Structures in tangent categories

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Outline

- What are tangent categories?
  - Definitions: intuitive and more precise.
  - Examples.
- What can one do within a tangent category?
  - Vector fields and their Lie bracket.
  - Vector spaces.
  - Vector bundles.
  - Differential forms.
  - Connections on a vector bundle.
Tangent categories (intuitively)

Definition (Rosický 1984, modified Cockett/Cruttwell 2013)

(Intuitively) A tangent category consists of a category $X$, which has, for each object $M$, an associated bundle over $M$, called $TM$, with the following properties:

- each $TM$ is an additive bundle over $M$, in a natural way;
- each $TM$ is a “vector” bundle over $M$, in a natural way;
- $T$ “preserves the structure of each bundle $TM$” in a natural way.
Tangent categories (more precisely)

Definition

More specifically, this means we have a functor $T : \mathbb{X} \to \mathbb{X}$, with:

- (projection) a natural transformation $p : T \to I$;
- (addition and zeroes) natural transformations $+: T_2 \to T$ and $0 : I \to T$;
- (vertical lift) a natural transformation $\ell : T \to T^2$ satisfying a certain universality property;
- (canonical flip) a natural transformation $c : T^2 \to T^2$;
- a number of coherence axioms.
Tangent category examples

(i) The canonical example: finite dimensional smooth manifolds.
(ii) Convenient manifolds (with the kinematic tangent bundle).
(iii) Any Cartesian differential category.
(iv) The infinitesimally linear objects in a model of synthetic differential geometry.
(v) Commutative ri(n)gs and its opposite (and other associated categories in algebraic geometry).
(vi) The category of \( C-\infty \) rings.
(vii) (Lack/Leung) A category of Weyl algebras.
(viii) (Rosický) If \( \mathbb{X} \) has tangent structure, then so does each slice \( \mathbb{X}/M \).
Vector fields and their Lie bracket

**Definition**

If \((\mathbf{X}, T)\) is a tangent category with an object \(M \in \mathbf{X}\), a **vector field** on \(M\) is a map \(M \xrightarrow{v} TM\) with \(pv = 1\).

If \(\mathbf{X}\) has negation, given two vector fields \(v_1, v_2 : M \to TM\), Rosický showed how to use the universal property of vertical lift to define the Lie bracket vector field \([v_1, v_2] : M \to TM\) so that the Jacobi identity

\[
[v_1, [v_2, v_3]] + [v_3, [v_1, v_2]] + [v_2, [v_3, v_1]] = 0
\]

is satisfied.
Vector Spaces/Differential objects

Vector spaces in tangent categories are represented by objects whose tangent bundle is trivial:

**Definition**

A **differential object** in a tangent category consists of a commutative monoid \((A, \sigma, \zeta)\) with a map \(\hat{p} : TA \to A\) such that

\[
\begin{array}{ccc}
A & \xleftarrow{\hat{p}} & TA \\
\downarrow{p} & & \downarrow{p} \\
A & \to & A
\end{array}
\]

is a product diagram, so that \(TA \cong A \times A\) (as well as some additional coherence axioms).

- \(\mathbb{R}^n\)'s in the category of smooth manifolds.
- The pullback of \(p : TM \to M\) along a point of \(M\).
- If \(T\) is representable with representing object \(D\), get an associated ring \(R\) which is differential (thus satisfying the “Kock-lawvere” axiom).
In general:

- a group bundle is a group in $\mathbb{X}/M$;
- a vector bundle is a vector space in $\mathbb{X}/M$;
- so a differential bundle should be a differential object in the canonical tangent category structure on $\mathbb{X}/M$. 
Vector/Differential bundles (more precisely)

**Definition**

A **differential bundle** in a tangent category consists of an additive bundle $q : E \to M$ with a map $\lambda : E \to TE$ so that $q : E \to M$ becomes a differential object in the slice tangent category $\mathbb{X}/M$.

