Double Categories

The best thing since slice categories

(https://www.mscs.dal.ca/∼pare/FMCS1.pdf)

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Double categories

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$$\begin{tikzcd}
\mathbb{A} : & \mathbf{A}_2 \arrow[r, shift left=3, "p_1"] \arrow[r, "\circ" description] \arrow[r, shift right=3, "p_2"'] & \mathbf{A}_1 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] & \mathbf{A}_0 \arrow[l, "1" description]
\end{tikzcd}$$

$\mathbb{A}$ has

- objects $A, A', \ldots$ the objects of $\mathbf{A}_0$
- morphisms $A \xrightarrow{f} A'$, the *objects* of $\mathbf{A}_1$
- composition $A \xrightarrow{f} A' \xrightarrow{f'} A'' = A \xrightarrow{f' \circ f} A''$
- identities $1_A : A \longrightarrow A$.

Double categories (cont.)

- $\mathbf{A}_0$ also has morphisms – another kind, internal
  - $A \xrightarrow{v} \bar{A}$
  - Composition $A \xrightarrow{v} \bar{A} \xrightarrow{\bar{v}} \tilde{A} = A \xrightarrow{\bar{v} \bullet v} \tilde{A}$
  - Identities $\mathrm{id}_A : A \longrightarrow A$

Double categories (cont.)

- $\mathbf{A}_0$ also has morphisms – another kind, internal
  - $A \overset{v}{\relbar\!\!\bullet\!\!\rightarrow} \bar{A}$
  - Composition $A \overset{v}{\relbar\!\!\bullet\!\!\rightarrow} \bar{A} \overset{\bar{v}}{\relbar\!\!\bullet\!\!\rightarrow} \tilde{A} = A \overset{\bar{v} \bullet v}{\relbar\!\!\bullet\!\!\rightarrow} \tilde{A}$
  - Identities $\mathrm{id}_A : A \relbar\!\!\bullet\!\!\rightarrow A$
- $\mathbf{A}_1$ has morphisms too – morphisms between external morphisms – cells

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
\arrow[r, phantom, "\Downarrow\alpha"] & \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B}
\end{tikzcd}$$

Double categories (cont.)

- $\mathbf{A}_0$ also has morphisms – another kind, internal
  - $A \overset{v}{\relbar\joinrel\bullet\joinrel\rightarrow} \bar{A}$
  - Composition $A \overset{v}{\relbar\joinrel\bullet\joinrel\rightarrow} \bar{A} \overset{\bar{v}}{\relbar\joinrel\bullet\joinrel\rightarrow} \tilde{A} = A \overset{\bar{v} \bullet v}{\relbar\joinrel\bullet\joinrel\rightarrow} \tilde{A}$
  - Identities $\mathrm{id}_A : A \relbar\joinrel\bullet\joinrel\rightarrow A$
- $\mathbf{A}_1$ has morphisms too – morphisms between external morphisms – cells

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\Downarrow \alpha"] & B \arrow[d, "w", "\bullet" marking] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B}
\end{tikzcd}$$

Double categories (cont.)

- Cells compose in $\mathbf{A}_1$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "w", "\bullet" marking] \\
\bar{A} \arrow[r, "\bar{f}"'] \arrow[d, "\bar{v}"', "\bullet" marking] \arrow[dr, phantom, "\Downarrow\bar{\alpha}"] & \bar{B} \arrow[d, "\bar{w}", "\bullet" marking] \\
\tilde{A} \arrow[r, "\tilde{f}"'] & \tilde{B}
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\bar{v}\bullet v"', "\bullet" marking] \arrow[dr, phantom, "\Downarrow\bar{\alpha}\bullet\alpha"] & B \arrow[d, "\bar{w}\bullet w", "\bullet" marking] \\
\tilde{A} \arrow[r, "\tilde{f}"'] & \tilde{B}
\end{tikzcd}$$

- Also have an "external" composition given by

$$\mathbf{A}_2 \xrightarrow{\circ} \mathbf{A}_1$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[r, "g"] \arrow[d, "w", "\bullet" marking] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "x", "\bullet" marking] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{B} \arrow[r, "\bar{g}"'] & \bar{C}
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "g\circ f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\Downarrow\beta\circ\alpha"] & C \arrow[d, "x", "\bullet" marking] \\
\bar{A} \arrow[r, "\bar{g}\circ\bar{f}"'] & \bar{C}
\end{tikzcd}$$

- $\circ$ and $\bullet$ are associative and unitary on arrows and cells

Double categories (cont.)

