 is that minimization is lax compositional. Thus, in order to compute the minimization of a composite of spans it is enough to compute the minimizations of the factors and then minimize the composite of these, that is

$$M(G'G) = MM(G'G) \xleftarrow{\cong} M(MG'MG)$$

using idempotence, the comparison from above, and that $M(\varphi)$ is always iso.

Expressions in the algebra $\mathsf{SpPL}\mathbf{Graph}$ can be used to model synchronous distributed systems. Minimization of the expression can be efficiently evaluated by minimizing sub-expressions which is useful in detecting deadlock in the original system. In order to discuss asynchronous systems such as the Dining Philosopher Problem we require an extension of the results above to the case of reflexive graphs which we pursue in the next section.

4 Reflexive graphs and branching bisimulation

For this section we require a specified 'null' edge at each node of any graph and require that morphisms, including labellings, preserve these. Thus, the label of the null edge is required to be the null label. The null label is generically denoted by '$-$' below. We will outline results for reflexive graphs that are analogous to those in Sections 2 and 3.

The specification of the null edge at each node of a graph $G : G_1 \underset{t_G}{\overset{s_G}{\rightrightarrows}} G_0$ is a mapping $G_0 \xrightarrow{r_G} G_1$ such that $s_G r_G = 1_{G_0} = t_G r_G$; thus G is a *reflexive graph*. Morphisms of reflexive graphs are exactly graph morphisms that preserve the reflexivity r_G, and hence the null edges. We denote the category of reflexive graphs and their morphisms by **RfGraph**. There is a forgetful functor **RfGraph** $\longrightarrow$ **Graph** which has a left adjoint (given by freely adjoining null edges). There is no difficulty in defining the concept of *pointed* reflexive graph and their morphisms, nor with *reachable* pointed reflexive graph. We denote the category of reachable pointed reflexive graphs and their morphisms by $\mathsf{RchPt}\mathbf{RfGraph}$.

When we consider a reflexive graph labelled by a reflexive graph A, that is an object (G, l) of $\mathbf{RfGraph}/A$, it is possible that a non-null edge may be labelled by a null edge. The intuition here is that the machine may 'idle' by consuming no input or producing no output through state transitions labelled with the null edge, so we are considering asynchronous systems. We will have to take account of this possibility by modifying the definitions of bisimulation and path-lifting morphisms needed for minimization. We do this by introducing a notion of *branching bisimulation* similar to that considered by van Glabbeek and Weijland [GW96], and reflexive path lifting morphisms. Branching bisimulation is stronger than the weak bisimulation notion often considered in concurrency theory, but we agree with van Glabbeek in arguing that it is more consistent with the requirements proposed by Hennessey and Milner [HM80] for observational, or weak, equivalence.

We first consider our version of branching bisimulation.

Definition 28 *Let A be a reflexive graph. Let $G = (G, l)$ and $H = (H, k)$ be objects of* $\mathbf{RfGraph}/A$. *A* fully branching bisimulation *from G to H is a relation R from G_0 to H_0*

which satisfies:
(i a) The projection of R on G_0 is onto
(i b) The projection of R on H_0 is onto
(ii a) For every edge $e : g \xrightarrow{a} g'$ in G and node h such that gRh there is a path in H:

$$h \xrightarrow{-} h_1 \xrightarrow{-} \dots h_k \xrightarrow{a} h' \quad k \geq 0$$

with $gRh_j, 1 \leq j \leq k$ and $g'Rh'$
(ii b) Similarly for edges in H.

We note that van Glabbeek and Weijland's branching bisimulation requires gRh_j only for $j = k$, but they prove that a maximal branching bisimulation is fully branching in our sense. Moreover, there is no loss of generality in having a null-labelled path only *before* the edge labelled a in H in (*ii a*) rather than, as is required for weak bisimulation, allowing also a path, say $l_1 \xrightarrow{-} l_2 \xrightarrow{-} \dots l_m \xrightarrow{a} h'$ with $g'Rl_j$ *after* $h_k \xrightarrow{a} l_1$. Indeed, one may clearly take l_1 for the h' whose existence is required.

Any reflexive graph has an underlying graph. It is clear that a bisimulation on the underlying graphs of labelled reflexive graphs G, H is a fully branching bisimulation, but a (fully) branching bisimulation need not be a bisimulation. It is easy to see that the identity relation, the composite of fully branching bisimulations, the union of fully branching bisimulations and the opposite of a fully branching bisimulation are all fully branching bisimulations. Thus we have the following analogue of Lemma 4:

Lemma 29 *(i) The union of fully branching bisimulations is a fully branching bisimulation, as is the identity relation*
(ii) There is a largest self fully branching bisimulation $\approx_G$ on any G in **RfGraph**/A. *$\approx_G$ is an equivalence relation on G.*

The proofs proceed exactly as in the **Graph** case. There is again an equivalence relation $\approx^e_G$ on edges of G defined exactly as in Section 2. With that relation defined, minimization is defined as follows:

Definition 30 *Let $G = (G, l)$ be an object of* **RfGraph**/A. *The* minimization *of G is the labelled graph $MG = (MG, l_M)$ in* **RfGraph**/A *where MG is the graph whose:*
- nodes MG_0 are the quotient set of the nodes G_0 of G by $\approx_G$;
- edges MG_1 are the quotient of the edges G_1 of G by $\approx^e_G$;
- the reflexivity r_{MG} sends a node $[g]$ in MG_0 to the $\approx^e_G$ class $[r_G(g)]$ of the null edge at g;
- the labelling is defined by $(l_M)_0([g]) = l_0(g)$ and $(l_M)_1([e]) = l_1(e)$ where $[g]$ is the $\approx_G$ class

of the node g in G and $[e]$ is the $\approx^e_G$ class of an edge e.
The projection morphism $\pi_G : G \longrightarrow MG$ *in* **RfGraph**$/A$ *is defined on nodes and edges of* G *by* $(\pi_G)_0(g) = [g]$, $(\pi_G)_1([e]) = e$.

As in Section 2 the resulting MG is a transition system, indeed it is a reflexive transition system. The projection morphism is well-defined and the minimization process is idempotent. Moreover, the projection morphism is an example of the following concept:

Definition 31 *Let G and H be objects of* **RfGraph**$/A$. *A* **RfGraph**$/A$ *morphism* $\varphi : G \longrightarrow H$ *is* reflexive path lifting *if whenever $e' : \varphi(g) = h \xrightarrow{a} h'$ is an a-labelled edge in H with source $\varphi(g)$ and target h', there are nodes $g_1, g_2, \ldots g_k, g', k \geq 0, g'$ in G such that $\varphi(g_1) = \varphi(g_2) = \ldots \varphi(g_k) = h$, $\varphi(g') = h'$ and a path $g \xrightarrow{-} g_1 \xrightarrow{-} \ldots g_k \xrightarrow{a} g'$ in G such that its null-labelled edges are mapped by φ to the* null *edge at $\varphi(g)$ and the image of the edge $g_k \xrightarrow{a} g'$ is e' (and hence $\varphi(g') = h'$).*

Similarly to the remarks after Definition 10, there is a correspondence between fully branching bisimulations and spans of reflexive path lifting morphisms, and a reflexive path lifting morphism determines a functional fully branching bisimulation.

Let $(A, *)$ be a pointed reflexive graph. Again, we do not require $(A, *)$ to be reachable. As in Section 2, we will write $\mathsf{PL}\mathbf{RfGraph}_A$ for the category in which an object $((G, g_0), l)$ is a pointed reflexive graph morphism $l : (G, g_0) \longrightarrow (A, *)$ from a reachable pointed reflexive graph (G, g_0) to $(A, *)$ and an arrow $\varphi : ((G, g_0), l) \longrightarrow (G', g'_0), l')$ is a reflexive path lifting morphism $\varphi : (G, g_0) \longrightarrow (G', g'_0)$ of reachable pointed reflexive graphs such that $l'\varphi = l$.

Notation $\mathsf{PL}\mathbf{RfGraph}_A$ has reachable pointed reflexive graphs labelled by $(A, *)$ as objects. Its morphisms are reflexive path lifting and label preserving.

Now the results of Section 2 can be repeated for the reflexive case. We summarize with the main points:

Proposition 32 *(i) If $G \xrightarrow{\varphi} H$ is a morphism of* $\mathsf{PL}\mathbf{RfGraph}_A$ *then* $MG \xrightarrow[\cong]{M\varphi} MH$.
(ii) M is the functor part of an idempotent monad (M, π) on $\mathsf{PL}\mathbf{RfGraph}_A$*, and so (M, π) has as Eilenberg-Moore algebras a full subcategory of* $\mathsf{PL}\mathbf{RfGraph}_A$ *denoted* $\mathsf{MPL}\mathbf{RfGraph}_A$ *(with objects of the form MG). (iii) There is an adjunction* $\mathsf{M}_A \dashv \mathsf{I}_A : \mathsf{MPL}\mathbf{RfGraph}_A \longrightarrow \mathsf{PL}\mathbf{RfGraph}_A$. ∎

Further, the results of Section 3 can also be extended to the reflexive case. The main point is the definition of a suitable bicategory of spans of reflexive graphs. In fact all that is needed is to add reflexivity to the definitions of Section 3 and then note that Lemma 25 extends to the definition of reflexive path lifting arrow used in this section. With that we define the bicategory $\mathsf{SpPL}\mathbf{RfGraph}$ which has as objects pointed reflexive graphs and whose hom category $\mathsf{SpPL}\mathbf{RfGraph}(A, B)$ for pointed reflexive graphs A and B is $\mathsf{PL}\mathbf{RfGraph}_{A\times B}$. Then we have:

Proposition 33 *The functors* $M_{(A,B)} : \mathsf{SpPL}\mathbf{RfGraph}(A, B) \longrightarrow \mathsf{SpPL}\mathbf{RfGraph}(A, B)$ *defined by* $M_{(A,B)} = M : \mathsf{PL}\mathbf{RfGraph}_{A\times B} \longrightarrow \mathsf{PL}\mathbf{RfGraph}_{A\times B}$ *extend to an idempotent lax monad on* $\mathsf{SpPL}\mathbf{RfGraph}$.

Example 34 As examples of morphisms in $\mathsf{SpPL}\mathbf{RfGraph}$, consider the labelled reflexive graphs $\mathcal{P}$ and $\mathcal{F}$ below. In both cases, the label alphabets on both left and right are the null label denoted $-$ and l and u standing for 'lock' and 'unlock' respectively. In the diagrams null edges are omitted and the left and right labellings are separated by $/$.

$$\begin{tikzcd}
1 \arrow[r, "l/-"] & 2 \arrow[d, "-/l"] \\
4 \arrow[u, "-/u"] & 3 \arrow[l, "u/-"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& 1 \arrow[dl, shift right, "l/-"'] \arrow[dr, shift left, "-/l"] & \\
2 \arrow[ur, shift right, "u/-"'] & & 3 \arrow[ul, shift left, "u/-"]
\end{tikzcd}$$

$\mathcal{P}$ $\qquad\qquad$ $\mathcal{F}$

The Dining Philosopher Problem arises from the idea that Philosophers $\mathcal{P}$ are seated around a table with Forks $\mathcal{F}$ between them. State 1 is the start state for both the Philosopher $\mathcal{P}$ and the Fork $\mathcal{F}$. According to the specification of the labelled graph $\mathcal{P}$ above, a Philosopher may pick up (or lock, l) the Fork to the left (the transition $l/-$) or right (transition $-/l$), and then put down (or unlock, u) the Fork to the left and then to the right. That is, the left fork must be picked up first (reaching state 2), then the right (so state 3 means left and right Forks have been picked up). Of course idling steps may always intervene. (At this point we imagine the Philosopher may eat, albeit with two Forks!) Then, the right fork must be put down first. A Fork is either free (unlocked) or in use (locked). State 1 of the Fork means it is free, 2 means it is locked (i.e. has been picked up) by the Philosopher to its left, and 3 means the Fork is locked by the Philosopher to its right.