1. If $A$ is a differential object, then for each object $M$, $\pi_2 : A \times M \to M$ is a differential bundle.
2. For each object $M$, $p : TM \to M$ is a differential bundle.
3. The pullback of a differential bundle $q : E \to M$ along any map $f : X \to M$ is a differential bundle.
4. If $q : E \to M$ is a differential bundle, $T(q) : TE \to TM$ is also.
A **morphism of differential bundles** between differential bundles \((q : E \rightarrow M), (q' : E' \rightarrow M')\) is simply a pair of maps \(f : E \rightarrow E', g : M \rightarrow M'\) making the obvious diagram commute.

A morphism of differential bundles \((f, g)\) is **linear** if it also preserves the lift, that is,

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow \lambda & & \downarrow \lambda' \\
T(E) & \xrightarrow{T(f)} & T(E')
\end{array}
\]

commutes.

**Note:** this does correspond to the ordinary definition of linear morphisms between vector bundles in the canonical example.
(Vector-valued) Differential forms

**Definition**

If $M$ is an object of $\mathbf{X}$ and $q : E \to M$ a differential bundle, a *$E$-valued differential $n$-form on $M$* consists of a map

$$\omega : T^n M \to E$$

which is “linear in each variable” and alternating.

In the case when the differential bundle is of the form $\pi_2 : A \times M \to M$ for some differential object $A$, these are ordinary differential forms - in particular in the canonical example, when $A = \mathbb{R}$. 
Connections (intuitively)

Intuitive idea: can “move tangent vectors between different tangent spaces”. Moving a tangent vector around a closed curve measures the “curvature” of the space. Connections have been expressed in many different ways:

- as a “horizontal subspace”;
- as a “connection map”;
- as a notion of “parallel transport”;
- as a “covariant derivative”.

Quoting Michael Spivak:

“I personally feel that the next person to propose a new definition of a connection should be summarily executed.”
Two fundamental maps

A differential bundle has two key maps involving $TE$ whose composite is the zero map:

$$
\begin{array}{ccc}
TE & \xrightarrow{\lambda} & \langle Tq,p \rangle \\
\downarrow & & \downarrow \\
E & \xrightarrow{\langle Tq,p \rangle} & TM \times_M E
\end{array}
$$
A connection consists of a linear section of $H$ of $\langle Tq, p \rangle$ called the horizontal lift...
which in addition has a linear retraction $K$ of $\lambda$ called the \textbf{connector}:
that satisfies the equations \( KH = 0 \) and
\[ (\lambda K \oplus 0p) + H\langle Tq, p \rangle = 1_{TE}. \]
Simple example

Any differential object $A$ is a differential bundle over 1 and these have a canonical connection given by:

- $K : TA \rightarrow A$ by $K(v, a) := v$ and
- $H : A \rightarrow TA$ by $H(a) := (0, a)$. 
Suppose \((X, T)\) is a tangent category with negation and \((q, \lambda)\) is a differential bundle.

**Proposition**

If \(H\) is a linear section of \(\langle T(q), p \rangle\), then \(q\) can be given the structure of a connection with horizontal lift \(H\).

**Proposition**

If \(K\) is a linear retract of \(\lambda\), and \(q\) has at least one section \(J\) of \(\langle T(q), p \rangle\), then \(q\) can be given the structure of a connection with connector \(K\).
The definition of a connection being flat in the literature is quite complicated, but by using the map $c$ we can make a very simple definition:

**Definition**

Say that a connection is **flat** if $cT(K)K = T(K)K$.

One can show this is equivalent to the standard definition (involving curvature) in the canonical example.
Affine and torsion-free connections

Torsion-free connections are connections on the tangent bundle for which the movement of tangent vectors does not “twist”. Again there is a simple definition of this in our setting:

**Definition**

When the connection is on a tangent bundle $p : TM \to M$, the connection is called **affine**. Say an affine connection is **torsion-free** if $cK = K$.

This again is again equivalent to the usual definition (involving the Lie bracket) in the canonical example.
Most categories related to differential or algebraic geometry are tangent categories.

The following are well-defined notions in any tangent category: vector fields, the Lie bracket, “vector” spaces and bundles, differential forms, and connections.

The definitions of differential object and bundle shed light on the nature of vector spaces and bundles in differential geometry.

The definition of connections, as well as their properties of being torsion-free and affine, shed light on connections in differential geometry.
References:
