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Lateral Separation

A reconsideration of the Axiom of Replacement

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From a 1922 essay by Adolf Fraenkel:

I. *The seven axioms of Zermelo do not suffice for establishing set theory.*

The following simple example serves to justify this assertion: consider Z_0, the set of positive integers that Zermelo defined and showed to exist; the power-set $\wp(Z_0)$ (the set of the subsets of Z_0) may then be indicated by Z_1, $\wp(Z_1)$ by Z_2, and so on. Then, as the review of Zermelo's axioms easily reveals, they do not permit the formation of the set $\{Z_0, Z_1, \ldots\}$, and therefore neither the union of this set. It follows that ... the axioms do not allow us to prove, e.g., the existence of sets having powers $\geq \aleph_\omega$.

This hitherto unremarked gap in Zermelo's account is to be filled either by the addition of a new axiom or by the broadening of one already available. For the example cited, one can get by with a (very extensive) broadening of Axiom VII [Infinity]. However, should we systematically devise farther-reaching, more universal counter-examples along the lines of that one example, we see that a *universal* requirement of a new kind must be erected: from such consideration we come right to the following Axiom:

Axiom of Replacement: Let M be an existing set and let each element of M be replaced by a "thing from domain D" (as Zermelo put it), then M gets transformed into yet another existing set.

To show the existence of the set taken as a example above, namely $\{Z_0, Z_1, \ldots\}$, one has only to replace the element 0 with Z_0, the element {0} [= 1] with Z_1, and so on — in accordance with the above Axiom. One can then apply the Axiom in an analogous way to the union of the now formed set and, so proceeding, manifestly obtain the freedom required in the formation of sets.

Moreover, for the special purposes of axiomatic set theory, it is desirable and possible to erect in place of the Axiom just formulated a less wide-ranging and more precise one; we then succeed in rendering superfluous the concept "replace" — a concept which essentially boils down to that of function, and which would require special treatment.

— *Mathematische Annalen*, v. 86, p. 230–231

Lateral Separation

The Axiom of Separation (a schema, of course) formally represents the species-genus relation essential to intellectual work. In our own tradition, at least, it was first formulated by Aristotle: a recognizable kind consists of a differentiation from a broader kind. In mathematical logic, the effective difference is expressed by φx, a predication akin to the traditional act of assigning a property to a subject:

$$\exists C \forall x[x \in C \leftrightarrow \exists A(x \in A \ \& \ \varphi x)]$$

However, in much intellectual work, we also predicate something in its relation to something else: x is spouse of y. Instead of the difference taking the form of φx, as in Separation, it takes the form of $(\varphi x,y)$.

Consider the set of Canadians. By Separation we may create a new set M from this C by assigning the difference Male as the φx in the formula: x is (also) male. But we might also wish to form a set F from that set C — a set of those spouses (of Canadians) that are non-Canadian, i.e. Foreigners. The difference is then expressed as a relation, a function of *two* variables, and the new set will consist of those lateral y's that, while married to Canadian x's, do not themselves have that nationality. Intuitively, there seems to be no reason why we could not procreate (separate out) such a set. Yet Separation, as originally formulated by Zermelo, does not provide for relational differentiation. How might we devise an Axiom that authorizes this much-needed lateral procreation? What all might be required of such an Axiom?

For one thing, once we have a Canadian, we must assume that, should this x be married, there is a definite way of determining the y (an individual or, if polygamy be allowed, a definite set) that will be gathered into the new set; otherwise the supposed new set F would not have definite membership and could not then count as a set. In logical terms, $\varphi(x,y)$ must be functional in x: given any x in the function $\varphi(x,y)$, any z that appears, differently named, in the place of y, will be identical to y. Once we have assured ourselves of this condition, namely

$$\forall x\forall y\forall z\{x \in C \,\&\, [(\varphi(x,y) \,\&\, \varphi(x,z)) \rightarrow z = y]\},$$

we also have a version of Separation that we may formulate with two variables:

$$\exists F\forall y\{y \in F \leftrightarrow \exists x[x \in C \,\&\, \varphi(x,y)]\}.$$

Even more formally, the Antecedent *implies* the Consequent, in this case with an understood universal quantifier on the C.

This Axiom was first proposed in 1922, independently and nearly simultaneously, by two set-theorists, Adolf Fraenkel and Thoralf Skolem. It is almost always referred to as the Axiom (Schema) of Replacement — misleadingly, since "replacement" is also the name of an operation bearing on logical equivalences — and, besides, the Axiom names a formative act whose *result*, an equivalence, may not even serve such logical replacement. I prefer to call it by a name reflecting the work it does: Lateral Separation.