- Interchange

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "\bullet" marking] \\
\bar{A} \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\Downarrow\bar{\alpha}"] & \bar{B} \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\Downarrow\bar{\beta}"] & \bar{C} \arrow[d, "\bullet" marking] \\
\tilde{A} \arrow[r] & \tilde{B} \arrow[r] & \tilde{C}
\end{tikzcd}$$

$$(\bar{\beta} \circ \bar{\alpha}) \bullet (\beta \circ \alpha) = (\bar{\beta} \bullet \beta) \circ (\bar{\alpha} \bullet \alpha)$$

- Also identity interchange laws

$$1_F \bullet 1_v = 1_{\bar{v} \bullet v} \qquad \mathrm{id}_g \circ \mathrm{id}_f = \mathrm{id}_{g \circ f} \qquad 1_{\mathrm{id}_A} = \mathrm{id}_{1_A}$$

Double categories

- So a double category has two kinds of morphisms $\longrightarrow$ and $\xrightarrow{\bullet}$ and cells $\Downarrow$ tying them together

  Many instances of this:

  - External/internal
  - Total/partial
  - Deterministic/stochastic
  - Classical/quantum
  - Linear/smooth
  - Classical/intuitionistic
  - Lax/oplax
  - Strong/weak
  - Horizontal/vertical

  Double categories formalize this

Double categories

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  - External/internal
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  Double categories formalize this

- Double categories are categories with two related kinds of morphisms 😉

The usual suspects

- $\mathbb{R}\mathrm{el}$ – Sets, functions, relations

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "R"', "\bullet" marking] \arrow[dr, phantom, "\leq"] & B \arrow[d, "S", "\bullet" marking] \\
C \arrow[r, "g"'] & D
\end{tikzcd} \qquad a \sim_R c \Rightarrow f(a) \sim_S g(c)$$

If **A** is a regular category we can also construct $\mathbb{R}\mathrm{el}(\mathbf{A})$

- $\square\mathbf{A}$ – **A** any category – the double category of commutative squares in **A**

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

There is a *subdouble category* of pullback squares $\mathbb{P}\mathrm{b}\square\mathbf{A}$

- $\mathbb{Q}\mathcal{A}$ – $\mathcal{A}$ is a 2-category – the double category of *quintets* in $\mathcal{A}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

Slices

A category, **A**, has a nerve $$\begin{tikzcd}\cdots & A_3 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & A_0 \arrow[l]\end{tikzcd}$$

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- Drop the bottom arrows and we get a new category $$\begin{tikzcd}A_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A_2 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & A_1\end{tikzcd}$$
    - objects are arrows of **A**
    - morphisms $(f) \xrightarrow{x} (g)$ are commutative triangles $$\begin{tikzcd}B \arrow[rr, "x"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

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  - objects are arrows of **A**
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$$\begin{tikzcd} B \arrow[rr, "x"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

- It is the disjoint union of all slices $\sum_A \mathbf{A}/A$

Slices

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- Drop the bottom arrows and we get a new category $$\begin{tikzcd} A_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A_2 \arrow[r, shift left=2] \arrow[r, shift right=2] & A_1 \arrow[l] \end{tikzcd}$$
  - objects are arrows of **A**
  - morphisms $(f) \xrightarrow{x} (g)$ are commutative triangles

$$\begin{tikzcd} B \arrow[rr, "x"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

- It is the disjoint union of all slices $\sum_A \mathbf{A}/A$
- By dropping the top arrows, we also get a category whose objects are again arrows of **A** but morphisms $\begin{tikzcd}(f) \arrow[r, "y", "\bullet" marking] & (\bar{f})\end{tikzcd}$ now are commutative triangles

$$\begin{tikzcd} & A \arrow[dd, "y"] \\ B \arrow[ur, "f"] \arrow[dr, "\bar{f}"'] & \\ & \bar{A} \end{tikzcd}$$

- We get the disjoint union of all coslices $\sum_B B/\mathbf{A}$

Slices (cont.)

