

Differential structure, tangent structure, and SDG

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Abstract

In 1984, J. Rosický gave an abstract presentation of the structure associated to tangent bundle functors in differential and algebraic geometry. By slightly generalizing this notion, we show that tangent structure is also fundamentally related to the more recently introduced Cartesian differential categories. In particular, tangent structure of a trivial bundle is precisely the same as Cartesian differential structure. We also provide a general result which shows how tangent structure arises from the manifold completion (in the sense of M. Grandis) of a differential restriction category. This construction includes all standard atlas-based constructions from differential geometry. Furthermore, we tighten the relationship, which Rosický had noted, between representable tangent structure and synthetic differential geometry, showing how such settings can be developed from a system of infinitesimal objects. We also show how infinitesimal objects give rise to dual tangent structure.

Taken together, these results show that tangent structures appropriately span a very wide range of definitions, from the syntactic and structural differentials arising in computer science and combinatorics, through the concrete manifolds of algebraic and differential geometry, and finally to the abstract definitions of synthetic differential geometry.

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1 Introduction

In [Rosický 1984], an abstract description of the properties of the tangent bundle functor on the category of smooth manifolds was proposed. In particular, the paper showed how a crucial structural component of differential geometry, the Lie bracket on vector fields, was already present in any category with such abstract tangent structure. Furthermore, Rosický showed that when tangent structure has its functor “representable”, in the sense of being of the form $(-)^D$ for some object D , then the resulting setting shared many of the important properties of synthetic differential geometry. Specifically, he showed that the representing object D was a commutative semigroup

with each element having $d^2 = 0$, and that there was a (not necessarily commutative) ring R with a map $D \rightarrow R$ which satisfies the Kock-Lawvere axiom. All this suggested that a category with tangent structure was an appropriate categorical doctrine in which an abstract theory of differential geometry could be developed.

In this paper, we extend Rosický’s work in a number of directions. We begin, in section 2, by slightly generalizing Rosický’s definition of tangent structure so that the bundles, on which the definition is based, are allowed to be commutative monoids rather than commutative groups. This is important as certain key examples arising from computer science and combinatorics do not have negatives. Removing negatives from the definition in this manner forces a further adjustment to the crucial equalizer condition in Rosický’s definition. Nonetheless, we show that if the tangent bundles have negation, our definition is equivalent to Rosický’s. We also indicate some basic examples of tangent structure and provide some easy consequences of the axioms.

In section 3, we discuss some of the general theory of an arbitrary category with tangent structure. We begin by defining vector fields and describe some of their properties. Next, we add the observation that tangent structure implies that the tangent functor and its square are monads, and describe how the Kleisli composition of the first monad is related to vector field addition. Rosický was able to define a Lie bracket on the vector fields in (group) tangent categories; we review some of the basic results concerning this important structure. In a future paper, we hope to return to these structural aspects and, in particular, show that every (group) tangent category naturally supports a notion of de Rham cohomology.

In section 4, we demonstrate the fundamental connection between tangent structure and the more recently introduced notion of Cartesian differential categories. Cartesian differential categories were first defined in [Blute et al. 2008]; their introduction was motivated not only by the wish to give a basic categorical semantics for the standard notion of differentiation in multivariable calculus, but also by the differentials arising in combinatorics [Bergeron et al. 1997], Computer Science [Abbot thesis, Abbott et al. 2003], and in linear logic [Ehrhard 2001, Ehrhard and Regnier 2003, Bucciarelli et al. 2010]. In contrast to the standard notion, these latter notions of differentiation do not have negatives.

In a category with (Cartesian) tangent structure we show that the full subcategory of “differential objects”, that is objects whose tangent bundle is “trivial”, form a Cartesian differential category. Indeed, the axioms of tangent structure on such objects amount precisely to the axioms of a Cartesian differential category. Moreover, we show that differential objects are fundamental, as they arise precisely from considering the tangent spaces of the objects in the tangent category. The relationship between these two theories is summarized by an adjoint between Cartesian differential categories and (Cartesian) tangent categories.

Section 5 recalls Rosický’s observations concerning the relationship between tangent structure and synthetic differential geometry. Synthetic differential geometry starts from the notion of a “ring of line type”. It is natural to look at Weil algebras as a mechanism for producing structure and thereby providing a basis for differential geometry: this approach can be seen both in [Kock 2006] and, more recently, from a more abstract perspective, in [Nishimura 2012]. Rosický realized, however, that demanding that tangent structure be representable already put one in the domain of synthetic differential geometry. This perspective then provides a minimalistic axiomatic approach to synthetic differential geometry. Here we expand on Rosický’s ideas, and identify several potentially different “rigs of line type” that occur in any instance of representable tangent structure. In particular, one of these rigs is commutative, improving upon Rosický’s observations, and tighten-

ing the relationship between synthetic differential geometry and representable tangent structure. We also give a definition of infinitesimal object that arises from considering representable tangent structure, and consider a notion of a “system of infinitesimals” (similar to Weil algebras) which can be used to generate representable tangent structure. Finally, we add an important theoretical observation, which shows that any category with representable tangent structure automatically has tangent structure on its opposite category.

In the final section, we discuss how tangent structure interacts with partial maps and manifolds. Our goal is to show that when one applies the abstract notion of “manifold completion” of [Grandis 1989] to a differential restriction category, the result has tangent structure. This captures one of the traditional constructions of the tangent bundle on the category of smooth manifolds.

Taken together, then, the results of this paper demonstrate the central role tangent structure plays in the abstract theory of differentiation.

2 Tangent structure

We begin by defining tangent categories. These are categories equipped with tangent structure: that is, a functor together with some natural transformations which satisfy certain coherence and limit conditions. This is essentially the structure introduced in [Rosický 1984], except that we generalize to allow additive rather than group bundles. As the idea of the definition is to axiomatize the properties of a tangent bundle functor, the reader may like to keep the category of smooth manifolds in mind as a motivating example.

2.1 Additive bundles

The tangent bundle TM of a smooth manifold M is a vector bundle over M . The axiomatization given in [Rosický 1984], however, does not ask for vector space structure as, for the basic theory, this is unnecessary. Instead, [Rosický 1984] simply asks that the bundles be commutative groups¹. Here, in order to include some of the syntactic examples motivated from computer science, we shall take these ideas one step further and only require that the bundles be commutative monoids. For brevity, we call these *additive* bundles. We also note that following our earlier works, all composition will be written in diagrammatic order and product indices will start at zero – although anomalously we shall talk of the first, second, third projection etc. meaning π_0 , π_1 , and π_2 respectively.