The Dining Philosopher Problem is discussed further in Example 55, but for now we note that the system consisting of n Dining Philosophers in a row, each with a Fork to the right is represented by the expression $(\mathcal{P}\mathcal{F})^n$. For example, the $\mathsf{SpPL}\mathbf{RfGraph}$ composite $\mathcal{P}\mathcal{F}$ expresses the possible actions of a Philosopher with a Fork to the right, and so on. Using the operation of *feedback* Fb (which we introduce in Section 7) we can represent the system of n Dining Philosophers 'in a circle' as $\mathsf{Fb}((\mathcal{P}\mathcal{F})^n)$.

5 The setting for minimal realization

For consideration of minimal realization in the sequel we will work with various restrictions on the categories and functors which we describe in this section. Our purpose here is to underline the generality of our setting. Up to Proposition 37 this is well-known material, some of which can be found in [KV97].

The base category for our discussion is the category of (directed) graphs **Graph** described at the beginning of Section 2.

The functor $V : \mathbf{Graph} \longrightarrow \mathbf{Set}$ which remembers only the nodes has both left and right adjoints denoted $D \dashv V \dashv H : \mathbf{Set} \longrightarrow \mathbf{Graph}$. For a set X, DX is the *discrete graph* on X with nodes X and no edges. HX is the *chaotic graph* on X with nodes X and exactly one edge in each direction between each pair of nodes. In this section we will be interested in the comma category $D/\mathbf{Graph}$. Thus, the objects of $D/\mathbf{Graph}$ are triples (X, f, G) with X a set, G a graph, and f a **Graph** arrow $f : DX \longrightarrow G$. We call them *multi-pointed graphs*. An object of $D/\mathbf{Graph}$ should be thought of as a graph G with a set X of entry points specified by f (not necessarily all distinct). An arrow $\varphi : (X, f, G) \longrightarrow (X', f', G')$ is a pair $\varphi = (\varphi^0, \varphi^1)$ where $\varphi^0 : X \longrightarrow X'$ is a mapping and $\varphi^1 : G \longrightarrow G'$ is a **Graph** arrow such that $\varphi^1 f = f' D\varphi^0$ in **Graph**.

To describe behaviours we will require the subcategory of $D/\mathbf{Graph}$ consisting of *trees*. The *category of trees* **Tree** is defined to be the full subcategory of $D/\mathbf{Graph}$ whose objects $(\{r\}, j, T)$ have a specified *root* node r in T_0, j is the name of r, and there is a unique path in the graph T from r to each other node of T. Thus a morphism of trees takes root nodes to root nodes and the unique path from the root to any node of the domain graph maps to the unique path to its image node.

We digress to consider some facts about *forests*. Intuitively a forest is just a set of trees. We define **Forest** to be the full subcategory of $D/\mathbf{Graph}$ whose objects are (X, f, F) where F is a graph with the property that for each node v of F there is a unique element r_v of X and unique path in F from $f(r_v)$ to v. As a consequence of the definition, f is one-one. We denote the inclusion functor $J : \mathbf{Forest} \longrightarrow D/\mathbf{Graph}$.

An equivalent way to describe forests is by the functor category $\mathbf{Forest}' = \mathbf{Set}^{0 \leftarrow 1 \leftarrow \cdots}$. An object F' of $\mathbf{Forest}'$ defines a forest F as follows. $F'(0)$ specifies the root nodes of the trees. The nodes F_0 of F are the disjoint union of the $F'(i), i \geq 0$. The children of the root nodes are the set $F'(1)$ with parentage given by the transition mapping $F'(1) \longrightarrow F'(0)$, and so on. So there is an edge from v in $F'(n)$ to v' in $F'(n+1)$ (note direction!) exactly when v is the image of v' under the transition mapping $F'(n+1) \longrightarrow F'(n)$. Each node v in F_0 has a unique path to it from a unique node in $F'(0)$ specified by the iterated transition function acting on v. Denote the inclusion function $j : F'(0) \longrightarrow F_0$. Then $(F'(0), j, F)$ is an object of $D/\mathbf{Graph}$ which is a forest as defined in the previous paragraph. Any arrow $\varphi : F_0 \longrightarrow F_1$ of $\mathbf{Forest}'$

evidently defines a unique morphism of the corresponding multi-pointed graphs. Conversely, suppose $\psi : (F_0(0), j_0, F) \longrightarrow (F_1(0), j_1, F')$ is a morphism between multi-pointed graphs that arise from objects F_0 and F_1 of **Forest**$'$. Since $\psi_0|_{F_0(0)} : F_0(0) \longrightarrow F_1(0)$, the definition of edges in the corresponding graph means we have $\psi_0|_{F_0(1)} : F_0(1) \longrightarrow F_1(1)$ and so on. The requirements for a $D/$**Graph** arrow mean that ψ determines an arrow from F to F' in the functor category **Forest**$'$. Since these correspondences are mutually inverse up to isomorphism, **Forest**$'$ is equivalent to **Forest**. Thus we have the well-known:

Proposition 35 *The category of forests* **Forest** *is a topos.* ∎

To a multi-pointed graph (X, f, G) (which we will often abuse notation by writing just f) there is associated a forest Uf, the *unfolding of* f. This is the forest (X, f, Uf_G) whose graph Uf_G consists of the trees built from paths in G originating from the root nodes specified by f. More precisely, and to establish notation, we will denote the nodes of the graph of Uf_G by $\langle xe_1e_2...e_n\rangle$, where $n \geq 0$, $v = f(x)$ is an entry node and $e_1, e_2, \ldots, e_n$ are the edges in a path in G with source v. The edges of Uf_G are denoted $\langle p\rangle e$ where $\langle p\rangle$ is a node of Uf_G and e is an edge whose source is the last node on the path p. If $\varphi = (\varphi^0, \varphi^1) : (X, f, G) \longrightarrow (X', f', G')$ is an arrow of $D/$**Graph** we define $U\varphi : (X, f, Uf_G) \longrightarrow (X', f', Uf'_{G'})$ by $(U\varphi)^0 = \varphi^0$, on nodes of Uf_G by $(U\varphi)^1_0(\langle xe_1 \ldots e_n\rangle) = \langle \varphi^0(x)\varphi^1_1(e_1)\ldots\varphi^1_1(e_n)\rangle$ and on edges by $(U\varphi)^1_1(\langle p\rangle e) = (U\varphi)^1_0(\langle p\rangle)\varphi^1_1(e)$. These define a functor $U : D/$**Graph** $\longrightarrow$ **Forest**. Now we have:

Proposition 36

$$J \dashv U : D/\mathbf{Graph} \longrightarrow \mathbf{Forest}$$

Proof. Suppose f is a multi-pointed graph and F is a forest. To any arrow φ (in **Forest**) from F to Uf we can define a unique arrow from JF to f by simply sending the edges of (the graph part of) JF to the edges determined by φ. Conversely, remembering that entry points must be preserved, it is easy to see that an arrow ψ from JF to f determines a unique arrow from the forest F to the unfolding of f. ∎

We should point out that for a multi-pointed graph (*even if finite*) which has a cycle reachable from a base point, the unfolding is a forest of countably infinite depth. For an example consider the second minimized graph in Example 7.

The adjunction $D \dashv V$ means that there is an isomorphism of categories between $D/$**Graph** and **Set**$/V$. Since V has a right adjoint it is a left exact functor and so **Set**$/V$ is a topos by the glueing construction [W74]. Thus $D/$**Graph** is also a topos. The functor

J above is not left exact since the terminal object in **Forest** ($\sim$ **Forest**$'$!) is the tree with exactly one node at each level, while the terminal object of $D/$**Graph** has a graph with exactly one node and one edge. However J does preserve pullbacks, so U is a *partial geometric morphism* as considered in [RW91].

Recall the category of *pointed graphs* Pt**Graph**. It is the full subcategory of $D/$**Graph** with objects $(1, p, G)$ where $p : D1 \longrightarrow G$ is the 'point'. The category of trees **Tree** defined above is a full subcategory of Pt**Graph**. Moreover:

Proposition 37 *The functors J and U restrict to J^{pt} :* **Tree** $\longrightarrow$ Pt**Graph** *and U^{pt} :* Pt**Graph** $\longrightarrow$ **Tree**, *and $J^{pt} \dashv U^{pt}$.* ∎

Given an object $f = (X, f, G)$ of $D/$**Graph** we construct the *reachable part of f*, $f_R = (X, f_R, G_R)$ as follows. G_R is the full subgraph of G whose nodes are those v such that there is a path to v from a node $v_0 = f(x)$ in the image of f. Such nodes are called *reachable nodes* of G. The function f_R is just the factorization of f through the inclusion of G_R in G and this clearly exists since any node in the image of f is trivially reachable.

Proposition 38 *The correspondence $f \mapsto f_R$ is the object part of an idempotent functor R on $D/$***Graph***. Define $\rho_f : f_R \longrightarrow f$ by the inclusion of G_R in G. Then ρ is a natural transformation from R to the identity functor on $D/$***Graph** *and (R, ρ) is an idempotent comonad.*

Proof. The idempotence of R as a functor is obvious. Using the fact that the putative counit for the monad is defined by the inclusion of G_R in G makes verification of the comonad equations trivial. ∎

The coalgebras for (R, ρ) denoted $D/$**Graph**$_R$ are graphs all of whose nodes are reachable by a path from an entry point. Note that J : **Forest** $\longrightarrow$ $D/$**Graph** factors through $D/$**Graph**$_R$ and that, for any f in $D/$**Graph**, Uf is reachable. It is evident that the comonad (R, ρ) restricts to the subcategory of pointed graphs Pt**Graph**. The coalgebras RchPt**Graph** for the restricted comonad is the subcategory of *reachable pointed graphs* also considered above in Section 2.

The category of multi-pointed graphs has pullbacks, so there is a bicategory of spans which we denote **Span**($D/$**Graph**). We note that (R, ρ) extends to an identity-on-objects colax functor (= bicategory comorphism) on the bicategory **Span**($D/$**Graph**). It is defined locally on a span $(X, f, G) \longleftarrow (Y, h, H) \longrightarrow (X', f', G')$ of multi-pointed graphs by taking the reachable part of the head graph (Y, h, H) and composing its inclusion $R(Y, h, H) \longrightarrow (Y, h, H)$ with the legs of the original span. The colax structure at the identity span is given by the counit of (R, ρ). To define the colaxity on a composite requires only the observation that the reachable part of a composite in $\mathbf{Span}(D/\mathbf{Graph})$ is contained in the composite of the reachable parts of the factors.

Furthermore, (R, ρ) is an idempotent colax comonad (in the sense of [CR91]) on the bicategory $\mathbf{Span}(D/\mathbf{Graph})$. As above in Proposition 27, since the local comonads are idempotent, we use Proposition 42 in [RSW98] to simplify the equations required for the colax comonad structure. In fact, all that need be shown is that the counit is compatible with the colaxity for composition, but both are defined by taking the reachable part.

Thus, there is a bicategory of coalgebras $\mathbf{Span}(D/\mathbf{Graph})_R$ with the same objects as $\mathbf{Span}(D/\mathbf{Graph})$ and local coalgebras given by the reachability comonad. Notice that composition in the coalgebras $\mathbf{Span}(D/\mathbf{Graph})_R$ is obtained by taking the *reachable part* of the usual composite in $\mathbf{Span}(D/\mathbf{Graph})$. It is worth pointing out again that the identity arrow on an object $G = (X, f, G)$ is actually the span whose legs are the inclusions of the reachable part of G. We denote by $\mathsf{SpRchPt}\mathbf{Graph}$ the locally full sub-bicategory of the coalgebras $\mathbf{Span}(D/\mathbf{Graph})_R$ determined by objects in $\mathsf{Pt}\mathbf{Graph}$ and 1-cells (spans) whose heads are reachable pointed graphs. Of course, $\mathsf{SpRchPt}\mathbf{Graph}$ is also describable as the bicategory of coalgebras for the reachability colax comonad restricted to spans of pointed graphs. The (not locally full) sub-bicategory of $\mathsf{SpRchPt}\mathbf{Graph}$ whose 2-cells are path lifting was denoted $\mathsf{SpPL}\mathbf{Graph}$ above.

6 Behaviour and minimal realization

In order to discuss minimal realization locally, we need a category of machines, a category of behaviours and a behaviour functor. In Section 2 we introduced the machine category we wish to consider, $\mathsf{PL}\mathbf{Graph}_A$. We need to define what we mean by behaviour of an object in $\mathsf{PL}\mathbf{Graph}_A$. As we will detail shortly, we consider that a certain equivalence class of synchronization trees is the appropriate behaviour for a reachable pointed graph labelled by A. Our objective in this section is to define a behaviour functor on $\mathsf{PL}\mathbf{Graph}_A$ and a right adjoint minimal realization functor to $\mathsf{PL}\mathbf{Graph}_A$ from the behaviours.

The theory of concurrency considers special labelled trees known as synchronization trees.

Definition 39 *[JNW96] Let A be an alphabet. A* synchronization tree *labelled by A with root r is a labelled transition system with initial state r whose underlying graph is a tree.*

Thus the underlying graph of the tree is reachable, acyclic and each non-initial state is the target of a unique edge. As also noted in [JNW96], a labelled transition system with an initial state unfolds to a synchronization tree, and the inclusion-unfolding adjunction of Section 5 extends to an adjunction between the category of transition systems labelled by A and the synchronization trees labelled by A. (Note that in [JNW96], the label set may vary along a transition system homomorphism and the image of a transition may 'idle' in the codomain.)

Let $(A, *)$ be a pointed graph. We denote by $\mathsf{PL}\mathbf{Tree}_A$ the full subcategory of $\mathsf{PL}\mathbf{Graph}_A$ with objects whose domains are trees. Thus the objects of $\mathsf{PL}\mathbf{Tree}_A$ are trees labelled by A with path lifting morphisms which are then necessarily onto on nodes and edges.

Proposition 40 *The functors J^{pt} and U^{pt} of Proposition 37 extend to*

$$J^{pt} \dashv U^{pt} : \mathsf{PL}\mathbf{Graph}_A \longrightarrow \mathsf{PL}\mathbf{Tree}_A$$

Proof. This requires only the observation that the labelled unfolding of a path lifting morphism of pointed graphs is a path lifting morphism of trees. ∎

Example 41 The unfolding of any object of $\mathsf{PL}\mathbf{Graph}_A$ is a synchronization tree labelled by A. Indeed, the definition of unfolding guarantees that there are never two edges with the same label between nodes of the unfolding — after all, a tree has at most one edge between two nodes! Of course, every synchronization tree arises, up to isomorphism, as its own unfolding.

The equivalence relation on objects of $\mathsf{PL}\mathbf{Tree}_A$ we have been alluding to is simply bisimilarity. As shown in [JNW96, Theorem 2], trees labelled by A are bisimilar if and only if they are linked by a span of path lifting arrows. Since $\mathsf{PL}\mathbf{Tree}_A$ has pullbacks, trees are linked by a span of path lifting arrows if and only if they are linked by a zig-zag of path lifting arrows if and only if they are in the same *connected component* of $\mathsf{PL}\mathbf{Tree}_A$. Now the behaviour we will define for an object of $\mathsf{PL}\mathbf{Graph}_A$ is an observational, that is bisimilarity, equivalence class of trees labelled by A and the class associated to an object (G, l) is precisely that of its unfolding synchronization tree. By the remarks at the beginning of this paragraph, this is the connected component of the unfolding of (G, l).

Recall the connected components functor defined on the category of categories: $\Pi_0 : \mathbf{Cat} \longrightarrow \mathbf{Set}$. It has a right adjoint D assigning the discrete category to a set, so for any category $\mathbf{C}$ there is a universal functor $P_{\mathbf{C}} : \mathbf{C} \longrightarrow D\Pi_0(\mathbf{C})$.

Notation We denote by $\mathbf{Beh}_A$ the (discrete) category $D\Pi_0(\mathsf{PL}\mathbf{Tree}_A)$ of behaviours.

We can now define the behaviour functor.

Definition 42 *The* behaviour functor E *on* $\mathsf{PL}\mathbf{Graph}_A$ *is defined by*

$$E = P_{\mathsf{PL}\mathbf{Tree}_A} U^{pt} : \mathsf{PL}\mathbf{Graph}_A \longrightarrow \mathbf{Beh}_A$$

Our next objective is to define a minimal realization functor N which is right adjoint to E. We will also find below that $\mathbf{Beh}_A$ is equivalent to $\mathsf{MPL}\mathbf{Graph}_A$, essentially via the behaviour and minimization functors. We note immediately the functor $E' = E\mathsf{I}_A$: $\mathsf{MPL}\mathbf{Graph}_A \longrightarrow \mathbf{Beh}_A$. Moreover, the projection arrow $\pi_G : G \longrightarrow MG$ shows that G and MG are in the same component of $\mathsf{PL}\mathbf{Graph}_A$, whence $EG = EMG = E\mathsf{I}_A\mathsf{M}_A G = E'\mathsf{M}_A G$ and so $E = E'\mathsf{M}_A$.