- We get a double category $\mathbb{S}\mathrm{lice}\,\mathbf{A}$
  - Objects are morphisms of $\mathbf{A}$
  - Horizontal arrows are slice morphisms (converging triangles)
  - Vertical arrows are coslice morphisms (diverging triangles)
  - Cells

$$\begin{tikzcd}
(f) \arrow[r, "x"] \arrow[d, "y"', "\bullet" marking] \arrow[dr, phantom, "\Downarrow"] & (g) \arrow[d, "z", "\bullet" marking] \\
(\bar{f}) \arrow[r, "\bar{x}"'] & (\bar{g})
\end{tikzcd}$$

are commutative tetrahedra: need $x = \bar{x}$, $z = y$ and

$$\begin{tikzcd}
B \arrow[r, "x"] \arrow[d, "\bar{f}"'] \arrow[dr, "f"'] & C \arrow[d, "g"] \arrow[dl, "\bar{g}"] \\
\bar{A} & A \arrow[l, "y"]
\end{tikzcd}$$

Spans

A a category with pullbacks
$\mathbb{S}$pan(**A**) has same objects as **A**

- horizontal arrows are morphisms of **A**
- vertical arrows $\begin{tikzcd} A \arrow[d, "\bullet" marking] \\ \bar{A} \end{tikzcd}$ are spans $\begin{tikzcd} & A \\ S \arrow[ur, "s_0"] \arrow[dr, "s_1"'] & \\ & \bar{A} \end{tikzcd}$
- cells $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "S"', "\bullet" marking] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "T", "\bullet" marking] \\ \bar{A} \arrow[r, "g"'] & \bar{B} \end{tikzcd}$ are commutative diagrams $\begin{tikzcd} & A \arrow[rr, "f"] & & B \\ S \arrow[ur, "s_0"] \arrow[rr, "\alpha"] \arrow[dr, "s_1"'] & & T \arrow[ur, "t_0"] \arrow[dr, "t_1"] & \\ & \bar{A} \arrow[rr, "g"'] & & \bar{B} \end{tikzcd}$
- vertical composition uses pullbacks

Weak double categories

$\mathbb{S}\mathrm{pan}(\mathbf{A})$ is not exactly a double category, it's a *weak double category*

- Same basic data and operations but vertical composition is only associative and unitary up to coherent globular isomorphism

$$\begin{tikzcd}
A_0 \arrow[r, equal] \arrow[d, "v_3\bullet(v_2\bullet v)"', "\bullet" marking] & A_0 \arrow[d, "(v_3\bullet v_2)\bullet v_1", "\bullet" marking] \\
A_3 \arrow[r, equal] \arrow[ur, phantom, "\overset{\cong}{\Longrightarrow}"] & A_3
\end{tikzcd}$$

- $\mathbb{S}\mathrm{pan}(\mathbf{Set}) = \mathbb{S}\mathrm{et}$

Another fundamental weak double category is $\mathbb{C}\mathrm{at}$

- Objects are small categories
- Horizontal arrows are functors
- Vertical arrows are profunctors
- Cells are natural transformations

The global or external theory of double categories

Double functors

A *double functor* $F : \mathbb{A} \longrightarrow \mathbb{B}$ consists of three functors F_0, F_1, F_2 making corresponding squares commute

$$\begin{tikzcd}
\mathbf{A}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "F_2"'] & \mathbf{A}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "F_1"] & \mathbf{A}_0 \arrow[l] \arrow[d, "F_0"] \\
\mathbf{B}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{B}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbf{B}_0 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v'", "\bullet" marking] \arrow[dr, phantom, "\longmapsto"] & FA \arrow[r, "Ff"] \arrow[d, "Fv"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FA' \arrow[d, "Fv'", "\bullet" marking] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{A}' & F\bar{A} \arrow[r, "F\bar{f}"'] & F\bar{A}'
\end{tikzcd}$$

preserves all compositions and identities

Example

A functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ induces a double functor $\square F : \square \mathbf{A} \longrightarrow \square \mathbf{B}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "g'"] \arrow[to=2-4, phantom, "\longmapsto"] & & FA \arrow[r, "Ff"] \arrow[d, "Fg"'] \arrow[dr, phantom, "F\alpha"] & FA' \arrow[d, "Fg'"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{A}' & & F\bar{A} \arrow[r, "F\bar{g}"'] & F\bar{A}'
\end{tikzcd}$$

