

Double Categories

The best thing since slice categories

(<https://www.mscs.dal.ca/~pare/FMCS1.pdf>)

Robert Paré

FMCS Tutorial
Mount Allison

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- A double category is a category object in **Cat** 😊

$$\mathbb{A} : \quad \mathbf{A}_2 \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{\circ} \\ \xrightarrow{p_2} \end{array} \mathbf{A}_1 \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{-1} \\ \xrightarrow{d_1} \end{array} \mathbf{A}_0$$

\mathbb{A} has

- objects A, A', \dots the objects of \mathbf{A}_0
- morphisms $A \xrightarrow{f} A'$, the *objects* of \mathbf{A}_1
- composition $A \xrightarrow{f} A' \xrightarrow{f'} A'' = A \xrightarrow{f' \circ f} A''$
- identities $1_A : A \rightarrow A$.

Double categories (cont.)

- \mathbf{A}_0 also has morphisms – another kind, internal

- $A \xrightarrow{\bar{v}} \bar{A}$

- Composition $A \xrightarrow{\bar{v}} \bar{A} \xrightarrow{\tilde{v}} \tilde{A} = A \xrightarrow{\tilde{v} \bullet v} \tilde{A}$

- Identities $\text{id}_A : A \xrightarrow{\bullet} A$

Double categories (cont.)

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 - $A \xrightarrow{\bullet v} \bar{A}$
 - Composition $A \xrightarrow{\bullet v} \bar{A} \xrightarrow{\bullet \bar{v}} \tilde{A} = A \xrightarrow{\bullet \bar{v} \bullet v} \tilde{A}$
 - Identities $\text{id}_A : A \xrightarrow{\bullet} A$
- \mathbf{A}_1 has morphisms too – morphisms between external morphisms – cells

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \Downarrow \alpha & \\ \bar{A} & \xrightarrow{\bar{f}} & \bar{B} \end{array}$$

Double categories (cont.)

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- $A \xrightarrow{\bullet} \bar{A}$

- Composition $A \xrightarrow{\bullet} \bar{A} \xrightarrow{\bullet} \tilde{A} = A \xrightarrow{\bullet \circ \bullet} \tilde{A}$

- Identities $\text{id}_A : A \xrightarrow{\bullet} A$

- \mathbf{A}_1 has morphisms too – morphisms between external morphisms – cells

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow v & \Downarrow \alpha & \downarrow w \\ \bar{A} & \xrightarrow{\bar{f}} & \bar{B} \end{array}$$

Double categories (cont.)

- Cells compose in \mathbf{A}_1

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow v & \Downarrow \alpha & \downarrow w \\
 \bar{A} & \xrightarrow{\bar{f}} & \bar{B} \\
 \downarrow \bar{v} & \Downarrow \bar{\alpha} & \downarrow \bar{w} \\
 \tilde{A} & \xrightarrow{\tilde{f}} & \tilde{B}
 \end{array}
 =
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow \bar{v} \bullet v & \Downarrow \bar{\alpha} \bullet \alpha & \downarrow \bar{w} \bullet w \\
 \tilde{A} & \xrightarrow{\tilde{f}} & \tilde{B}
 \end{array}$$

- Also have an “external” composition given by

$$\mathbf{A}_2 \xrightarrow{\circ} \mathbf{A}_1$$

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 \downarrow v & \Downarrow \alpha & \downarrow w & \Downarrow \beta & \downarrow x \\
 \bar{A} & \xrightarrow{\bar{f}} & \bar{B} & \xrightarrow{\bar{g}} & \bar{C}
 \end{array}
 =
 \begin{array}{ccc}
 A & \xrightarrow{g \circ f} & C \\
 \downarrow v & \Downarrow \beta \circ \alpha & \downarrow x \\
 \bar{A} & \xrightarrow{\bar{g} \circ \bar{f}} & \bar{C}
 \end{array}$$

- \circ and \bullet are associative and unitary on arrows and cells

Double categories (cont.)