Definition 2.1 *If A is an object in a category \mathbb{X} , then an **additive bundle over A** consists of a commutative monoid in the slice category \mathbb{X}/A . Explicitly, this consists of the following data:*

- a map $X \xrightarrow{p} A$ such that pullback powers of p exist, that is the pullback of n copies of $X \xrightarrow{p} A$ for each $n \in \mathbb{N}$ exists; denote these by X_n , with structure maps $\pi_i : X_n \rightarrow X$ (with $i \in \{0, \dots, n-1\}$);
- maps $+$: $X_2 \rightarrow X$ and $0 : A \rightarrow X$, with $+p = \pi_0 p = \pi_1 p$ and $0p = 1$ such that this operation

¹Rosický does note that an R -linear structure on the bundles can be recovered by considering the natural endomorphisms of the tangent functor.

is associative, commutative, and unital; that is, each of the following diagrams commute:

$$\begin{array}{ccc}
\begin{array}{ccc}
X_3 & \xrightarrow{\langle\langle\pi_0,\pi_1\rangle+, \pi_2\rangle} & X_2 \\
\langle\pi_0, \langle\pi_1, \pi_2\rangle+\rangle \downarrow & & \downarrow + \\
X_2 & \xrightarrow{+} & X
\end{array} &
\begin{array}{ccc}
X_2 & & \\
\langle\pi_1, \pi_0\rangle \downarrow & \searrow + & \\
X_2 & \xrightarrow{+} & X
\end{array} &
\begin{array}{ccc}
X & & \\
\langle p_0, 1\rangle \downarrow & \searrow 1 & \\
X_2 & \xrightarrow{+} & X
\end{array}
\end{array}$$

Note that in the presence of equalizers, one can recover the group of units of an additive bundle from the equalizer:

$$X_G \xrightarrow{e} X_2 \xrightarrow[0]{+} X.$$

This gives two inclusions $u = e\pi_0, -u = e\pi_1 : X_G \rightarrow X$; the symmetry of X_2 induces the negation $X_G \xrightarrow{-} X_G$.

Definition 2.2 Suppose that $p : X \rightarrow A$ and $q : Y \rightarrow B$ are additive bundles. An **additive bundle morphism** consists of a pair of maps $f : X \rightarrow Y, g : A \rightarrow B$ so that the following diagrams commute:

$$\begin{array}{ccc}
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
p \downarrow & & \downarrow q \\
A & \xrightarrow{g} & B
\end{array} &
\begin{array}{ccc}
X_2 & \xrightarrow{\langle\pi_0 f, \pi_1 f\rangle} & Y_2 \\
+\downarrow & & \downarrow + \\
X & \xrightarrow{f} & Y
\end{array} &
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
0 \downarrow & & \downarrow 0 \\
X & \xrightarrow{f} & Y
\end{array}
\end{array}$$

The first diagram says that the pair is a map in the arrow category; the second that the map preserves addition, and the last that it preserves zeroes. Clearly any additive bundle morphism will restrict to a morphism between the group of units.

2.2 Definition of tangent categories

With additive bundles defined, we can now, as promised, give the slightly generalized version of the definition in [Rosický 1984] of a tangent structure. (Note that in 3.10, we show that if the bundles in the definition are commutative groups, then our definition of tangent structure is equivalent to Rosický's).

Definition 2.3 A category \mathbb{X} has **tangent structure**, $\mathbb{T} = (T, p, 0, +, \ell, c)$ in case:

- (**tangent functor**) $T : \mathbb{X} \rightarrow \mathbb{X}$ is a functor with a natural transformation $p_M : TM \rightarrow M$, such that pullback powers of p_M exist for each M , and T^n preserves these pullback powers for each $n \in \mathbb{N}$;
- (**tangent bundle**) there are natural transformations $+ : T_2M \rightarrow TM$ (where T_2M is the pullback of p_M over itself) and $0_M : M \rightarrow TM$ making each $p : TM \rightarrow M$ an additive bundle;
- (**vertical lift**) there is a natural transformation $T \xrightarrow{\ell} T^2$ such that for each M

$$(\ell_M, 0_M) : (p : TM \rightarrow M, +, 0) \rightarrow (Tp : T^2M \rightarrow TM, T(+), T(0))$$

is an additive bundle morphism;

- (**canonical flip**) there is a natural transformation $T^2 \xrightarrow{c} T^2$ such that for each M

$$(c_M, 1) : (Tp : T^2M \longrightarrow TM, T(+), T(0)) \longrightarrow (p_T : T^2M \longrightarrow TM, +_T, 0_T)$$

is an additive bundle morphism;

- (**coherence of ℓ and c**) we have $c^2 = 1$, $\ell c = \ell$, and the following diagrams commute:

$$\begin{array}{ccccc} T & \xrightarrow{\ell} & T^2 & & T^3 & \xrightarrow{T(c)} & T^3 & \xrightarrow{c_T} & T^3 & & T^2 & \xrightarrow{\ell_T} & T^3 & \xrightarrow{T(c)} & T^3 \\ \ell \downarrow & & \downarrow T(\ell) & & c_T \downarrow & & & & \downarrow T(c) & & c \downarrow & & & & \downarrow c_T \\ T^2 & \xrightarrow{\ell_T} & T^3 & & T^3 & \xrightarrow{T(c)} & T^3 & \xrightarrow{c_T} & T^3 & & T^2 & \xrightarrow{T(\ell)} & T^3 & & T^3 \end{array}$$

- (**universality of vertical lift**) the following is an equalizer diagram:

$$T_2M \xrightarrow{v := \langle \pi_0 \ell, \pi_1 0_T \rangle T(+)} T^2M \begin{array}{c} \xrightarrow{T(p)} \\ \xrightarrow{pp0} \end{array} TM$$

(i.e. v is the equalizer of $T(p)$ and $pp0$).

We shall refer to the pair (\mathbb{X}, \mathbb{T}) as a **tangent category**.

The canonical example of a tangent category is the category of finite-dimensional smooth manifolds and smooth maps between them. The tangent bundle functor is formed by “thickening” a manifold at each point by its tangent space: p is then the projection down to the original space. Clearly, these tangent bundles are additive and, in fact, groups. The canonical flip is a result of the symmetry of second partial derivatives $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, while the vertical lift is a result of the derivative being a linear operator. The universality of the vertical lift is perhaps the most important property of tangent structure; however, to provide intuition for it is also the most difficult: it is discussed further in section 2.5.

That finite-dimensional smooth manifolds with their tangent bundle is an example of a tangent category can be verified directly. However, in the final section, we give a general result which includes this as an example. Specifically, we show that manifolds built out of any differential restriction category form a tangent category. The proof will be split into two parts: first of all, in section 4, we show that any Cartesian differential category has tangent structure; this gives the local description of the required natural transformations. In section 6.3, we then show that tangent structure lifts to the manifold completion. This also shows that the category of convenient manifolds and smooth maps between them, with the kinematic tangent bundle of [Kriegl and Michor 1997], has tangent structure.

In sections 4 and 5, we will discuss two important classes of examples of tangent structure. In Section 4, we show that any Cartesian differential category has tangent structure. The relationship between the behaviour of differentiation and tangent structure is, in this interpretation, quite explicit. Moreover, we describe how, on objects with structure similar to those found in Cartesian differential categories, the tangent structure axioms give the Cartesian differential axioms. In Section 5, we explore what may be viewed as the other end of the spectrum, that is the models given by synthetic differential geometry. When restricted to suitable objects, any model of synthetic

differential geometry gives an example of tangent structure, with $TM = M^D$. Moreover, representable tangent structure is a model of a generalized synthetic differential geometry. These models illustrate rather explicitly the source of some of the coherence conditions of tangent structure.

An important and immediate effect of the canonical flip is to allow one to switch between the two different additive structures on the second tangent bundle T^2M . Note that T^2M is automatically an additive bundle over TM via the maps $(p_T, +_T, 0_T)$. But, in fact, since we assume T preserves the pullbacks used in defining addition, T^2M is also an additive bundle over TM in a different way, via the maps $(T(p), T(+), T(0))$. The canonical flip $c : T^2 \rightarrow T^2$ is an additive bundle morphism between these structures, and is in fact an additive bundle isomorphism, since $c^2 = 1$. Thus, the canonical flip allows one to pass back and forth between the two different additive structures on T^2M .