From this version we may derive not only the original Axiom of Separation (by introducing a φxy that reduces to φx), but also the Pair Theorem (which otherwise requires its own Axiom).

§1. *Deriving the* **Axiom of Separation**

$$\vdash\ \exists C\forall x[x \in C \leftrightarrow \exists A(x \in A \,\&\, \varphi x)]$$

Let $\varphi(x,y)$ be $x = y \,\&\, \psi y$. We may then prove the Antecedent of the Axiom:

1.	$x \in A \,\&\, [(x = y \,\&\, \psi y) \,\&\, (x = z \,\&\, \psi z)]$	Assume / $\therefore\ y = z$
2.	$x = y \,\&\, x = z$	1 Comm. and Simpl.
3.	$y = z$	2 =-Exch.
4.	$(1) \rightarrow (3)$	RCP
5.	$\forall x\forall y\forall z[x \in A \,\&\, [(x = y \,\&\, \psi y) \,\&\, (x = z \,\&\, \psi z) \rightarrow y = z]$	4 $\forall$-intros.

Note that $x \in A$ is a fellow traveler for the sake of form: the proof

here makes no use of it. We proceed with the Consequent:

6.	$\exists B\forall y\{(y \in B) \leftrightarrow \exists x[(x \in A) \,\&\, x = y \,\&\, \psi y]\}$	5 Axiom, MP
7.	$\forall y\{(y \in B) \leftrightarrow \exists x[(x \in A) \,\&\, x = y \,\&\, \psi y]\}$	6 $\exists$-elim.
8.	$\{(z \in B) \leftrightarrow \exists x[(x \in A) \,\&\, x = z \,\&\, \psi z]\}$	7 $\forall$-elim.

Separation requires that the free variable (z) name an element in A. To continue, we must break up this last formula, extract $x = z$ and get everything into the variable z.

9.	$z \in B$	Assume / $\therefore$ line 13
10.	$\exists x[(x \in A) \,\&\, x = z \,\&\, \psi z]$	8,9 Repl.
11.	$(a \in A) \,\&\, a = z \,\&\, \psi z$	10 $\exists$-elim.
12.	$z \in A \,\&\, \psi z$	11 Simpl., =-Exch.,Conj.
13.	$\exists A[z \in A \,\&\, \psi z]$	12 $\exists$-intro.
14.	$z \in B \rightarrow \exists A[z \in A \,\&\, \psi z]$	9—13 RCP

The proof of the converse is facilitated by a **lemma** (see §3, #2):

$$\vdash\ \forall z\{\exists x[(x \in A) \,\&\, x = z] \leftrightarrow z \in A\},$$

the derivation of which is routine. To continue:

15.	$\exists A[z \in A \,\&\, \psi z]$	Assume / $\therefore\ z \in B$
16.	$z \in A \,\&\, \psi z$	$\exists$-elim.
17.	$\exists x[(z \in A) \,\&\, x = z] \leftrightarrow z \in A$	Lemma, $\forall$-elim.
18.	$\exists x[(z \in A) \,\&\, x = z] \,\&\, \psi z$	17,16 Repl.
19.	$z \in B$	18,8 Repl.
20.	$\exists A(z \in A \,\&\, \psi z) \rightarrow (z \in B)$	15—19 RCP
	$\therefore\ \exists C\forall x[x \in C \leftrightarrow \exists A(x \in A \,\&\, \varphi x)]$	14,20 Q-intros. (renaming)

Mathematicians often do not insert the $\exists$ on the variable A — yet what needs to be established during many proofs invoking Separation is precisely the existence of this set.

From this simpler Separation we may prove the Finitude Theorem, $\sim\exists U\forall x[x \in U]$ (there is no Universal Set: *Logic Ancient and Modern*, p. 606). Now, if in the Axiom we let $\varphi(x,y)$ be $y \neq x$ we would violate its Antecedent by allowing an indefinite number of y's for any one x. And if we skipped over this fact and concluded the Consequent anyway, we would obtain:

$$\exists B\forall y\{(y \in B) \leftrightarrow \exists x[(x \in A) \,\&\, y \neq x]\},$$

whereupon B would be the absolute complement of A and we could derive the union of the two as the Universal Set denied existence by the Finitude Theorem.

Another example of a faulty $\varphi(x,y)$ is $x \subseteq y$: by letting $A = \{\varnothing\}$ one may again derive $\exists U \forall x\{(x \in U)\}$, the universal set, something that set theory was devised to prevent.