- Interchange

$$\begin{array}{ccccc}
 A & \longrightarrow & B & \longrightarrow & C \\
 \downarrow & & \downarrow & & \downarrow \\
 \bullet & & \bullet & & \bullet \\
 \downarrow & \Downarrow \alpha & \downarrow & \Downarrow \beta & \downarrow \\
 \bar{A} & \longrightarrow & \bar{B} & \longrightarrow & \bar{C} \\
 \downarrow & & \downarrow & & \downarrow \\
 \bullet & & \bullet & & \bullet \\
 \downarrow & \Downarrow \bar{\alpha} & \downarrow & \Downarrow \bar{\beta} & \downarrow \\
 \tilde{A} & \longrightarrow & \tilde{B} & \longrightarrow & \tilde{C}
 \end{array}$$

$$(\bar{\beta} \circ \bar{\alpha}) \bullet (\beta \circ \alpha) = (\bar{\beta} \bullet \beta) \circ (\bar{\alpha} \bullet \alpha)$$

- Also identity interchange laws

$$1_F \bullet 1_V = 1_{V \bullet F}$$

$$\text{id}_g \circ \text{id}_f = \text{id}_{g \circ f}$$

$$1_{\text{id}_A} = \text{id}_{1_A}$$

Double categories

- So a double category has two kinds of morphisms \rightarrow and $\xrightarrow{\bullet}$ and cells \Downarrow tying them together

Many instances of this:

- External/internal
- Total/partial
- Deterministic/stochastic
- Classical/quantum
- Linear/smooth
- Classical/intuitionistic
- Lax/oplax
- Strong/weak
- Horizontal/vertical

Double categories formalize this

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Double categories formalize this

- Double categories are categories with two related kinds of morphisms



The usual suspects

- $\mathbb{R}el$ – Sets, functions, relations

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 R \downarrow & \leq & \downarrow S \\
 C & \xrightarrow{g} & D
 \end{array}
 \quad a \sim_R c \Rightarrow f(a) \sim_S g(c)$$

If \mathbf{A} is a regular category we can also construct $\mathbb{R}el(\mathbf{A})$

- $\square \mathbf{A}$ – \mathbf{A} any category – the double category of commutative squares in \mathbf{A}

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 h \downarrow & & \downarrow k \\
 C & \xrightarrow{g} & D
 \end{array}$$

There is a *subdouble category* of pullback squares $\mathbb{P}b\square \mathbf{A}$

- $\mathbb{Q}\mathcal{A}$ – \mathcal{A} is a 2-category – the double category of *quintets* in \mathcal{A}

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 h \downarrow & \alpha \swarrow & \downarrow k \\
 C & \xrightarrow{g} & D
 \end{array}$$

Slices

A category, \mathbf{A} , has a nerve $\cdots A_3 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} A_2 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} A_1 \begin{array}{c} \rightrightarrows \\ \leftleftarrows \\ \rightrightarrows \end{array} A_0$

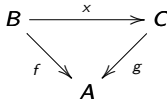
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- Drop the bottom arrows and we get a new category $A_3 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} A_2 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} A_1$

- objects are arrows of \mathbf{A}

- morphisms $(f) \xrightarrow{x} (g)$ are commutative triangles



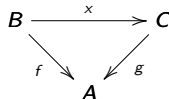
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- It is the disjoint union of all slices $\sum_A \mathbf{A}/A$

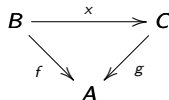
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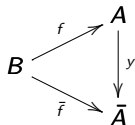
- Drop the bottom arrows and we get a new category $A_3 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} A_2 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} A_1$

- objects are arrows of \mathbf{A}

- morphisms $(f) \xrightarrow{x} (g)$ are commutative triangles



- It is the disjoint union of all slices $\sum_A \mathbf{A}/A$
- By dropping the top arrows, we also get a category whose objects are again arrows of \mathbf{A} but morphisms $(f) \xrightarrow{y} (\bar{f})$ now are commutative triangles



- We get the disjoint union of all coslices $\sum_B B/\mathbf{A}$

Slices (cont.)