Proposition

Every double functor $\square \mathbf{A} \longrightarrow \square \mathbf{B}$ is of this form

Proof.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "1_{A'}"] \arrow[to=2-4, phantom, "\longmapsto"] & & FA \arrow[r, "Ff"] \arrow[d, "F'f"'] \arrow[dr, phantom, "F\alpha"] & FA' \arrow[d, "1_{FA'}"] \\
A' \arrow[r, "1_{A'}"'] & A' & & FA' \arrow[r, "1_{FA'}"'] & FA'
\end{tikzcd}$$

□

Questions

Homework: What are double functors $\mathbb{S}\mathrm{lice}\,\mathbf{A} \longrightarrow \mathbb{S}\mathrm{lice}\,\mathbf{B}$ like?

Open question: What are double functors $\mathbb{P}\mathrm{b}\square\,\mathbf{A} \longrightarrow \mathbb{P}\mathrm{b}\square\,\mathbf{B}$ like?

Spans

- For categories **A** and **B** with pullbacks, a pullback preserving functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ gives

$$\mathbb{S}\mathrm{pan}\, F : \mathbb{S}\mathrm{pan}\, \mathbf{A} \longrightarrow \mathbb{S}\mathrm{pan}\, \mathbf{B}$$

$$\begin{tikzcd}
A \arrow[r, "f"] & A' & & FA \arrow[r, "Ff"] & FA' \\
S \arrow[u, "s_0"] \arrow[d, "s_1"'] \arrow[r, "\sigma"] & S' \arrow[u, "s_0'"'] \arrow[d, "s_1'"] & \longmapsto & FS \arrow[u, "Fs_0"] \arrow[d, "Fs_1"'] \arrow[r, "F\sigma"] & FS' \arrow[u, "Fs_0'"'] \arrow[d, "Fs_1'"] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{A}' & & F\bar{A} \arrow[r, "F\bar{f}"'] & F\bar{A}'
\end{tikzcd}$$

- Preservation of vertical composition comes from preservation of pullbacks and only holds up to coherent isomorphism
- $\mathbb{S}\mathrm{pan}\, \mathbf{A}$ and $\mathbb{S}\mathrm{pan}\, \mathbf{B}$ are weak double categories and $\mathbb{S}\mathrm{pan}\, F$ is a weak, or pseudo, double functor

Lax double functors

A *lax double functor* $F : \mathbb{A} \longrightarrow \mathbb{B}$ has the same data as a strict one

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v'", "\bullet" marking] & & FA \arrow[r, "Ff"] \arrow[d, "Fv"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FA' \arrow[d, "Fv'", "\bullet" marking] \\
\bar{A} \arrow[r, "\bar{f}"'] & \bar{A}' \arrow[r, phantom, "\longmapsto"] & {} & F\bar{A} \arrow[r, "F\bar{f}"'] & F\bar{A}'
\end{tikzcd}$$

- preserves horizontal composition
- for vertical composition there are given globular comparison cells

$$\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, "Fv"', "\bullet" marking] & FA \arrow[dd, "F(\bar{v}\bullet v)", "\bullet" marking] \\
F\bar{A} \arrow[d, "F\bar{v}"', "\bullet" marking] \arrow[r, phantom, "\overset{\phi(\bar{v},v)}{\Rightarrow}"] & {} \\
F\tilde{A} \arrow[r, equal] & F\tilde{A}
\end{tikzcd}
\text{ and }
\begin{tikzcd}
FA \arrow[r, equal] \arrow[d, "\mathrm{id}_{FA}"', "\bullet" marking] \arrow[dr, phantom, "\overset{\phi(A)}{\Rightarrow}"] & FA \arrow[d, "F(\mathrm{id}_A)", "\bullet" marking] \\
FA \arrow[r, equal] & FA
\end{tikzcd}$$

satisfying the "usual" coherence conditions

- For an *oplax double functor*, $\phi(\bar{v}, v)$ and $\phi(A)$ go in the opposite direction
- For a *pseudo double functor* they are isomorphisms

Examples

- Any functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ ($\mathbf{A}, \mathbf{B}$ with pullbacks) gives an *oplax normal* double functor

$$\mathbb{S}\mathrm{pan}\, F : \mathbb{S}\mathrm{pan}\, \mathbf{A} \longrightarrow \mathbb{S}\mathrm{pan}\, \mathbf{B}$$