Let $P(T) = P_{\mathsf{PL}\mathbf{Tree}_A}(T)$ be the component of an A labelled tree T in $\mathsf{PL}\mathbf{Tree}_A$. For each component c, choose a tree T_c in $P(T)$.

Definition 43 *The* minimal realization functor N : $\mathbf{Beh}_A \longrightarrow \mathsf{PL}\mathbf{Graph}_A$ *is defined by* $N(P(T)) = MJ^{pt}(T_c)$.

The definition is independent of the choice of T_c since minimization M is (essentially) constant on components of $\mathsf{PL}\mathbf{Graph}_A$.

For the rest of this section we omit the superscripts on U and J. The functorial property we expect of minimal realization is the following:

Proposition 44 *There is an adjunction* $E \dashv N : \mathbf{Beh}_A \longrightarrow \mathsf{PL}\mathbf{Graph}_A$.

Proof. First, suppose that we have a $\mathsf{PL}\mathbf{Graph}_A$ arrow $G \longrightarrow N(P(T)) = MJ(T_c)$. We want to show that there is an arrow $E(G) \longrightarrow P(T)$. Indeed, we have $E(G) = P(UG) = P(T)$ since we have arrows

$$UG \longrightarrow U\mathsf{I}_A\mathsf{M}_A J(T_c) \longleftarrow T_c$$

where the second arrow is the unit for $\mathsf{M}_A J \dashv U\mathsf{I}_A$. Hence $E(G) = P(UG) = P(T_c) = P(T)$. In the other direction, suppose that $E(G) = P(T)$ (since $\mathbf{Beh}_A$ is discrete). We need to find an arrow $G \longrightarrow N(P(T))$. By definition of E we have $P(UG) = E(G) = P(T) = P(T_c)$, so in $\mathsf{PL}\mathbf{Tree}_A$ we have a span $UG \longleftarrow T' \longrightarrow T_c$. Using the counit for $J \dashv U$ to provide an arrow $JUG \longrightarrow G$ and applying MJ to the span just mentioned, in $\mathsf{PL}\mathbf{Graph}_A$ we have

$$G \xrightarrow{\pi_G} M(G) \xleftarrow{\cong} MJU(G) \xleftarrow{\cong} MJ(T') \xrightarrow{\cong} MJ(T_c) = N(P(T))$$

providing $G \longrightarrow N(P(T))$ as required. It remains to show that these passages are mutually inverse. Beginning from $E(G) \xrightarrow{=} P(T)$ we defined an arrow $G \longrightarrow N(P(T))$, and thence an arrow $E(G) \longrightarrow P(T)$ which must be the original equality arrow by discreteness. For the other required equation, starting from $G \xrightarrow{\varphi} N(P(T)) = MJ(T_c)$, the first process produces the cospan $UG \xrightarrow{U\varphi} UMJ(T_c) \longleftarrow T_c$. By pulling back we get a span $UG \longleftarrow T' \longrightarrow T_c$ to which the second process applies yielding an arrow $G \longrightarrow MJ(T_c) = N(P(T))$. This latter arrow has a minimized object as codomain, and so by Lemma 20 it must equal φ. ∎

Notice that $N(P(T)) = MJ(T_c)$ is minimized, so we can factor N through $\mathsf{MPL}\mathbf{Graph}_A$ as $N = \mathsf{I}_A N'$ where $N'(P(T)) = \mathsf{M}_A J(T_c)$.

Remark 45 The situation is summed up in the following diagram of categories and functors in which we have M_A and E are left adjoints, $\mathsf{I}_A N' = N$, $E = E'\mathsf{M}_A$.

$$\begin{tikzcd}
 & \mathsf{PL}\mathbf{Graph}_A \arrow[dl, shift right, "\mathsf{M}_A"'] \arrow[dr, shift left, "E"] & \\
\mathsf{MPL}\mathbf{Graph}_A \arrow[ur, shift right, "\mathsf{I}_A"'] \arrow[rr, shift left, "E'"] & & \mathbf{Beh}_A \arrow[ul, shift left, "N"'] \arrow[ll, shift left, "N'"]
\end{tikzcd}$$

It is also important to note that

Proposition 46 $\mathbf{Beh}_A$ *and* $\mathsf{MPL}\mathbf{Graph}_A$ *are equivalent.*

Proof. For T in $\mathsf{PL}\mathbf{Tree}_A$ we have

$$E'N'(P(T)) = E'(\mathsf{M}_A J(T_c)) = E\mathsf{I}_A\mathsf{M}_A J(T_c) = P(U\mathsf{I}_A\mathsf{M}_A J(T_c))$$

but since we have the unit $T_c \longrightarrow U\mathsf{I}_A\mathsf{M}_A J(T_c)$, the component of $U\mathsf{I}_A\mathsf{M}_A J(T_c)$ is $P(T_c) = P(T)$ and $E'N'$ is the identity.

In the other direction, for G in $\mathsf{MPL}\mathbf{Graph}_A$ we have

$$N'E'(G) = N'(P(U\mathsf{I}_A G)) = \mathsf{M}_A J((U\mathsf{I}_A G)_c)$$

Now since $U\mathsf{I}_A G$ and $(U\mathsf{I}_A G)_c$ are linked by a span in $\mathsf{PL}\mathbf{Tree}_A$ we have $\mathsf{M}_A J((U\mathsf{I}_A G)_c) \cong \mathsf{M}_A JU\mathsf{I}_A(G)$ and the counit for $\mathsf{M}_A J \dashv U\mathsf{I}_A$ provides $\mathsf{M}_A JU\mathsf{I}_A(G) \cong G$, so $N'E'(G) \cong G$. ∎

With the results of Section 3, this proposition shows that our minimal realization is also compositional. We have a diagram of bicategories, local adjunctions defined by the (equivalent) lax monads M and NE, and the (bi)equivalence in the bottom row of:

$$
\begin{tikzcd}
 & \mathsf{SpPL}\mathbf{Graph} \arrow[dl, shift right, "\mathsf{M}"'] \arrow[dr, shift left, "E"] & \\
\mathsf{SpMPL}\mathbf{Graph} \arrow[ur, shift right, "\mathsf{I}"'] \arrow[rr, shift left, "E'"] & & \mathsf{Sp}\mathbf{Beh} \arrow[ul, shift left, "N"] \arrow[ll, shift left, "N'"]
\end{tikzcd}
$$

where $\mathsf{Sp}\mathbf{Beh}$ is the locally discrete bicategory with the same objects as $\mathsf{SpPL}\mathbf{Graph}$ and $\mathsf{Sp}\mathbf{Beh}(A, B) = \mathbf{Beh}_{A\times B}$. Thus we have reached the goal announced in the Introduction.