- We get a double category **Slice** **A**
 - Objects are morphisms of **A**
 - Horizontal arrows are slice morphisms (converging triangles)
 - Vertical arrows are coslice morphisms (diverging triangles)
 - Cells

$$\begin{array}{ccc} (f) & \xrightarrow{x} & (g) \\ \downarrow y & \Downarrow & \downarrow z \\ (\bar{f}) & \xrightarrow{\bar{x}} & (\bar{g}) \end{array}$$

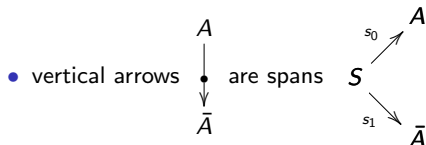
are commutative tetrahedra: need $x = \bar{x}$, $z = y$ and

$$\begin{array}{ccc} B & \xrightarrow{x} & C \\ \bar{f} \downarrow & \begin{array}{c} f \swarrow \\ \bar{g} \searrow \end{array} & \downarrow g \\ \bar{A} & \xleftarrow{y} & A \end{array}$$

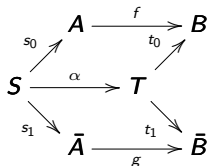
Spans

\mathbf{A} a category with pullbacks
 $\mathbb{S}\text{pan}(\mathbf{A})$ has same objects as \mathbf{A}

- horizontal arrows are morphisms of \mathbf{A}



- cells $\begin{array}{ccc} A & \xrightarrow{f} & B \\ s \downarrow & \Rightarrow \alpha & \downarrow T \\ \bar{A} & \xrightarrow{g} & \bar{B} \end{array}$ are commutative diagrams



- vertical composition uses pullbacks

Weak double categories

$\text{Span}(\mathbf{A})$ is not exactly a double category, it's a *weak double category*

- Same basic data and operations but vertical composition is only associative and unitary up to coherent globular isomorphism

$$\begin{array}{ccc} A_0 & \equiv & A_0 \\ \downarrow & & \downarrow \\ v_3 \bullet (v_2 \bullet v) & \xRightarrow{\cong} & (v_3 \bullet v_2) \bullet v_1 \\ \downarrow & & \downarrow \\ A_3 & \equiv & A_3 \end{array}$$

- $\text{Span}(\mathbf{Set}) = \mathbf{Set}$

Another fundamental weak double category is \mathbf{Cat}

- Objects are small categories
- Horizontal arrows are functors
- Vertical arrows are profunctors
- Cells are natural transformations

The global or external theory of double categories

Double functors

A double functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ consists of three functors F_0, F_1, F_2 making corresponding squares commute

$$\begin{array}{ccccc}
 \mathbf{A}_2 & \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} & \mathbf{A}_1 & \begin{array}{c} \leftleftarrows \\ \leftleftarrows \\ \leftleftarrows \end{array} & \mathbf{A}_0 \\
 \downarrow F_2 & & \downarrow F_1 & & \downarrow F_0 \\
 \mathbf{B}_2 & \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} & \mathbf{B}_1 & \begin{array}{c} \leftleftarrows \\ \leftleftarrows \\ \leftleftarrows \end{array} & \mathbf{B}_0
 \end{array}$$

$$\begin{array}{ccc}
 A \xrightarrow{f} A' & & FA \xrightarrow{Ff} FA' \\
 \downarrow v \quad \alpha \quad \downarrow v' & \longmapsto & \downarrow Fv \quad F\alpha \quad \downarrow Fv' \\
 \bar{A} \xrightarrow{\bar{f}} \bar{A}' & & F\bar{A} \xrightarrow{F\bar{f}} F\bar{A}'
 \end{array}$$

preserves all compositions and identities

Example

A functor $F : \mathbf{A} \rightarrow \mathbf{B}$ induces a double functor $\square F : \square \mathbf{A} \rightarrow \square \mathbf{B}$

$$\begin{array}{ccc}
 A \xrightarrow{f} A' & & FA \xrightarrow{Ff} FA' \\
 g \downarrow & \alpha & \downarrow g' \\
 \bar{A} \xrightarrow{\bar{f}} \bar{A}' & \longmapsto & F\bar{A} \xrightarrow{F\bar{f}} F\bar{A}' \\
 & & Fg \downarrow \quad F\alpha \quad \downarrow Fg'
 \end{array}$$

Proposition

Every double functor $\square \mathbf{A} \rightarrow \square \mathbf{B}$ is of this form

Proof.

$$\begin{array}{ccc}
 A \xrightarrow{f} A' & & FA \xrightarrow{Ff} FA' \\
 f \downarrow & \alpha & \downarrow 1_{A'} \\
 A' \xrightarrow{1_{A'}} A' & \longmapsto & FA' \xrightarrow{1_{FA'}} FA' \\
 & & F'f \downarrow \quad F\alpha \quad \downarrow 1_{FA'}
 \end{array}$$



Questions

Homework: What are double functors $\mathbb{S}\text{lice } \mathbf{A} \rightarrow \mathbb{S}\text{lice } \mathbf{B}$ like?