- Given a double category $\mathbb{A}$ and an object A of $\mathbb{A}$ we get a hom functor

$$\mathbb{A}(A, -) : \mathbb{A} \longrightarrow \mathbb{S}\mathrm{et}$$

$$X \longmapsto \mathbb{A}(A, X) = \{A \xrightarrow{f} X \mid f \text{ horizontal}\}$$

$$\begin{tikzcd}
 & & \mathbb{A}(A, X) \\
X \arrow[d, "x"', "\bullet" marking] & \mathbb{A}(A, x) \arrow[ur] \arrow[dr] & \\
\bar{X} & & \mathbb{A}(A, \bar{X})
\end{tikzcd}$$

$$X \overset{x}{\longrightarrow} \bar{X} \longmapsto$$

where $\mathbb{A}(A, x)$ is the set of cells of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}"', "\bullet" marking] \arrow[dr, phantom, "\xi"] & X \arrow[d, "x", "\bullet" marking] \\
A \arrow[r, "\bar{f}"'] & \bar{X}
\end{tikzcd}$$

- $\mathbb{A}(A, -)$ is lax

Transformations

- Given double functors $F, G : \mathbb{A} \longrightarrow \mathbb{B}$, what should a transformation $F \longrightarrow G$ be?

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$$\begin{tikzcd}
\mathbf{A}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, bend right, "F_2"'] \arrow[d, bend left, "G_2"] & \mathbf{A}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, bend right, "F_1"'] \arrow[d, bend left, "G_1"] \arrow[dl, dashed] & \mathbf{A}_0 \arrow[l] \arrow[d, bend right, "F_0"'] \arrow[d, bend left, "G_0"] \arrow[dl, dashed] \\
\mathbf{B}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{B}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbf{B}_0 \arrow[l]
\end{tikzcd}$$

Transformations

- Given double functors $F, G : \mathbb{A} \longrightarrow \mathbb{B}$, what should a transformation $F \longrightarrow G$ be?

$$\begin{tikzcd}[column sep=large, row sep=huge]
\mathbf{A}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, bend right, "F_2"'{name=F2}] \arrow[d, bend left, "G_2"{name=G2}] & \mathbf{A}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, bend right, "F_1"'{name=F1}] \arrow[d, bend left, "G_1"{name=G1}] & \mathbf{A}_0 \arrow[l] \arrow[d, bend right, "F_0"'{name=F0}] \arrow[d, bend left, "G_0"{name=G0}] \\
\mathbf{B}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{B}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbf{B}_0 \arrow[l]
\arrow[Rightarrow, from=F2, to=G2, shorten=2pt]
\arrow[Rightarrow, from=F1, to=G1, shorten=2pt]
\arrow[Rightarrow, from=F0, to=G0, shorten=2pt]
\end{tikzcd}$$

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$$\begin{tikzcd}
\mathbf{A}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, bend right=40, "F_2"'] \arrow[d, bend left=40, "G_2"] \arrow[d, phantom, "\Rightarrow"] & \mathbf{A}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, bend right=40, "F_1"'] \arrow[d, bend left=40, "G_1"] \arrow[d, phantom, "\Rightarrow"] & \mathbf{A}_0 \arrow[l] \arrow[d, bend right=40, "F_0"'] \arrow[d, bend left=40, "G_0"] \arrow[d, phantom, "\Rightarrow"] \\
\mathbf{B}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{B}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbf{B}_0 \arrow[l]
\end{tikzcd}$$

- The first was external; this is internal!

Transformations

- Given double functors $F, G : \mathbb{A} \longrightarrow \mathbb{B}$, what should a transformation $F \longrightarrow G$ be?

$$\begin{tikzcd}
\mathbf{A}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, bend right, "F_2"'] \arrow[d, bend left, "G_2"] \arrow[d, phantom, "\Rightarrow"] & \mathbf{A}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, bend right, "F_1"'] \arrow[d, bend left, "G_1"] \arrow[d, phantom, "\Rightarrow"] & \mathbf{A}_0 \arrow[l] \arrow[d, bend right, "F_0"'] \arrow[d, bend left, "G_0"] \arrow[d, phantom, "\Rightarrow"] \\
\mathbf{B}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{B}_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbf{B}_0 \arrow[l]
\end{tikzcd}$$

- The first was external; this is internal!