We record some other useful consequences of the definition of tangent structure.

Proposition 2.4 *If \mathbb{T} is tangent structure on \mathbb{X} then:*

- (i) *for each $f : M \rightarrow N$ the pair $(Tf, f) : (p_M, +_M, 0_M) \rightarrow (p_N, +_N, 0_N)$ is a morphism of bundles, furthermore, demanding this is equivalent to demanding that p_M , $+_M$, and 0_M be natural;*
- (ii) *$(c, 1) : (T(p), T(+), T(0)) \rightarrow (p_T, +_T, 0_T)$ is an isomorphism of bundles;*
- (iii) *$(\ell, 0) : (p, +, 0) \rightarrow (p_T, +_T, 0_T)$ is a bundle morphism;*
- (iv) *$cp_T = T(p)$, and $cT(p) = p_T$;*
- (v) *$T(+)= \langle T(\pi_0)c, T(\pi_1)c \rangle +_T c$.*

PROOF:

(i) Naturality of p , $+$, and 0 implies that, for each $f : M \rightarrow N$ the following diagrams commute:

$$\begin{array}{ccccc}
 TM & \xrightarrow{Tf} & TN & & T_2M \xrightarrow{\langle \pi_0 Tf, \pi_1 Tf \rangle} T_2N & & M \xrightarrow{f} N \\
 p_M \downarrow & & \downarrow p_N & & +_M \downarrow & & \downarrow +_N & & 0_M \downarrow & & \downarrow 0_N \\
 M & \xrightarrow{f} & N & & TM & \xrightarrow{Tf} & TN & & TM & \xrightarrow{Tf} & TN
 \end{array}$$

which are precisely the diagram required to make (Tf, f) an additive bundle morphism.

(ii) Since $(c, 1)(c, 1) = (1, 1)$, $(c, 1)$ is an additive bundle isomorphism between these objects.

(iii) Compose $(\ell, 0)$ with $(c, 1)$.

(iv) As $(c, 1) : (T(p), T(+), T(0)) \rightarrow (p_T, +_T, 0_T)$ is a bundle morphism $cp_T = T(p)$. But then since $c^2 = 1$, we also have $cT(p) = p_T$.

(v) Since c is an additive bundle morphism, we have

$$T(+)= \langle T(\pi_0)c, T(\pi_1)c \rangle +_T c,$$

the result then follows by applying c to both sides and using $c^2 = 1$.

□

Note that even though T has a natural transformation $\ell : T \rightarrow T^2$ satisfying coassociativity, and a natural transformation $p : T \rightarrow I$, it is not the case that (T, p, ℓ) is a comonad. The fact that $(\ell, 0)$ is an additive bundle morphism tells us $\ell T(p) = p0$, rather than the identity. Surprisingly, however, we shall see in section 3.2 that there is a multiplication $\mu : T^2 \rightarrow T$ which makes $(T, 0, \mu)$ into a monad.

A straightforward observation from [Rosický 1984] provides a basic source of additional examples:

Proposition 2.5 *If (\mathbb{X}, \mathbb{T}) is a tangent category and A is an object of \mathbb{X} , then the slice category \mathbb{X}/A has tangent structure \mathbb{T}' , where*

$$T'(X, f) := (TX, pf) \text{ and } T'f = Tf.$$

See propositions 5.17 and 5.19 for further ways of generating new tangent structure from old.

The following result indicates that one can interchange the two additions on the second tangent bundle, when the expressions are well-defined, and therefore that the additions are equal if they share a neutral element:

Lemma 2.6 *(Interchange of addition) In a tangent category, for $v_1, v_2, v_3, v_4 : X \rightarrow T^2M$:*

(i) *We can interchange addition,*

$$\langle\langle v_1, v_2 \rangle T(+), \langle v_3, v_4 \rangle T(+)\rangle +_T = \langle\langle v_1, v_3 \rangle +_T, \langle v_2, v_4 \rangle +_T \rangle T(+)$$

whenever both sides are defined;

(ii) *When $v_1 T(p0) = v_1 p0$ and $v_2 T(p0) = v_2 p0$ then*

$$\langle v_1, v_2 \rangle T(+) = \langle v_1, v_2 \rangle +_T$$

whenever both sides are defined.

PROOF:

(i) The requirement that both sides of this equation be defined amounts explicitly to requiring that the following equations hold:

$$v_1 T(p) = v_2 T(p), v_3 T(p) = v_4 T(p), v_1 p_T = v_3 p_T, \text{ and } v_2 p_T = v_4 p_T.$$

We have:

$$\begin{aligned} \langle\langle v_1, v_2 \rangle T(+), \langle v_3, v_4 \rangle T(+)\rangle +_T &= \langle\langle v_1, v_2 \rangle, \langle v_3, v_4 \rangle \rangle T_2(+) +_T \\ &= \langle\langle v_1, v_2 \rangle, \langle v_3, v_4 \rangle \rangle +_{T_2} T(+) \text{ (naturality of } +) \\ &= \langle\langle v_1, v_3 \rangle +_T, \langle v_2, v_4 \rangle +_T \rangle T(+) \text{ (see below).} \end{aligned}$$

For the last step, using the naturality of $+$, $+_{T_2} T(\pi_0) = T_2(\pi_0) +_T$ and $+_{T_2} T(\pi_1) = T_2(\pi_1) +_T$, we have:

$$\begin{aligned} \langle\langle v_1, v_2 \rangle, \langle v_3, v_4 \rangle \rangle +_{T_2} &= \langle\langle v_1, v_2 \rangle, \langle v_3, v_4 \rangle \rangle \langle T_2(\pi_0) +_T, T_2(\pi_1) +_T \rangle \\ &= \langle\langle v_1, v_2 \rangle, \langle v_3, v_4 \rangle \rangle \langle \langle \pi_0 T(\pi_1), \pi_1 T(\pi_0) \rangle +_T, \langle \pi_0 T(\pi_1), \pi_1 T(\pi_1) \rangle +_T \rangle \\ &= \langle\langle v_1, v_3 \rangle +_T, \langle v_2, v_4 \rangle +_T \rangle. \end{aligned}$$

- (ii) The requirement that both sides of this equation be defined amounts explicitly to requiring $v_1p = v_2p$ and $v_1T(p) = v_2T(p)$. The proof uses the Eckmann-Hilton argument to show the additions are the same:

$$\begin{aligned}
\langle v_1, v_2 \rangle T(+) &= \langle \langle v_2p0v_1 \rangle +, \langle v_2, v_1p0 \rangle + \rangle T(+) \quad (\text{interchange}) \\
&= \langle \langle v_2p0, v_2 \rangle T(+), \langle v_1, v_1p0 \rangle T(+) \rangle + \\
&= \langle \langle v_2T(p0), v_2 \rangle T(+), \langle v_1, v_1T(p0) \rangle T(+) \rangle + \\
&= \langle v_2, v_1 \rangle +
\end{aligned}$$

□

2.3 Morphisms of tangent structure

There will be several instances where morphisms of tangent structure will become important. In fact, we shall need both a strong and a weak version of these morphisms.