Open question: What are double functors $\mathbb{P}\text{b}\square \mathbf{A} \rightarrow \mathbb{P}\text{b}\square \mathbf{B}$ like?

- For categories **A** and **B** with pullbacks, a pullback preserving functor $F : \mathbf{A} \rightarrow \mathbf{B}$ gives

$$\text{Span } F : \text{Span } \mathbf{A} \rightarrow \text{Span } \mathbf{B}$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 s_0 \uparrow & & \uparrow s'_0 \\
 S & \xrightarrow{\sigma} & S' \\
 s_1 \downarrow & & \downarrow s'_1 \\
 \bar{A} & \xrightarrow{\bar{f}} & \bar{A}'
 \end{array}
 & \mapsto &
 \begin{array}{ccc}
 FA & \xrightarrow{Ff} & FA' \\
 F_{s_0} \uparrow & & \uparrow F_{s'_0} \\
 FS & \xrightarrow{F\sigma} & FS' \\
 F_{s_1} \downarrow & & \downarrow F_{s'_1} \\
 F\bar{A} & \xrightarrow{F\bar{f}} & F\bar{A}'
 \end{array}
 \end{array}$$

- Preservation of vertical composition comes from preservation of pullbacks and only holds up to coherent isomorphism
- $\text{Span } \mathbf{A}$ and $\text{Span } \mathbf{B}$ are weak double categories and $\text{Span } F$ is a weak, or pseudo, double functor

Lax double functors

A lax double functor $F : \mathbb{A} \multimap \mathbb{B}$ has the same data as a strict one

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 \downarrow v & \alpha & \downarrow v' \\
 \bar{A} & \xrightarrow{\bar{f}} & \bar{A}'
 \end{array} & \longmapsto &
 \begin{array}{ccc}
 FA & \xrightarrow{Ff} & FA' \\
 \downarrow Fv & F\alpha & \downarrow Fv' \\
 F\bar{A} & \xrightarrow{F\bar{f}} & F\bar{A}'
 \end{array}
 \end{array}$$

- preserves horizontal composition
- for vertical composition there are given globular comparison cells

$$\begin{array}{ccc}
 FA = FA & & FA = FA \\
 \downarrow Fv & & \downarrow \text{id}_{FA} \\
 F\bar{A} & \xrightarrow{\phi(\bar{v}, v)} & \bullet \\
 \downarrow F\bar{v} & & \downarrow F(\text{id}_A) \\
 F\tilde{A} = F\tilde{A} & & FA = FA
 \end{array}
 \text{ and }
 \begin{array}{ccc}
 FA = FA & & FA = FA \\
 \downarrow \text{id}_{FA} & \xrightarrow{\phi(A)} & \downarrow F(\text{id}_A) \\
 FA = FA & & FA = FA
 \end{array}$$

satisfying the “usual” coherence conditions

- For an *oplax double functor*, $\phi(\bar{v}, v)$ and $\phi(A)$ go in the opposite direction
- For a *pseudo double functor* they are isomorphisms

Examples

- Any functor $F : \mathbf{A} \rightarrow \mathbf{B}$ (\mathbf{A}, \mathbf{B} with pullbacks) gives an *oplax normal* double functor

$$\text{Span } F : \text{Span } \mathbf{A} \rightarrow \text{Span } \mathbf{B}$$

- Given a double category \mathbb{A} and an object A of \mathbb{A} we get a hom functor

$$\mathbb{A}(A, -) : \mathbb{A} \rightarrow \text{Set}$$

$$X \mapsto \mathbb{A}(A, X) = \{A \xrightarrow{f} X \mid f \text{ horizontal}\}$$

$$\begin{array}{ccc}
 & & \mathbb{A}(A, X) \\
 & & \nearrow \\
 X & \xrightarrow{\quad} & \mathbb{A}(A, x) \\
 \downarrow x & & \searrow \\
 \bar{X} & & \mathbb{A}(A, \bar{X})
 \end{array}$$