Theorem

$\mathbb{D}$oub *(strict double categories) is cartesian closed*

Horizontal transformation

Definition

A *horizontal transformation* $t : F \longrightarrow G$ is given by the following:

- For every A in $\mathbb{A}$ a horizontal arrow $tA : FA \longrightarrow GA$
- For every $v : A \longrightarrow\!\!\bullet\!\!\rightarrow \bar{A}$ in $\mathbb{A}$ a cell

$$\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "Fv"'] & GA \arrow[d, "Gv"] \\
F\bar{A} \arrow[r, "t\bar{A}"'] \arrow[ur, phantom, "tv"] & G\bar{A}
\end{tikzcd}$$

satisfying

- Horizontal naturality (for arrows and cells)
- Vertical functoriality (two conditions)

A *vertical transformation* is the transpose notion, horizontal and vertical are switched

Modifications

There are also *double modifications*

Definition

A modification

$$\begin{tikzcd}
F \arrow[r, "t"] \arrow[d, "\phi"', "\bullet" marking] \arrow[dr, phantom, "\mu"] & G \arrow[d, "\psi", "\bullet" marking] \\
\bar{F} \arrow[r, "\bar{t}"'] & \bar{G}
\end{tikzcd}$$

is given by

$$\begin{tikzcd}
 & & FA \arrow[r, "tA"] \arrow[dd, "\phi A"', "\bullet" marking] \arrow[ddr, phantom, "\mu A"] & GA \arrow[dd, "\psi A", "\bullet" marking] \\
A \arrow[r, mapsto, "\mu"] & {} & & \\
 & & \bar{F}A \arrow[r, "\bar{t}A"'] & \bar{G}A
\end{tikzcd}$$

satisfying the “obvious” conditions, determined by cartesian closedness in the strict case

Bicategories

- A bicategory $\mathcal{B}$ gives a weak double category $\mathbb{V}\mathrm{ert}\,\mathcal{B}$ – in fact a bicategory is the same as a weak double category all of whose horizontal arrows are identities
- A lax (oplax) morphism of bicategories $F : \mathcal{B} \longrightarrow \mathcal{C}$ gives a lax (oplax) double functor $\mathbb{V}\mathrm{ert} F : \mathbb{V}\mathrm{ert}\,\mathcal{B} \longrightarrow \mathbb{V}\mathrm{ert}\,\mathcal{C}$
- If $F, G : \mathcal{B} \longrightarrow \mathcal{C}$ are oplax, then a horizontal transformation

  $$\mathbb{V}\mathrm{ert}\, F \longrightarrow \mathbb{V}\mathrm{ert}\, G$$

  is an ICON (Lack)
- A vertical transformation is a pseudo natural transformation

The internal theory of double categories

Companions

Definition

$A \xrightarrow{f} B$ and $\begin{tikzcd}A \arrow[r, "v", "\bullet" marking] & B\end{tikzcd}$ are *companions* if there are given cells (binding cells)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\eta"] & A \arrow[d, "v", "\bullet" marking] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\eta"] & A \arrow[r, "f"] \arrow[d, "v", "\bullet" marking] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \arrow[dr, phantom, "="] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_f"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
A \arrow[r, "f"'] & B \arrow[r, "1_B"'] & B & A \arrow[r, "f"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\eta"] & A \arrow[d, "v", "\bullet" marking] \\
A \arrow[r, "f"'] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}
\cdot = \cdot
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_v"] & A \arrow[d, "v", "\bullet" marking] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

Conjoints

Definition

$A \xrightarrow{f} B$ and $B \xrightarrow[\bullet]{u} A$ are *conjoints* if there are given cells (conjunctions)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "u", "\bullet" marking] & B \arrow[r, "1_B"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
A \arrow[r, "1_A"'] & A & A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that $\beta\alpha = \mathrm{id}_f$ and $\alpha \bullet \beta = 1_u$

Definition

$A \xrightarrow{f} B$ is *left adjoint* to $B \xrightarrow{g} A$ if it is so in $\mathcal{H}or\, \mathbb{A}$

$A \xrightarrow[\bullet]{v} B$ is *left adjoint* to $B \xrightarrow[\bullet]{u} A$ if it is so in $\mathcal{V}ert\, \mathbb{A}$

Theorem

(1) *If f has a companion (conjoint) it is unique up to globular isomorphism*

(2) *If f has companion (conjoint) v and g has companion (resp. conjoint) w then gf has companion $w \bullet v$ (resp. conjoint $v \bullet w$)*

(3) *Any two of the following conditions imply the third*

- *v is a companion for f*
- *w is a conjoint for f*
- *v is left adjoint to w in $\mathcal{V}ert\,\mathbb{A}$*

Proof.

Exercise!

□

Examples

- In $\mathbb{R}\mathrm{el}$ every function $f : A \longrightarrow B$ determines a relation $f_* : A \xrightarrow{\bullet} B$, its graph

  $$\{(a, b) \mid f(a) = b\}$$

  f_* is the companion of f
  The opposite relation $f^* : B \xrightarrow{\bullet} A$ is the conjoint of f
- In $\mathbb{S}\mathrm{pan}(\mathbf{A})$ every morphism $f : A \longrightarrow B$ has a companion and conjoint

$$\begin{tikzcd}
 & A & & & B \\
A \arrow[ur, "1_A"] \arrow[dr, "f"'] & & \text{and} & A \arrow[ur, "f"] \arrow[dr, "1_A"'] & \\
 & B & & & A
\end{tikzcd}$$

- In $\mathbb{C}\mathrm{at}$, every functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ determines two profunctors F_* and F^*, its companion and conjoint

Double limits

- The original motivation for studying double categories was to understand 2-dimensional limits
- There is now a well-developed theory of double limits
- Consider a couple of examples
- Tabulators

  Given a vertical arrow $\begin{tikzcd} A \arrow[r, "v", "\bullet" marking] & B \end{tikzcd}$ in $\mathbb{A}$ its *tabulator*, if it exists, is an object T and a cell τ

$$\begin{tikzcd} & A \arrow[dd, "\bullet" marking, "v"] \\ T \arrow[ur, "t_0"] \arrow[dr, "t_1"'] \arrow[r, phantom, "\tau"] & {} \\ & B \end{tikzcd}$$

i.e.

$$\begin{tikzcd} T \arrow[r, "t_0"] \arrow[d, "\bullet" marking, "\mathrm{id}_T"'] \arrow[dr, phantom, "\tau"] & A \arrow[d, "\bullet" marking, "v"] \\ T \arrow[r, "t_1"'] & B \end{tikzcd}$$

such that

for any other cell

$$\begin{tikzcd}
 & A \arrow[dd, "v", "\bullet"{marking}] \\
C \arrow[ur, "c_0"] \arrow[dr, "c_1"'] \arrow[r, phantom, "\gamma"] & \\
 & B
\end{tikzcd}$$

there exists a unique horizontal morphism $c : C \longrightarrow T$ such that $\gamma = \tau c$, i.e.

$$\begin{tikzcd}
C \arrow[r, "c_0"] \arrow[d, "\mathrm{id}_C"', "\bullet"{marking}] \arrow[dr, phantom, "\gamma"] & A \arrow[d, "v", "\bullet"{marking}] & = & C \arrow[r, "c"] \arrow[d, "\mathrm{id}_C"', "\bullet"{marking}] \arrow[dr, phantom, "\mathrm{id}_c"] & T \arrow[r, "t_0"] \arrow[d, "\mathrm{id}_T", "\bullet"{marking}] \arrow[dr, phantom, "\tau"] & A \arrow[d, "v", "\bullet"{marking}] \\
C \arrow[r, "c_1"'] & B & & C \arrow[r, "c"'] & T \arrow[r, "t_1"'] & B
\end{tikzcd}$$

There is a 2-dimensional universal property to keep in mind, the *tetrahedron condition*:

Every commutative tetrahedron
$$\begin{tikzcd}
C \arrow[r] \arrow[d, "u"', "\bullet"{marking}] \arrow[dr] & A \arrow[d, "v", "\bullet"{marking}] \\
D \arrow[r] \arrow[ur] & B
\end{tikzcd}$$
factors uniquely as
$$\begin{tikzcd}
C \arrow[dd, "u"', "\bullet"{marking}] \arrow[dr] & & A \arrow[dd, "v", "\bullet"{marking}] \\
 & T \arrow[ur] \arrow[dr] \arrow[r, phantom, "\tau"] & \\
D \arrow[ur] & & B
\end{tikzcd}$$

- A tabulator is *effective* if
  - t_0 has a conjoint t_0^*
  - t_1 has a companion t_{1*}
  - the induced cell

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "t_0^*"', "\bullet" marking] & A \arrow[dd, "v", "\bullet" marking] \\
T \arrow[d, "t_{1*}"', "\bullet" marking] \arrow[r, phantom, "\overset{\sim}{\tau}"] & \\
B \arrow[r, equal] & B
\end{tikzcd}$$

    is an isomorphism

- $\mathbb{R}\mathrm{el}(\mathbf{A})$, $\mathbb{S}\mathrm{pan}\,\mathbf{A}$, $\mathbb{C}\mathrm{at}$ have effective tabulators

Given a relation $R : A \xrightarrow{\bullet} B$ in $\mathbb{S}\mathrm{et}$, e.g. congruence mod p, we can tabulate it

$T(R) = \{(a, b) | a \sim_R b\}$ and
$$\begin{tikzcd}
 & A \arrow[dd, "R", "\bullet" marking] \\
T(R) \arrow[ur, "t_0"] \arrow[dr, "c"'] \arrow[r, phantom, "\tau"] & \\
 & B
\end{tikzcd}$$
is the tabulator

Binary products

Given A and B in $\mathbb{A}$, their product is an object $A \times B$ with two horizontal morphisms p_1, p_2

$$\begin{tikzcd}
 & A \\
A \times B \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\
 & B
\end{tikzcd}$$

which has the universal property for horizontal morphisms, i.e. it is a product in $\mathbf{Hor}(\mathbb{A})$.

It also has a 2-dimensional universal property: for cells

$$\begin{tikzcd} C \arrow[rd] \arrow[dd, "u"', "\bullet" marking] & \\ {} \arrow[r, phantom, "\alpha"] & A \\ D \arrow[ru] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[rd] \arrow[dd, "u"', "\bullet" marking] & \\ {} \arrow[r, phantom, "\beta"] & B \\ D \arrow[ru] & \end{tikzcd}$$

there exists a unique cell

$$\begin{tikzcd} C \arrow[rd] \arrow[dd, "u"', "\bullet" marking] & \\ {} \arrow[r, phantom, "{(\alpha,\beta)}"] & A \times B \\ D \arrow[ru] & \end{tikzcd}$$

such that $p_1(\alpha, \beta) = \alpha$ and $p_2(\alpha, \beta) = \beta$

An intermediate condition is to require the 2-dimensional condition only for globular cells, i.e. $u = \text{id}$

Binary products (cont.)

$\mathbb{A}$ has binary products if

(1) Every A, B has a horizontal product

(2) Every $\begin{tikzcd} A \arrow[r, "v", "\bullet" marking] & C \end{tikzcd}$, $\begin{tikzcd} B \arrow[r, "w", "\bullet" marking] & D \end{tikzcd}$ has a product

$$\begin{tikzcd}
A \times B \arrow[r, "p_1"] \arrow[d, "v \times w"', "\bullet" marking] \arrow[dr, phantom, "\pi_1"] & A \arrow[d, "v", "\bullet" marking] & & A \times B \arrow[r, "p_2"] \arrow[d, "v \times w"'] \arrow[dr, phantom, "\pi_2"] & B \arrow[d, "w"] \\
C \times D \arrow[r, "p_1"'] & C & , & C \times D \arrow[r, "p_2"'] & D
\end{tikzcd}$$

We get a lax functor

$$(\ ) \times (\ ) : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$$

- $(\ ) \times (\ )$ is normal ($\mathrm{id}_A \times \mathrm{id}_B \simeq \mathrm{id}_{A \times B}$) if and only if products have the 2-dimensional universal property
- We usually require that $(\ ) \times (\ )$ be pseudo

Examples

A has binary products

if and only if $\square\mathbf{A}$ has binary products

if and only if $\mathbb{S}$pan(**A**) has binary products

However not $\mathbb{P}\mathrm{b}\square(\mathbf{A})$ nor $\mathbb{S}\mathrm{lice}(\mathbf{A})$

$\mathbb{C}\mathrm{at}$ also has binary products

They are all pseudo

Note: The same holds for infinite products *but* for $\mathbb{C}\mathrm{at}$ they are merely lax