For a fruitful $\varphi(x,y)$, Fraenkel let $y = \wp(x)$ and $A = \omega$, the set of positive integers. Once having proved the existence of the set A with the Axiom of Infinity, one may axiomatically infer the existence of a new transfinite set — the "size" of which Cantor called $\aleph_1$ — its predecessor being $\aleph_0$ (the "size" of ω). Since $\aleph_1 = 2^{\aleph_\circ}$, Cantor and others aspired to prove that there was no transfinite cardinal between these two (the Continuum Hypothesis).

§2. *Deriving the* **Pair Theorem**

$$\vdash \exists B \forall z[(z \in B) \leftrightarrow (z = x \vee z = y)]$$

What we need is a set A such that it indeed has two and only two elements — already a pair. And the set $\wp\wp\varnothing$ fits this description, having only the elements $\varnothing$ and $\{\varnothing\}$. We may then insert for the function $\varphi(u,v)$ the function

$$(u = \varnothing \ \&\ v = x) \vee (u = \{\varnothing\} \ \&\ v = y).$$

We then prove that this formula "is functional in u":

$(u = \varnothing \ \&\ v = x) \vee (u = \{\varnothing\} \ \&\ v = y)$ Assume / $\therefore$ $(y = x)$

Informally: Assume $y \neq x$. Show by Constructive Dilemma that $v \neq x$ and $v \neq y$, which leads quickly to a contradiction. Thus this Antecedent of the Axiom is a logical truth. We may then proceed with the Consequent, first listing the five assertions with which the proof will proceed:

α. $\exists B \forall v\{v \in B \leftrightarrow \exists u(u \in \wp\wp\varnothing \ \&\ [(u = \varnothing \ \&\ v = x) \vee (u = \{\varnothing\} \ \&\ v = y)])\}$

β. $\varnothing \in \wp\wp\varnothing$ — $\varnothing \in$ every power set

γ. $\{\varnothing\} \in \wp\wp\varnothing$ — $\varnothing \in \{\varnothing\}$ by inspection, $\wp\varnothing = \{\varnothing\}$, etc.

δ. $u \in \wp\wp\varnothing \rightarrow (u = \varnothing \vee u = \{\varnothing\})$ — β,γ Pre-theoretical pairing!

ε. $\varnothing \neq \{\varnothing\}$ — $\varnothing$ has no element, $\{\varnothing\}$ has an element

Strategy: get $(u \in \wp\wp\varnothing \ \&\ [(u = \varnothing \ \&\ v = x) \vee (u = \{\varnothing\} \ \&\ v = y)])$ to imply $(v = x \vee v = y)$, then obtain the converse. Finally, replace this equivalence in the Consequent of the Axiom.

1.	$\exists u(u \in \wp\wp\varnothing \ \&\ [(u = \varnothing \ \&\ v = x) \vee (u = \{\varnothing\} \ \&\ v = y)])\}$	Assume
2.	$(u \in \wp\wp\varnothing \ \&\ [(u = \varnothing \ \&\ v = x) \vee (u = \{\varnothing\} \ \&\ v = y)])$	∃-elim.
3.	$u \in \wp\wp\varnothing$	2 Simpl.
4.	$(u = \varnothing \vee u = \{\varnothing\})$	δ,3 MP
5.	$(u = \varnothing \ \&\ v = x) \vee (u = \{\varnothing\} \ \&\ v = y)$	2 Simpl.
6.	$u = \varnothing$	Assume
7.	$(\varnothing = \varnothing \ \&\ v = x) \vee (\varnothing = \{\varnothing\} \ \&\ v = y)$	6,5 =-Exch.
8.	$(\varnothing = \varnothing \ \&\ v = x)$	ε,7 =-intro., DS
9.	$v = x$	8 Simpl.
10.	$v = x \vee v = y$	9 Add.
11.	$(u = \varnothing) \rightarrow (v = x \vee v = y)$	6—10 RCP
12.	$(u = \{\varnothing\}) \rightarrow (v = x \vee v = y)$	Procedure of 6–11
13.	$(v = x \vee v = y)$	4,11,12 ∨-elim.
14.	$(1) \rightarrow (v = x \vee v = y)$	1,13 RCP
15.	$\sim\exists u(u \in \wp\wp\varnothing \ \&\ [(u = \varnothing \ \&\ v = x) \vee (u = \{\varnothing\} \ \&\ v = y)])$	Assume
16.	$\forall u \sim(u \in \wp\wp\varnothing \ \&\ [(u = \varnothing \ \&\ v = x) \vee (u = \{\varnothing\} \ \&\ v = y)])$	15 QN
17.	$\sim(\varnothing \in \wp\wp\varnothing \ \&\ [(\varnothing = \varnothing \ \&\ v = x) \vee (\varnothing = \{\varnothing\} \ \&\ v = y)])$	16 ∀-elim.
18.	$\varnothing \notin \wp\wp\varnothing \vee \sim[(\varnothing = \varnothing \ \&\ v = x) \vee (\varnothing = \{\varnothing\} \ \&\ v = y)])$	17 DeM
19.	$\sim[(\varnothing = \varnothing \ \&\ v = x) \vee (\varnothing = \{\varnothing\} \ \&\ v = y)]$	β,18 DS
20.	$\sim(\varnothing = \varnothing \ \&\ v = x) \ \&\ \sim(\varnothing = \{\varnothing\} \ \&\ v = y)$	19 DM
21.	$\sim(\varnothing = \varnothing \ \&\ v = x)$	20 Simpl.
22.	$\sim(\varnothing = \varnothing) \vee \sim(v = x)$	21 DeM
23.	$\sim(v = x)$	22 =-intro., DS
24.	$\sim(v = y)$	Procedure of 16–23
25.	$\sim(v = x) \ \&\ \sim(v = y)$	23,24 Conj.
26.	$\sim(v = x \vee v = y)$	25 DeM
27.	$(15) \rightarrow (26)$	RCP
28.	$(v = x \vee v = y) \rightarrow (1)$	27 Contrapos.
29.	$(1) \leftrightarrow (v = x \vee v = y)$	14,28 def. of ↔
∴	$\exists B \forall z[(z \in B) \leftrightarrow (z = x \vee z = y)]$	α,29 Repl. (renaming)

§3. *Some remarks*

(1) After bearing these two children (themselves extremely fecund), the Axiom confines itself to forming sets and enabling inferences in the demonstration of a few set-theoretical truths.

(2) Separation is just a special case of Lateral Separation. But note that its derivation depends on the **lemma**, itself proved only on the principle $\sim\sim p \rightarrow p$ (decried by intuitionists like Brouwer).

(3) The derivation of the Pair Theorem here bypasses the Pair Axiom, which reads more primitively as a uni-conditional:

$$\forall y \forall z \forall x[(x = y \vee x = z) \rightarrow \exists A(x \in A)].$$

This form suggests what thinkers since Kant have sometimes called a transcendental act, one yielding a synthetic *a priori* truth. On this version, then, the Pair Theorem *follows* upon the Pair Axiom.

(4) Although the proof of the Pair Theorem makes use of an in-house pair, *any* existent pair serves the purpose. E.g., call M an existent married couple with sole elements Dick and Jane. We have $u \in M \rightarrow (u = d \vee u = j)$ for the above δ. Then, too, $d \neq j$, $d \in M$, $j \in M$. And for $\varphi(u,v)$ we take $(u = d \;\&\; v = x) \vee (u = j \;\&\; v = y)$.

Mathematicians will of course resist introducing an empirical set into a formal proof. The inference to obtain line 4 (and again to obtain line 12) seems preferable to the parallel inference from $u \in M$ to $(u = d \vee u = j)$ in the "empirical" proof. Yet *both* inferences rely on "inspection" — on intellectual intuition of the truth of δ.

The need for a "seed" instance or "catalyst" to derive the Pair Theorem may remind us that the mathematical proofs of a number of Aristotelian syllogistic forms (later called Darapti, Felapton, and Fesapo) require that we postulate an existence for the middle term (formally, $\exists xMx$ — or, for Bramatip, $\exists xPx$).

(5) The heart of the Axiom reads as a uni-conditional: given any non-empty set A of x's and a function $\varphi(x,y)$ relating each x to a unique y, we may form another set B of those y's:

$$\exists B \forall y\{\exists x[x \in A \;\&\; \varphi(x,y)] \rightarrow y \in B\}.$$

Yet the Axiom reads as a bi-conditional — as it must, and not only

for permitting the derivation of simple Separation and the bi-conditional of Pairing. Thus we also have:

$$\exists B \forall y\{y \in B \rightarrow \exists x[x \in A \;\&\; \varphi(x,y)]\},$$

or: Starting with a set B of y's we must be able to find a set A of x's and formulate a $\varphi(x,y)$ functional in x.

Unlike the first, the generating conditional, where we are free to devise the separating function, this second, the parentage conditional, requires that we discover both the womb and the seed.

Imagine having a set of letters from our alphabet, a set that makes no sense in ordinary English. If the set is not randomly ordered, it might be encoded — in which case we should be able (with effort, ingenuity, and luck) to discover the parent set (letters perhaps even making sense in English) and the parent function which, when reinserted into the full formula of the Axiom, accounts for the generation of the encoded set (message).

The *generating* half of the Axiom is essential to encrypting, while the *parentage* half is essential to deciphering.