where $\mathbb{A}(A, x)$ is the set of cells of the form

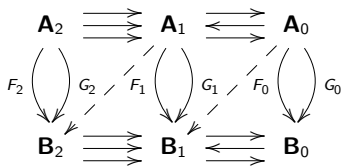
$$\begin{array}{ccc}
 A & \xrightarrow{f} & X \\
 \text{id} \downarrow & \xi & \downarrow x \\
 A & \xrightarrow{\bar{f}} & \bar{X}
 \end{array}$$

- $\mathbb{A}(A, -)$ is lax

- Given double functors $F, G : \mathbb{A} \rightarrow \mathbb{B}$, what should a transformation $F \rightarrow G$ be?

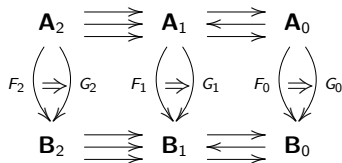
Transformations

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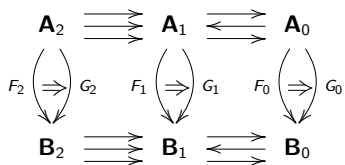
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- The first was external; this is internal!

Transformations

- Given double functors $F, G : \mathbb{A} \rightarrow \mathbb{B}$, what should a transformation $F \rightarrow G$ be?

$$\begin{array}{ccccc} \mathbf{A}_2 & \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \leftarrow \\ \rightrightarrows \end{array} & \mathbf{A}_1 & \begin{array}{c} \rightrightarrows \\ \leftarrow \\ \rightrightarrows \\ \leftarrow \\ \rightrightarrows \end{array} & \mathbf{A}_0 \\ F_2 \left(\begin{array}{c} \curvearrowright \\ \Rightarrow \\ \curvearrowleft \end{array} \right) G_2 & & F_1 \left(\begin{array}{c} \curvearrowright \\ \Rightarrow \\ \curvearrowleft \end{array} \right) G_1 & & F_0 \left(\begin{array}{c} \curvearrowright \\ \Rightarrow \\ \curvearrowleft \end{array} \right) G_0 \\ \mathbf{B}_2 & \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \leftarrow \\ \rightrightarrows \end{array} & \mathbf{B}_1 & \begin{array}{c} \rightrightarrows \\ \leftarrow \\ \rightrightarrows \\ \leftarrow \\ \rightrightarrows \end{array} & \mathbf{B}_0 \end{array}$$

- The first was external; this is internal!

Theorem

Doub (strict double categories) is cartesian closed

Horizontal transformation

Definition

A horizontal transformation $t : F \longrightarrow G$ is given by the following:

- For every A in \mathbb{A} a horizontal arrow $tA : FA \longrightarrow GA$
- For every $v : A \longrightarrow \bar{A}$ in \mathbb{A} a cell

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ \downarrow Fv & & \downarrow Gv \\ F\bar{A} & \xrightarrow{t\bar{A}} & G\bar{A} \end{array} \quad tv$$

satisfying

- Horizontal naturality (for arrows and cells)
- Vertical functoriality (two conditions)

A vertical transformation is the transpose notion, horizontal and vertical are switched

Modifications

There are also *double modifications*

Definition

A modification

$$\begin{array}{ccc} F & \xrightarrow{t} & G \\ \phi \downarrow & \mu & \downarrow \psi \\ \bar{F} & \xrightarrow{\bar{t}} & \bar{G} \end{array}$$

is given by

$$A \quad \xrightarrow{\mu} \quad \begin{array}{ccc} FA & \xrightarrow{t^A} & GA \\ \phi^A \downarrow & \mu^A & \downarrow \psi^A \\ \bar{F}A & \xrightarrow{\bar{t}^A} & \bar{G}A \end{array}$$

satisfying the “obvious” conditions, determined by cartesian closedness in the strict case