Remark 47 (i) There is no obstruction to extending the results of this section to reflexive graphs, reflexive path lifting morphisms and fully branching bisimulation. This would require modification of the codomain of the unfolding functor to be a category of reflexive graphs that are trees except that they have a reflexive node. Behaviours would then be reflexive path lifting components (fully branching bisimulation equivalence classes) of these.
(ii) We could extend our model to include final states and recover language recognizers. In that case bisimulations should respect final states in the sense that a final state may only be bisimilar to a final state and a non-final state to a non-final one. Morphisms must then also preserve and reflect the final states. Once again we can define a minimization using the quotient by a maximal self bisimulation of the extended type. With this setup we can recover the classical minimal realization of Nerode as follows. For an alphabet A, a language is a subset of the words over A. It can be recognized by the tree which has an edge labelled by each letter of A at each node and final states given by selecting the nodes which correspond to paths labelled by words in the language. Of course this tree is the unfolding of *any* deterministic labelled graph (automaton) which recognizes the language. Now the minimization described above, when applied to this tree, gives the classical minimal realization of Nerode.

7 Other structures

In this section we consider the tensor structure on spans of graphs and its relationship to our minimal realization functors and apply this to the concept of feedback and the Dining Philosopher Problem. We also consider extension of the results of the previous section to multi-pointed graphs and forests.

Definition 48 *The* tensor *of 1-cells* $G : A \xleftarrow{l} G \xrightarrow{r} B$ *and* $G' : C \xleftarrow{l'} G' \xrightarrow{r'} D$ *in* $\mathsf{SpPL}\mathbf{Graph}$ *is the span*

$$G \otimes G' : A \times C \xleftarrow{l''} (G \times G')_R \xrightarrow{r''} B \times D$$

where l'' is the inclusion of $(G \times G')_R$ in $G \times G'$ followed by $l \times l'$ and similarly for r''.

Definition 49 *For any pointed graph A, we denote by η_A the span*

$$\eta_A : 1 \longleftarrow A_R \xrightarrow{\overline{\Delta}_R} A \times A$$

where $\overline{\Delta}_R$ is the inclusion of the reachable part followed by the diagonal. Similarly, we denote by ϵ_A the span

$$\epsilon_A : A \times A \xleftarrow{\overline{\Delta}_R} A_R \longrightarrow 1$$

Before considering our application we note:

Proposition 50 *For any pointed graph A, in* $\mathsf{SpPL}\mathbf{Graph}$ *we have*

$$M_{(A,A)} 1_A = 1_A \qquad M_{(1,A\times A)} \eta_A = \eta_A \qquad M_{(A\times A,1)} \epsilon_A = \epsilon_A$$

Proof. These are all similar. The uniqueness of labels (on both sides for 1_A, on the right for η_A, on the left for ϵ_A) means that the largest self bisimulation in each case is the identity relation and so minimization is trivial. ∎

Proposition 51 *For morphisms* $G : A \xleftarrow{l} G \xrightarrow{r} B$ *and* $G' : C \xleftarrow{l'} G' \xrightarrow{r'} D$ *in* $\mathsf{SpPL}\mathbf{Graph}$ *there is a comparison arrow*

$$\mu_{G.G'} : MG \otimes MG' \longrightarrow M(G \otimes G')$$

Proof. This merely requires the observation that a node on the left is a pair of equivalence classes in (the reachable part of) the product of the quotient of G and G' by their largest self bisimilarities which may be sent to the equivalence class of the same pair in $M(G \otimes G')$. ∎

Definition 52 *The* feedback *of a 1-cell* $G : A \times U \xleftarrow{l} G \xrightarrow{r} B \times U$ *in* $\mathsf{SpPL}\mathbf{Graph}(A \times U, B \times U)$ *is* $\mathsf{Fb_R}(G)$ *in* $\mathsf{SpPL}\mathbf{Graph}(A, B)$ *defined as the composite (in* $\mathsf{SpPL}\mathbf{Graph}$*):*

$$\mathsf{Fb_R}(G) = (1_B \otimes \epsilon_U)(G \otimes 1_U)(1_A \otimes \eta_U)$$

We have subscripted R in the definition to emphasize that the operations involved in the expression for feedback involve taking reachable parts. The definition of feedback is the same for $\mathbf{Span}(\mathbf{Graph})$, here modified to take account of path lifting morphisms. See [KSW97b] for examples and more details. For an account of the universal properties of feedback, in particular the relationship with traced monoidal categories, see [KSW02].

Proposition 53 *For* G *as in the definition, there is a comparison arrow*

$$\mathsf{Fb_R} M(G) \longrightarrow M(\mathsf{Fb_R}(G))$$

Proof. Combine the definition of feedback with Propositions 50, 51 and the fact that M is lax to obtain:

$$\mathsf{Fb_R} M(G) = (1_B \otimes \epsilon_U)(MG \otimes 1_U)(1_A \otimes \eta_U) \cong$$
$$(M1_B \otimes M\epsilon_U)(MG \otimes M1_U)(M1_A \otimes M\eta_U) \longrightarrow M(1_B \otimes \epsilon_U)M(G \otimes 1_U)M(1_A \otimes \eta_U) \longrightarrow$$
$$M((1_B \otimes \epsilon_U)(G \otimes 1_U)(1_A \otimes \eta_U)) = M(\mathsf{Fb_R}(G))$$ ∎

Corollary 54

$$M\mathsf{Fb_R} M(G) \cong M\mathsf{Fb_R}(G)$$

∎

All of these results extend to the reflexive case, so we can consider:

Example 55 (**The Dining Philosophers**) A *deadlock state* of an asynchronous system is one for which the only available transitions are null. Deadlock states are preserved by our reflexive path lifting morphisms in the sense of Section 4, hence in searching for deadlocks it is often useful to calculate minimization.

A compositional model checking algorithm based on this idea is the following :
Given a system presented as an expression in the operations of composition, tensor and feedback, calculate M of the system by successively evaluating an operation and calculating

M of the result. Record the quotient morphisms π which arise in the calculation. When M of the system is obtained, find all deadlock states, and then trace inverse images which are deadlocks along the calculated π's.

When we apply the algorithm to the Dining Philosopher Problem, namely to detect deadlock in a circle of Philosophers and Forks as introduced above (see Example 34), we obtain the following results:

- $M(\mathcal{P} \cdot \mathcal{F})$ is a five node graph with 15 edges
- $M(\mathcal{P} \cdot \mathcal{F} \cdot \mathcal{P} \cdot \mathcal{F})$ may then be evaluated as $M(M(\mathcal{P} \cdot \mathcal{F}) \cdot M(\mathcal{P} \cdot \mathcal{F}))$ with 8 nodes and 36 edges
- when $M(\mathcal{P} \cdot \mathcal{F} \cdot \mathcal{P} \cdot \mathcal{F} \cdot \mathcal{P} \cdot \mathcal{F})$ is similarly evaluated we find that

$$M(\mathcal{P} \cdot \mathcal{F} \cdot \mathcal{P} \cdot \mathcal{F} \cdot \mathcal{P} \cdot \mathcal{F}) \cong M(\mathcal{P} \cdot \mathcal{F} \cdot \mathcal{P} \cdot \mathcal{F})$$

- hence $M((\mathcal{P} \cdot \mathcal{F})^n) \cong M((\mathcal{P} \cdot \mathcal{F})^2)$ for all $n \geq 2$.

Thus the time complexity to calculate $M((\mathcal{P} \cdot \mathcal{F})^n)$ is linear. Note that similar results hold for $M(\mathcal{F} \cdot \mathcal{P})$, and so on.

The last stage of computing the minimization of the Dining Philosophers is to calculate $\mathsf{Fb}(M((\mathcal{P} \cdot \mathcal{F})^n))$ and minimize the result yielding the terminal graph as expected.

Retrieving the deadlocks requires tracing back through π's considering only deadlock states along the way. Note that the single state of the final minimization is a deadlock, but only one of its inverse images in $\mathsf{Fb}M((\mathcal{P} \cdot \mathcal{F})^n)$ is so. In tracing back along the π's at each stage there is only one deadlock.

Finally, we also wish to extend the minimization, behaviour, minimal realization theory above from pointed to multi-pointed graphs and (for behaviour) from trees to forests. The material in Section 5 allows us to do this immediately, and we simply outline the results.

Let A be a multi-pointed graph. The category of multi-pointed graphs labelled by A is $(D/\mathbf{Graph})/A$. For a pair of objects of $(D/\mathbf{Graph})/A$ we continue to define bisimulation as in Section 2. Bisimulations again enjoy properties that provide a maximal self bisimulation. Recall the discussion of reachability for multi-pointed graphs in Section 5. Reachability means existence of a path from at least one entry point and we defined the coalgebra category $D/\mathbf{Graph}_R$. Since the notion of path lifting morphism also makes sense for multi-pointed graphs labelled by A, we have a category whose objects are reachable multi-pointed graphs labelled by A and whose arrows are path lifting that we will denote $\mathsf{PLD}\mathbf{Graph}_A$. (As always A need not be reachable.) We obtain a minimization functor on $\mathsf{PLD}\mathbf{Graph}_A$ just as in Section 2. Once again, the minimization is an idempotent monad on $\mathsf{PLD}\mathbf{Graph}_A$ with algebras denoted $\mathsf{MPLD}\mathbf{Graph}_A$.

To make the minimization just described compositional, we need a composition of spans from $\mathsf{PLD}\mathbf{Graph}_A$. As in Section 3, this derives from composition in $\mathbf{Span}((D/\mathbf{Graph})/A)$, the latter being a bicategory since $D/\mathbf{Graph}$, as mentioned above, is a topos so the slice category $(D/\mathbf{Graph})/A$ certainly has pullbacks. Restricting to appropriate head graphs and path-lifting two-cells, we obtain the bicategory $\mathsf{SpPLD}\mathbf{Graph}$ with objects multi-pointed graphs and hom categories defined by $\mathsf{SpPLD}\mathbf{Graph}(A,B) = \mathsf{PLD}\mathbf{Graph}_{A\times B}$. Minimization is again a lax monad on $\mathsf{SpPLD}\mathbf{Graph}$ and provides a compositional minimization for reachable multi-pointed labelled graphs.

We can also immediately extend the minimal realization constructions of Section 6 with now bisimilarity classes of labelled forests for behaviour categories as summed up in:

$$\begin{tikzcd}
 & \mathsf{PLD}\mathbf{Graph}_A \arrow[dl, shift right, "\mathsf{M}_A"'] \arrow[dr, shift left, "E"] & \\
\mathsf{MPLD}\mathbf{Graph}_A \arrow[ur, shift right, "\mathsf{I}_A"'] \arrow[rr, shift left, "E'"] & & \mathsf{D}\mathbf{Beh}_A \arrow[ul, shift left, "N"'] \arrow[ll, shift left, "N'"]
\end{tikzcd}$$

We leave it to the reader to formulate the analogue of the last diagram in Section 6.

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