Definition 2.7 *Suppose that $(\mathbb{X}, T, p, 0, +, \ell, c)$ and $(\mathbb{X}', T', p', 0', +', \ell', c')$ are tangent categories. A **morphism of tangent categories** consists of a functor $F : \mathbb{X} \rightarrow \mathbb{X}'$ and a natural transformation $\alpha : TF \rightarrow FT'$ such that the following diagrams commute:*

$$\begin{array}{ccc}
TF \xrightarrow{\alpha} FT' & & T_2F \xrightarrow{\alpha_2F} FT'_2 \\
\downarrow pF & \downarrow Fp' & \downarrow +F \\
F & & TF \xrightarrow{\alpha} FT' \\
\downarrow 0F & \downarrow F0' & \downarrow F+' \\
TM \xrightarrow{\alpha} FT' & & FT'_2 \\
\downarrow \ell & \downarrow F\ell' & \downarrow c \\
T^2F \xrightarrow{(T\alpha)(\alpha T)} FT'^2 & & T^2F \xrightarrow{(T\alpha)(\alpha T)} FT'^2
\end{array}$$

Say that the morphism is **strong** if α is invertible, and F preserves the equalizers and pullbacks of the tangent structure $(\mathbb{X}, T, p, 0, +, \ell, c)$. Let **Tan** denote the category of (small) tangent categories and their morphisms.

It is often useful to look at the subcategory of **Tan** consisting of tangent structures on a single category \mathbb{X} , in which the morphisms all have functor the identity; denote this category by **Tan_X**.

In this category, a tangent substructure of T is given by a morphism, $\alpha : T' \rightarrow T$, of tangent structures in which every component α_M and $T'(\alpha_M)$ is monic. This ensures that the map $T'^2M \xrightarrow{T'(\alpha)\alpha} T^2M$ is monic. This is implied if $T(\alpha)$ is monic as $T'(\alpha)\alpha = \alpha T(\alpha)$. All the coherence conditions of a substructure are forced on the data restricted to the substructure except the requirements that T' preserve the pullbacks, and the universality of the vertical lift: these must be checked independently. An example of such a tangent substructure is given in Section 3.3, where we show (under certain conditions) that every tangent structure has an associated tangent substructure in which the additive bundles are groups.

In fact, every category has a trivial tangent structure, $\mathbb{I} = (I, 1, 1, 1, 1, 1)$ given by the identity functor. It is easily checked that there is always a unique morphism of tangent structures $0 : \mathbb{I} \rightarrow \mathbb{T} = (T, p, 0, +, \ell, c)$ so that the identity functor always gives the initial tangent structure in $\mathbf{Tan}_{\mathbb{X}}$. However, it also gives the final tangent structure, as $p : \mathbb{T} \rightarrow \mathbb{I}$ is also a morphism.

One might then expect that the functor T_n would give the n -fold product of (\mathbb{X}, \mathbb{T}) in $\mathbf{Tan}_{\mathbb{X}}$. However, this is not the case. These functors have an obvious projection $\pi_i p : T_n \rightarrow I$. It is clear that the pullback of m copies of T_n along this projection equals T_{nm} , and there is an addition map

$$T_{2n} \xrightarrow{\langle (\pi_i, p_{i+n})^+ \rangle_{i < n}} T_n$$

which satisfies the required coherences. One can also define a canonical flip. For $n \geq 2$, however, there is no vertical lift of the form we consider here: the obvious extension of the vertical lift for T gives a map from T_{n^2} to $(T_n)^2$. The structure of these functors, and other “Weil functors”, such as T^2 , are clearly of great interest, but are beyond the scope of this paper (see also [Nishimura 2012]).

Finally, the endomorphisms of a single tangent structure (\mathbb{X}, T) in $\mathbf{Tan}_{\mathbb{X}}$ are often of great interest. For example, [Kólár 1984] shows that for the canonical example (finite-dimensional smooth manifolds with their tangent bundle), such endomorphisms give a ring isomorphic to \mathbb{R} . Similarly, as we shall see in section 5.3, the endomorphisms of an instance of representable tangent structure give a rig which satisfies the Kock-Lawvere axiom.

2.4 Cartesian tangent structure

Categories with tangent structure are frequently equipped with products, but we must make additional assumptions to ensure that the tangent structure interacts well with them.

Definition 2.8 *A Cartesian tangent category is a tangent category which has finite products and for which T preserves these products, and the product functor, with the natural isomorphism $\alpha : T(X) \times T(Y) \rightarrow T(X \times Y)$ (inverse of the canonical transformation for the product), is a strong morphism of tangent structure.*

A morphism of Cartesian tangent structures is a morphism of tangent structures (F, α) for which F preserves products. Let $\mathbf{cartTan}$ be the resulting category of Cartesian tangent structures and their morphisms.

In this paper we shall almost exclusively be concerned with Cartesian tangent categories.

It is worth noting that if \mathbb{X} has Cartesian tangent structure it does not follow that the tangent structure on \mathbb{X}/A , given by Proposition 2.5, will be Cartesian. For this to be the case we must demand that the object A have $p : TA \rightarrow A$ an isomorphism: one might say such an object, with respect to the tangent structure, is “still”. It is quite possible for a tangent structure to have every object still in this sense: in this case, of course, the tangent structure will essentially be the trivial structure \mathbb{I} .

Tangent structures on Cartesian categories allow one to apply the tangent functor in each variable separately. Let s be the canonical map

$$T(A \times B) \xrightarrow{s} TA \times TB;$$

if T is Cartesian tangent structure, this map has an inverse, which we use in the following definition.

Definition 2.9 Suppose (\mathbb{X}, \mathbb{T}) is a Cartesian tangent category, and $f : A \times B \rightarrow C$. We define

$$T_A(f) := TA \times B \xrightarrow{1 \times 0} TA \times TB \xrightarrow{s^{-1}} T(A \times B) \xrightarrow{Tf} TC$$

and

$$T_B(f) := A \times TB \xrightarrow{0 \times 1} TA \times TB \xrightarrow{s^{-1}} T(A \times B) \xrightarrow{Tf} TC.$$

We can recover Tf from these “partial tangents” much as one can recover a differential from partial differentials:

Proposition 2.10 *If f is as above, then*

$$Tf = \langle s(1 \times p)T_A f, s(p \times 1)T_B f \rangle +$$

PROOF: First, note the pairing map into the pullback is well-defined, as the two maps are equal when post-composed by p :

$$\begin{aligned} s^{-1}(1 \times p)T_A f p &= s^{-1}(1 \times p)(1 \times 0)sT(f)p \\ &= s^{-1}(1 \times p0)sp f \text{ (by naturality of } p) \\ &= s^{-1}(1 \times p0)(p \times p)f \\ &= s^{-1}(p \times p)f \text{ (since } 0p = 1) \end{aligned}$$

and similarly when the other maps is post-composed by p .

To show that Tf can be recovered as described, consider:

$$\begin{aligned} \langle s^{-1}(1 \times p)T_A f, s^{-1}(p \times 1)T_B f \rangle + &= \langle s^{-1}(1 \times p0)Tf, s^{-1}(p0 \times 1)Tf \rangle + \text{(as above)} \\ &= \langle s^{-1}(1 \times p0), s^{-1}(p0 \times 1) \rangle \langle \pi_0 Tf, \pi_1 Tf \rangle + \\ &= \langle s^{-1}(1 \times p0), s^{-1}(p0 \times 1) \rangle + T(f) \text{ (naturality of } +) \\ &= T(f) \text{ (since addition is unital)} \end{aligned}$$

as required. □

2.5 Universality of vertical lift

The universality of the vertical lift is essential to obtaining key behavioural properties of tangent structure. In particular, it plays a fundamental role in defining the Lie bracket of vector fields (definition 3.13), showing that tangent spaces have differential structure (theorem 4.15), and showing that representable tangent structure satisfies the Kock-Lawvere axiom (theorem 5.12). Thus, the universality of the vertical lift, while being perhaps the least intuitive aspect of tangent structure, is also perhaps the most important. In this section, we record some observations about this universality which will be useful later.

Recall that the definition of tangent structure uses the following morphism

$$T_2M \xrightarrow{v := \langle \pi_0 \ell, \pi_1 0_T \rangle T(+)} T^2M.$$

in which the lift ℓ is hidden. The behaviour of this morphism is fundamental to understanding the axiom and we begin by giving a useful lemma on the projections of v .

Lemma 2.11 $vp_T = \pi_1$ and $vT(p) = \pi_0 p 0$.

PROOF: For the first claim:

$$\begin{aligned}
vp_T &= \langle \pi_0 \ell, \pi_1 0_T \rangle T(+) p_T \\
&= \langle \pi_0 \ell p_T, \pi_1 0_T p_T \rangle + (\text{naturality of } p) \\
&= \langle \pi_0 \ell c p_T, \pi_1 \rangle + \\
&= \langle \pi_0 \ell T(p), \pi_1 \rangle + \\
&= \langle \pi_0 p 0, \pi_1 \rangle + (\text{since } l \text{ is an additive bundle morphism}) \\
&= \pi_1 (\text{unit of addition}).
\end{aligned}$$

For the second claim:

$$\begin{aligned}
vT(p) &= \langle \pi_0 \ell, \pi_1 0_T \rangle T(+) T(p) \\
&= \langle \pi_0 \ell, \pi_1 0_T \rangle T(+p) \\
&= \langle \pi_0 \ell, \pi_1 0_T \rangle T(\pi_1 p) \\
&= \langle \pi_0 \ell, \pi_1 0_T \rangle T(\pi_1) T(p) \\
&= \pi_0 \ell T(p) \\
&= \pi_0 p 0 (\ell \text{ is an additive bundle morphism}).
\end{aligned}$$

□

The universality of the vertical lift can be expressed in several different ways:

Lemma 2.12 *In the presence of the earlier axioms, the universality of vertical lift may be equivalently expressed by demanding either of the following:*

(i) *for any map $f : X \rightarrow T^2M$ which equalizes² $T(p)$ and $pp0$, there is a unique map $\{f\} : X \rightarrow TM$ such that*

$$f = \langle \{f\} \ell, f p_T 0_T \rangle T(+)$$

(ii) *the square*

$$\begin{array}{ccc}
T_2M & \xrightarrow{v} & T^2M \\
\pi_0 p \downarrow & & \downarrow T(p) \\
M & \xrightarrow{0} & TM
\end{array}$$

is a pullback.

PROOF:

²Here and throughout the paper, we use “ f equalizes g and h ” to mean that the map f , when composed with the subsequent maps g and h , produces the same map; that is, $fg = fh$. We reserve “equalizer” for those maps which satisfy this property and, in addition, induce a universal cone for the equalizer diagram.

(i) Since v is the equalizer, we have a unique map $f|_v : X \rightarrow T_2M$ such that $f|_v v = f$. We set $\{f\} := f|_v \pi_0$ and have:

$$\begin{aligned}
f &= f|_v v \\
&= f|_v \langle \pi_0 \ell, \pi_1 0_T \rangle T(+) \\
&= \langle f|_v \pi_0 \ell, f|_v \pi_1 0_T \rangle T(+) \\
&= \langle f|_v \pi_0 \ell, f|_v v p_T 0_T \rangle T(+) \text{ (by lemma 2.11)} \\
&= \langle f|_v \pi_0 \ell, f p_T 0_T \rangle T(+) \\
&= \langle \{f\} \ell, f p_T 0_T \rangle T(+)
\end{aligned}$$

The uniqueness of $\{f\}$ follows from the uniqueness of $f|_v$ as

$$f = \langle \{f\} \ell, f p_T 0_T \rangle T(+) = \langle \{f\}, f p_T \rangle \langle \pi_0 \ell, \pi_1 0_T \rangle T(+) = \langle \{f\}, f \rangle v$$

so that $\langle \{f\}, f p_T \rangle = f|_v$.

(ii) Note first that the square commutes as:

$$\begin{aligned}
vT(p) &= \langle \pi_0 \ell, \pi_1 0 \rangle T(+) T(p) = \langle \pi_0 \ell, \pi_1 0 \rangle T(\pi_1) T(p) \\
&= \pi_0 \ell T(p) = \pi_0 p 0
\end{aligned}$$

Assuming the square is a pullback, if g equalizes $T(p)$ and $pp0$ then setting $f = gpp$ the square

$$\begin{array}{ccc}
X & \xrightarrow{g} & T^2(M) \\
f \downarrow & & \downarrow T(p) \\
M & \xrightarrow{0} & T(M)
\end{array}$$

commutes. So there is a unique map $g|_v : X \rightarrow T_2(M)$ as required.

Conversely, suppose that this square commutes (so $gT(p) = f0$) and v is the equalizer of $T(p)$ and $pp0$ then, as $gT(p) = f0 = f0p0 = gT(p)p0 = gpp0$, g equalizes $T(p)$ and $pp0$ and so there is a unique map $g|_v : X \rightarrow T_2(M)$ with $g|_v v = g$. It remains only to show $g|_v \pi_0 p = f$ to establish that the square is a pullback. However, as 0 is monic it suffices to prove $g|_v \pi_0 p 0 = f0$ which follows as $f0 = gT(p) = g|_v v T(p) = g|_v \pi_0 p 0$.

□

We will use the operation $\{-\}$ of this lemma quite frequently in what follows. In particular, note that we may always write $f|_v$ as $\langle \{f\}, f p_T \rangle$.

Before going further, we should justify why the axiom we have called the ‘‘universality of vertical lift’’ does give importation on the vertical lift $\ell : T(A) \rightarrow T^2(A)$. The following consequence of the axiom helps to isolate this:

Lemma 2.13 *In any tangent category the following is a (triple) equalizer diagram:*

$$T(A) \xrightarrow{\ell} T^2(A) \begin{array}{c} \xrightarrow{T(p)} \\ \xrightarrow{p} \\ \xrightarrow{pp0} \end{array} T(A)$$

PROOF: We first show that ℓ equalizes $T(p)$, p_T , and $T(p)p_0$. Since $(\ell, 0)$ is a bundle morphism, we have

$$\begin{array}{ccc} T(A) & \xrightarrow{\ell} & T^2(A) \\ p \downarrow & & \downarrow T(p) \\ M & \xrightarrow{0} & T(A) \end{array}$$

We have:

$$\begin{aligned} \ell T(p)p_0 &= p_0 p_0 = p_0 \\ \ell T(p) &= p_0 \\ \ell p_T &= \ell c p_T = \ell T(p) = p_0. \end{aligned}$$

Showing that ℓ equalizes the three maps.