- A bicategory \mathcal{B} gives a weak double category $\text{Vert } \mathcal{B}$ – in fact a bicategory is the same as a weak double category all of whose horizontal arrows are identities
- A lax (oplax) morphism of bicategories $F : \mathcal{B} \rightarrow \mathcal{C}$ gives a lax (oplax) double functor $\text{Vert } F : \text{Vert } \mathcal{B} \rightarrow \text{Vert } \mathcal{C}$
- If $F, G : \mathcal{B} \rightarrow \mathcal{C}$ are oplax, then a horizontal transformation

$$\text{Vert } F \rightarrow \text{Vert } G$$

is an ICON (Lack)

- A vertical transformation is a pseudo natural transformation

The internal theory of double categories

Companions

Definition

$A \xrightarrow{f} B$ and $A \xrightarrow{v} B$ are *companions* if there are given cells (binding cells)

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow v & \epsilon & \downarrow \text{id}_B \\
 B & \xrightarrow{1_B} & B
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 A & \xrightarrow{1_A} & A \\
 \downarrow \text{id}_A & \eta & \downarrow v \\
 A & \xrightarrow{f} & B
 \end{array}$$

such that

$$\begin{array}{ccc}
 A & \xrightarrow{1_A} & A & \xrightarrow{f} & B \\
 \downarrow \text{id}_A & \eta & \downarrow v & \epsilon & \downarrow \text{id}_B \\
 A & \xrightarrow{f} & B & \xrightarrow{1_B} & B
 \end{array}
 =
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow \text{id}_A & \text{id}_f & \downarrow \text{id}_B \\
 A & \xrightarrow{f} & B
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 A & \xrightarrow{1_A} & A \\
 \downarrow \text{id}_A & \eta & \downarrow v \\
 A & \xrightarrow{f} & B \\
 \downarrow v & \epsilon & \downarrow \text{id}_B \\
 B & \xrightarrow{1_B} & B
 \end{array}
 \cdot
 \begin{array}{ccc}
 A & \xrightarrow{1_A} & A \\
 \downarrow v & \text{id}_v & \downarrow v \\
 B & \xrightarrow{1_B} & B
 \end{array}$$

Conjoints

Definition

$A \xrightarrow{f} B$ and $B \xrightarrow{u} A$ are *conjoints* if there are given cells (conjunctions)

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \text{id} \downarrow \bullet & \alpha & \downarrow \bullet u \\ A & \xrightarrow{1_A} & A \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{1_B} & B \\ u \downarrow \bullet & \beta & \downarrow \bullet \text{id}_B \\ A & \xrightarrow{f} & B \end{array}$$

such that $\beta\alpha = \text{id}_f$ and $\alpha \bullet \beta = 1_u$

Definition

$A \xrightarrow{f} B$ is *left adjoint* to $B \xrightarrow{g} A$ if it is so in $\mathcal{H}or \mathbb{A}$

$A \xrightarrow{v} B$ is *left adjoint* to $B \xrightarrow{u} A$ if it is so in $\mathcal{V}ert \mathbb{A}$

Theorem

- (1) *If f has a companion (conjoint) it is unique up to globular isomorphism*
- (2) *If f has companion (conjoint) v and g has companion (resp. conjoint) w then gf has companion $w \bullet v$ (resp. conjoint $v \bullet w$)*
- (3) *Any two of the following conditions imply the third*
 - *v is a companion for f*
 - *w is a conjoint for f*
 - *v is left adjoint to w in $\mathcal{V}ert \mathbb{A}$*

Proof.

Exercise!



Examples

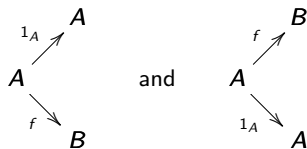
- In $\mathbb{R}el$ every function $f : A \rightarrow B$ determines a relation $f_* : A \multimap B$, its graph

$$\{(a, b) \mid f(a) = b\}$$

f_* is the companion of f

The opposite relation $f^* : B \multimap A$ is the conjoint of f

- In $\mathbb{S}pan(\mathbf{A})$ every morphism $f : A \rightarrow B$ has a companion and conjoint

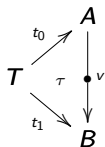


- In $\mathbb{C}at$, every functor $F : \mathbf{A} \rightarrow \mathbf{B}$ determines two profunctors F_* and F^* , its companion and conjoint

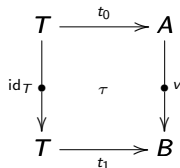
Double limits

- The original motivation for studying double categories was to understand 2-dimensional limits
- There is now a well-developed theory of double limits
- Consider a couple of examples
- Tabulators

Given a vertical arrow $A \xrightarrow{\nu} B$ in \mathbb{A} its *tabulator*, if it exists, is an object T and a cell τ

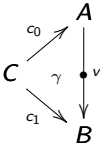


i.e.

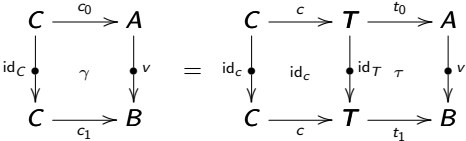


such that

for any other cell

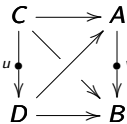


there exists a unique horizontal morphism $c : C \rightarrow T$ such that $\gamma = \tau c$, i.e.

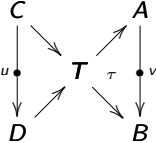


There is a 2-dimensional universal property to keep in mind, the *tetrahedron condition*:

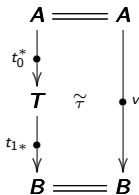
Every commutative tetrahedron



factors uniquely as



- A tabulator is *effective* if
 - t_0 has a conjoint t_0^*
 - t_1 has a companion t_{1*}
 - the induced cell



is an isomorphism

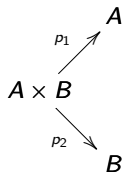
- $\mathbf{Rel}(\mathbf{A})$, $\mathbf{Span} \mathbf{A}$, \mathbf{Cat} have effective tabulators

Given a relation $R : A \multimap B$ in \mathbf{Set} , e.g. congruence mod ρ , we can tabulate it

$$T(R) = \{(a, b) \mid a \sim_R b\} \text{ and } T(R) \begin{array}{c} \nearrow t_0 \\ \bullet \\ \searrow c \\ \downarrow R \\ B \end{array} \text{ is the tabulator}$$

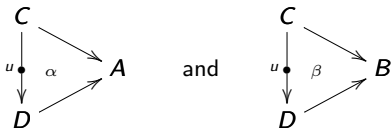
Binary products

Given A and B in \mathbb{A} , their product is an object $A \times B$ with two horizontal morphisms p_1 , p_2

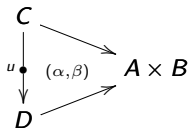


which has the universal property for horizontal morphisms, i.e. it is a product in $\mathbf{Hor}(\mathbb{A})$.

It also has a 2-dimensional universal property: for cells



there exists a unique cell



such that $p_1(\alpha, \beta) = \alpha$ and $p_2(\alpha, \beta) = \beta$

An intermediate condition is to require the 2-dimensional condition only for globular cells, i.e. $u = \text{id}$

Binary products (cont.)

\mathbb{A} has binary products if

- (1) Every A, B has a horizontal product
- (2) Every $A \xrightarrow{v} C, B \xrightarrow{w} D$ has a product

$$\begin{array}{ccc}
 A \times B & \xrightarrow{p_1} & A \\
 \downarrow v \times w & \pi_1 & \downarrow v \\
 C \times D & \xrightarrow{p_1} & C
 \end{array}
 ,
 \begin{array}{ccc}
 A \times B & \xrightarrow{p_2} & B \\
 \downarrow v \times w & \pi_2 & \downarrow w \\
 C \times D & \xrightarrow{p_2} & D
 \end{array}$$

We get a lax functor

$$(\) \times (\) : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$$

- $(\) \times (\)$ is normal ($\text{id}_A \times \text{id}_B \simeq \text{id}_{A \times B}$) if and only if products have the 2-dimensional universal property
- We usually require that $(\) \times (\)$ be pseudo

Examples

\mathbf{A} has binary products

if and only if $\square\mathbf{A}$ has binary products

if and only if $\mathbb{S}\text{pan}(\mathbf{A})$ has binary products

However not $\mathbb{P}\text{b}\square(\mathbf{A})$ nor $\mathbb{S}\text{lice}(\mathbf{A})$

$\mathbb{C}\text{at}$ also has binary products

They are all pseudo

Note: The same holds for infinite products *but* for $\mathbb{C}\text{at}$ they are merely lax