Now suppose we have another map $f : X \rightarrow T^2 M$ which equalizes these three maps: in particular, f equalizes $T(p)$ and pp_0 . Thus, we have a unique map $\langle \{f\}, f p_T \rangle$ (using part (i) of 2.12 above) such that:

$$\begin{array}{ccccc} X & & & & \\ & \searrow f & & & \\ & & T_2(A) & \xrightarrow{v} & T^2(A) & \xrightarrow{T(p)} & T(A) \\ & \swarrow \langle \{f\}, f p \rangle & & & & \xrightarrow{pp_0} & \\ & & & & & & \end{array}$$

We claim that $\{f\} : X \rightarrow TM$ is the required unique map

$$\begin{array}{ccccc} X & & & & \\ & \searrow f & & & \\ & & T(A) & \xrightarrow{\ell} & T^2(A) & \xrightarrow{T(p)} & T(A) \\ & \swarrow \{f\} & & & & \xrightarrow{p} & \\ & & & & & \xrightarrow{pp_0} & \end{array}$$

because

$$\begin{aligned} f &= \langle \{f\}, f \rangle v = \langle \{f\} \ell, f p_0 \rangle T(+) = \langle \{f\} \ell, f p p_0 \rangle T(+) \\ &= \langle \{f\} \ell, f p p_0 T(0) \rangle T(+) = \langle \{f\} \ell, f T(p) T(0) \rangle T(+) = \langle \{f\} \ell, \{f\} \ell T(p) T(0) \rangle T(+) = \{f\} \ell. \end{aligned}$$

□

In particular, as ℓ is an equalizer it is certainly monic.

We record some further manipulations for the operation $\{-\}$. In particular we shall use its interaction with the two additions, the tangent functor, and composition:

Lemma 2.14 For $f, g : X \rightarrow T^2(A)$ which equalize $T(p)$ and pp_0 and have $fT(p)p = gT(p)p$:

- (i) for any $k : Z \rightarrow X$, $k\{f\} = \{kf\}$;
- (ii) $\{f\}p = fT(p)p$ when the left hand side is defined;

(iii) $\{00_T\} = 0$;

(iv) $\langle\{f\}, \{g\}\rangle_+ = \langle\langle f, g \rangle T(+)\rangle$ when either side is defined;

(v) $\langle\{f\}, \{g\}\rangle_+ = \langle\langle f, g \rangle_+\rangle$ when both sides are defined;

(vi) $T(\{f\}) = \{T(f)c_T T(c)\}c$ when the left side is defined;

(vii) for $h : A \longrightarrow B$, $\{f\}T(h) = \{fT^2(h)\}$ when the left side is defined.

PROOF:

(i) This follows immediately as $\{f\}$ is a map to a limit: this can be seen explicitly in the following calculation $k\{f\} = kf|_v \pi_0 = (kf)|_v \pi_0 = \{kf\}$.

(ii) We have:

$$\begin{aligned}
 fT(p)p &= \langle\{f\}\ell, fp_T 0_T\rangle T(+)\langle T(p)p \rangle \\
 &= \langle\{f\}\ell, fp_T 0_T\rangle T(\pi_0)\langle T(p)p \rangle \text{ (+ is a bundle morphism)} \\
 &= \{f\}\ell T(p)p \\
 &= \{f\}p0p \text{ ((}\ell, 0\text{) is a bundle morphism)} \\
 &= \{f\}p \text{ (since } 0p = 1\text{)}.
 \end{aligned}$$

(iii) We need to show 0 has the same universal property as 00_T :

$$\begin{aligned}
 \langle 0\ell, 00_T p_T 0_T \rangle T(+) &= \langle 0T(0), 00_T \rangle T(+) \text{ (}\ell \text{ is additive, and } 0p = 1\text{)} \\
 &= 0\langle T(0), 0_T \rangle T(+) \\
 &= 00_T \text{ (addition of a zero term)}.
 \end{aligned}$$

(iv) If $\langle\{f\}, \{g\}\rangle_+$ is defined then $fT(p) = fpp0$, $gT(p) = gpp0$, and (using (ii)) $fpp = \{f\}p = \{g\}p = gpp$. Thus, $gT(p) = gpp0 = fpp0 = fT(p)$ so that $\langle f, g \rangle T(+)$ is defined. Also $\langle f, g \rangle T(+)\langle T(p)p \rangle = fT(p) = fT(p)p0 = \langle f, g \rangle T(+)\langle T(p)p \rangle = \langle f, g \rangle pp0$ so that $\langle\{f, g\}T(+)\rangle$ is defined.

Conversely if $\langle\langle f, g \rangle T(+)\rangle$ is defined then $fT(p) = gT(p)$ and $\langle f, g \rangle T(+)\langle T(p)p \rangle = \langle f, g \rangle T(+)\langle pp0 \rangle$. However, this means $fT(p) = \langle f, g \rangle T(+)\langle T(p)p \rangle = \langle f, g \rangle T(+)\langle pp0 \rangle = \langle f, g \rangle T(+)\langle T(p)p \rangle = fT(p)p0 = fpp0$. Thus, $\{f\}$ is defined and similarly $\{g\}$ is defined. Finally $\{f\}p = fpp = fT(p)p = gT(p)p = gpp = \{g\}p$ so that $\langle\{f\}, \{g\}\rangle_+$ is defined.

Thus, we have shown that if either side is defined the other side is as well.

It remains to show $\langle\{f\}, \{g\}\rangle_+$ has the same universal property as $\langle\langle f, g \rangle T(+)\rangle$:

$$\begin{aligned}
 &\langle\langle\{f\}, \{g\}\rangle_+ \ell, \langle f, g \rangle T(+)\langle pT \rangle T(+)\rangle \\
 &= \langle\langle\{f\}\ell, \{g\}\ell\rangle T(+), \langle fp0, gp0 \rangle T(+)\rangle T(+) \text{ (}\ell \text{ is a morphism of bundles)} \\
 &= \langle\langle\{f\}\ell, fp0\rangle T(+), \langle\{g\}\ell, gp0\rangle T(+)\rangle T(+) \text{ (associativity and commutativity)} \\
 &= \langle f, g \rangle T(+).
 \end{aligned}$$

(v) Note that either side of this equation can be defined when the other is not! Thus, there are side conditions which must be checked when using this identity. We start by making these precise.

Going from left to right: not only must we require that $\langle\{f\}, \{g\}\rangle_+$ is defined but also that $fp = gp$ in order for $\langle\{f, g\}\rangle_+$ to be defined. To see this note that saying $\langle\{f\}, \{g\}\rangle_+$ is defined means $fT(p) = fpp0$, $gT(p) = gpp0$ and $fpp = gpp$ and requiring $fp = gp$ means $\langle f, g \rangle_+$ is certainly defined. But also $\langle f, g \rangle_+ + T(p) = \langle fT(p), gT(p) \rangle_+ = \langle fpp0, gpp0 \rangle_+ = \langle fpp0, fpp0 \rangle_+ = fpp0 = \langle f, g \rangle_+ + pp0$ so that $\langle\{f, g\}\rangle_+$ is defined.

Going from right to left we require that in addition to $\langle\{f, g\}\rangle_+$ being defined that $fT(p) = fpp0$ or $gT(p) = gpp0$. To see this note that $\langle\{f, g\}\rangle_+$ being defined means $fp = gp$ and so $fpp0 = gpp0$, furthermore, assuming (without loss) that $fT(p) = fpp0$ gives

$$\begin{aligned} gT(p) &= \langle gpp0, gT(p) \rangle_+ = \langle fpp0, gT(p) \rangle_+ = \langle fT(p), gT(p) \rangle_+ + \\ &= \langle f, g \rangle_+ + T(p) = \langle f, g \rangle_+ + pp0 = gpp0 \end{aligned}$$

so that $\{f\}$ and $\{g\}$ are defined. Finally, $\{f\}p = fpp = gpp = \{g\}p$ so that $\langle\{f\}, \{g\}\rangle_+$ is defined.

The following variation on the previous calculation for universality gives the equality:

$$\begin{aligned} &\langle\langle\{f\}, \{g\}\rangle_+ + \ell, \langle f, g \rangle_+ + p0\rangle T(+) \\ &= \langle\langle\{f\}\ell, \{g\}\ell\rangle_+, fp0\rangle T(+) \quad (\ell \text{ is a morphism of bundles}) \\ &= \langle\langle\{f\}\ell, \{g\}\ell\rangle_+, \langle fp0, gp0 \rangle_+\rangle T(+) \quad (\text{as } fp = gp \text{ and adding of zero}) \\ &= \langle\langle\{f\}\ell, fp0\rangle T(+), \langle\{g\}\ell, gp0\rangle T(+)\rangle + \quad (\text{interchange from lemma 2.6}) \\ &= \langle f, g \rangle_+ . \end{aligned}$$

(vi) We first need to check that $T(f)c_T T(c)$ equalizes the required maps when f does:

$$\begin{aligned} T(f)c_T T(c)T(p_T) &= T(f)c_T T(cp_T) = T(f)c_T T(T(p)) \\ &= T(f)c_T T^2(p) = T(f)T^2(p)c_T \quad (\text{naturality of } c) \\ &= T(fT(p))c = T(fT(p)p0)c \quad (\text{by assumption on } f) \\ &= T(f)T^2(p)T(p)T(0)c = T(f)T^2(p)T(p)0_T \\ &= T(f)T^2(p)c_{p_T}0_T = T(f)c_T T^2(p)p_T0_T \quad (\text{naturality of } c) \\ &= T(f)c_T T(cp_T)p_T0_T = T(f)c_T T(c)T(p_T)p_T0_T. \end{aligned}$$

We now show that $T(\{f\})c$ has the same universal property as $\{T(f)c_T T(c)\}$:

$$\begin{aligned}
& T(f)c_T T(c) \\
&= T(\langle \{f\} \ell, f p_T 0_T \rangle T(+)) c_T T(c) \text{ (by the universal property of } \{f\}) \\
&= \langle T(\{f\})T(\ell), T(f)T(p_T)T(0_T) \rangle T^2(+)_c T(c) \text{ (} T \text{ preserves pullback for } T_2) \\
&= \langle T(\{f\})T(\ell), T(f)T(p_T)T(0_T) \rangle c_{T_2} T^2(+)_T(c) \text{ (naturality of } c) \\
&= \langle T(\{f\})T(\ell)c_T, T(f)T(p_T)T(0_T)c_T \rangle T^2(+)_T(c) \\
&= \langle T(\{f\})T(\ell)c_T, T(f)T(p_T)0_{T^2} \rangle T(T(+)_c) \text{ (} T(0)c = 0_T) \\
&= \langle T(\{f\})T(\ell)c_T, T(f)T(p_T)0_{T^2} \rangle T(\langle \pi_0 c, \pi_1 c \rangle +) \text{ (} c \text{ is additive)} \\
&= \langle T(\{f\})T(\ell)c_T, T(f)T(p_T)0_{T^2} \rangle \langle T(\pi_0)T(c), T(\pi_1)T(c) \rangle T(+) \text{ (} T \text{ preserves pullback)} \\
&= \langle T(\{f\})T(\ell)c_T T(c), T(f)T(p_T)0_{T^2} T(c) \rangle T(+) \\
&= \langle T(\{f\})c \ell_T, T(f)T(p_T)c 0_{T^2} \rangle T(+) \text{ (coherence equation and naturality of } 0) \\
&= \langle T(\{f\})c \ell_T, T(f)c_T p_{T^2} c 0_{T^2} \rangle T(+) \text{ (} c p_T = T(p)) \\
&= \langle T(\{f\})c \ell_T, T(f)c_T T(c) p_{T^2} 0_{T^2} \rangle T(+) \text{ (naturality of } p).
\end{aligned}$$

(vii) We first need to check that $fT^2(h)$ equalizes the required maps when f does:

$$\begin{aligned}
fT^2(h)T(p) &= fT(T(h)p) = fT(ph) \text{ (naturality of } p) \\
&= fT(p)T(h) = fT(p)p_0 T(h) = fT(p)ph_0 \text{ (naturality of } 0) \\
&= fT(p)T(h)p_0 \text{ (naturality of } p) \\
&= fT(ph)p_0 = fT(T(h)p)p_0 \text{ (naturality of } p) \\
&= fT^2(h)T(p)p_0.
\end{aligned}$$

Now, we need to show that $\{f\}T(h)$ has the same universal property as $\{fT^2(h)\}$:

$$\begin{aligned}
\langle \{f\}T(h)\ell, fT^2(h)p_T 0_T \rangle T(+) &= \langle \{f\} \ell T^2(h), f p_T 0_T T^2(h) \rangle T(+) \text{ (naturality of } \ell \text{ and } p_0) \\
&= \langle \{f\} \ell, f p_T 0_T \rangle T(T_2(h))T(+) \\
&= \langle \{f\} \ell, f p_T 0_T \rangle T(T_2(h)+) \\
&= \langle \{f\} \ell, f p_T 0_T \rangle T(+T(h)) \text{ (naturality of } +) \\
&= \langle \{f\} \ell, f p_T 0_T \rangle T(+)T^2(h) \\
&= fT^2(h).
\end{aligned}$$

□

Finally, we observe:

Lemma 2.15 T^n preserves the equalizer diagram for the universality of vertical lift.

PROOF: The canonical flip can be used to bring the equalization of $T^n(T(p))$ and $T^n(pp_0)$ to the top level where one can use the fact that T^n preserves pullback powers of p_M to show that the

required map is an equalizer. The basic step of lifting the equalization is given by the following serially commuting diagram:

$$\begin{array}{ccccc}
T_2(TM) & \xrightarrow{v} & T^3M & \xrightarrow{T(p)} & T^2M \\
\cong \downarrow & & \downarrow T(c)c & \xrightarrow{pp0} & \downarrow c \\
T(T_2M) & \xrightarrow{T(v)} & T^3M & \xrightarrow{T^2(p)} & T^2M \\
& & & \xrightarrow{T(pp0)} &
\end{array}$$

□

3 Some basic theory of tangent structure

3.1 Vector Fields

The notion of a vector field is of central importance in differential geometry, and generalizes to tangent structures in a natural way.

Definition 3.1 *If M is an object of a tangent category (\mathbb{X}, T) , a section of p_M , that is any $w : M \rightarrow T(M)$ with $wp = 1_M$, is a **vector field on M** . Denote the set of vector fields on M by $\chi(M)$.*

The existence of the map $v : T_2(M) \rightarrow T^2(M)$ allows one to define the so-called “Liouville vector field” on any object of the form $T(M)$ by the map

$$T(M) \xrightarrow{\langle 1, 1 \rangle v} T^2(M).$$

This is a vector field since, by lemma 2.11, $\langle 1, 1 \rangle v p_T = \langle 1, 1 \rangle \pi_1 = 1$. Furthermore, every object M always has a zero vector field $0_M : M \rightarrow T(M)$, vector fields inherit additive structure from T , and this structure is preserved in two different ways:

Proposition 3.2 *If M is an object of a tangent category (\mathbb{X}, T) with vector fields $w_1, w_2 \in \chi(M)$, then*

- (i) $\chi(M)$ has the structure of a commutative monoid, with $w_1 + w_2 := \langle w_1, w_2 \rangle +$ and $0 := 0_M$;
- (ii) for any map $f : M \rightarrow N$, $(w_1 + w_2)T(f) = w_1T(f) + w_2T(f)$ and $0T(f) = 0$;
- (iii) $T(w_1 + w_2) = (T(w_1)c +_T T(w_2)c)c$ and $T(0) = 0_{Tc}$.

PROOF:

(i) Immediate.

(ii) For the second claim, we have:

$$\begin{aligned}
\langle w_1, w_2 \rangle +_M T(f) &= \langle w_1, w_2 \rangle T_2(f) +_N \text{ (by naturality of } +) \\
&= \langle w_1, w_2 \rangle \langle \pi_0 T(f), \pi_1 T(f) \rangle +_N \text{ (by definition of } T_2) \\
&= \langle w_1 T(f), w_2 T(f) \rangle +_N \\
&= w_1 T(f) + w_2 T(f).
\end{aligned}$$

The result for 0 follows similarly.

(iii) Here is the calculation:

$$\begin{aligned}
T(w_1 + w_2) &= T(\langle w_1, w_2 \rangle +) \\
&= T(\langle w_1, w_2 \rangle)T(+) \\
&= T(\langle w_1, w_2 \rangle)\langle T(\pi_0)c, T(\pi_1)c \rangle +_T c \text{ (since } c \text{ is an additive bundle isomorphism)} \\
&= \langle T(\langle w_1, w_2 \rangle \pi_0)c, T(\langle w_1, w_2 \rangle \pi_1)c \rangle +_T c \\
&= \langle T(w_1)c, T(w_2)c \rangle +_T c \\
&= (T(w_1)c +_T T(w_2)c)c.
\end{aligned}$$

The result for 0 is immediate as c is a morphism of bundles. □

As we shall see in the next section, the addition of vector fields is a specific case of composition in the Kleisli category for the monad on tangent bundles.

3.2 Monad structure of T and T^2

In this section, we show that the axioms for tangent structure imply not only that T is a monad but also that T^2 is a monad. The Kleisli category \mathbb{X}_T , in particular, contains, the vector fields and their addition.

We begin with an important map which “forgets” a double tangent vector in T^2 :

Lemma 3.3 *The map*

$$T^2(M) \xrightarrow{u_M := \langle Tp, p_T \rangle} T_2M$$

is a natural transformation from T^2 to T_2 .

PROOF: Let $M \xrightarrow{f} N$ be an arbitrary map, and consider

$$\begin{aligned}
(u_M)(T_2f) &= \langle Tp, p_T \rangle \langle \pi_0 T(f), \pi_1 T(f) \rangle \text{ by definition of } T_2(f), \\
&= \langle T(p)T(f), p_T T(f) \rangle = \langle T(pf), p_T T(f) \rangle \\
(T^2f)(u_N) &= T^2(f)\langle Tp, p_T \rangle = \langle T^2(f)T(p), T^2(f)p_T \rangle \\
&= \langle T(T(f)p), p_T T(f) \rangle = \langle T(pf), p_T T(f) \rangle \text{ by naturality of } p;
\end{aligned}$$

so that the two are equal, as required. Alternatively, one can immediately see this as $u = \Delta(p \times T(p))$. □

In the context of synthetic differential geometry, a right inverse for u is considered to provide an affine connection [Kock and Reyes 1979]: we do not pursue that idea here.

We can now give the monad structure of T .

Proposition 3.4 *If (\mathbb{X}, \mathbb{T}) is a tangent category, then T is a monad, with unit $M \xrightarrow{0} TM$, and multiplication μ_1 given by the composite*

$$T^2M \xrightarrow{u} T_2M \xrightarrow{+} TM.$$

PROOF: By definition, the 0 and + are natural, and we have shown u is natural above. Thus, we only need to check the unit and associativity axioms. For one unit axiom, consider

$$\begin{aligned}
T(0)u+ &= T(0)\langle T(p), p \rangle + \\
&= \langle T(0)T(p), T(0)p \rangle + \\
&= \langle T(0p), p0 \rangle + \text{ by functoriality of } T \text{ and naturality of } p, \\
&= \langle T(1), p0 \rangle + \text{ by coherence for } p, \\
&= \langle 1, p0 \rangle + \\
&= 1 \text{ by the unit axiom for the addition of tangent vectors.}
\end{aligned}$$

For the other unit axiom, consider

$$\begin{aligned}
0\langle T(p), p \rangle + &= \langle 0T(p), 0p \rangle + \\
&= \langle 0cp, 1 \rangle + \text{ by coherence of } c \text{ and } p, \\
&= \langle T(0)p, 1 \rangle + \text{ since } c \text{ is an additive bundle morphism,} \\
&= \langle p0, 1 \rangle + \text{ by naturality of } p, \\
&= 1 \text{ by the unit axiom for the addition of tangent vectors.}
\end{aligned}$$

For the associativity, we need to show the commutativity of the outside of the following diagram:

$$\begin{array}{ccccc}
T^3M & \xrightarrow{T(u)} & TT_2M & \xrightarrow{T(+)} & T^2M \\
\downarrow u_T & \swarrow u_3 & \downarrow w & \nearrow c & \downarrow u \\
& & T_2TM & \xrightarrow{+_T} & T^2M \\
& & \downarrow +_T & & \downarrow + \\
T_2TM & & T_3M & \xrightarrow{+_r} & T_2M \\
\downarrow +_T & & \downarrow +_l & & \downarrow + \\
T^2M & \xrightarrow{u} & T_2M & \xrightarrow{+} & TM
\end{array}$$

where we have added the dashed arrows, and

$$u_3 = \langle T(p_T)T(p), p_{T^2}T(p), p_{T^2}p_T \rangle, \quad w = \langle T(\pi_0)c, T(\pi_1)c \rangle.$$

The bottom right diagram commutes by the associativity of tangent vector addition, and the top right diagram since c preserves addition. Thus, we only need to show the commutativity of the left region and the middle region.

To show the left region commutes, we begin by expanding the right composite of that region:

$$\begin{aligned}
(u_3)(+_l) &= \langle T(p_T)T(p), p_{T^2}T(p), p_{T^2}p_T \rangle \langle \langle \pi_0, \pi_1 \rangle +, \pi_2 \rangle \\
&= \langle \langle T(p_T)T(p), p_{T^2}T(p) \rangle +, p_{T^2}p_T \rangle
\end{aligned}$$

Now, since these are two maps into T_2 , to show they are equal, it suffices to show they are equal when post-composed by π_0 and π_1 . Indeed, if we consider

$$\begin{aligned}
(u_T)(+_T)(u)\pi_0 &= \langle T(p_T), p_{T^2} \rangle (+_T) \langle T(p), p_T \rangle \pi_0 \\
&= \langle T(p_T), p_{T^2} \rangle (+_T) T(p) \\
&= \langle T(p_T), p_{T^2} \rangle T_2(p) (+) \text{ by naturality of } +, \\
&= \langle T(p_T), p_{T^2} \rangle \langle \pi_0 T(p), \pi_1 T(p) \rangle (+) \text{ by definition of } T_2(f), \\
&= \langle T(p_T) T(p), p_{T^2} T(p) \rangle (+) \\
&= (u_3)(+_l)\pi_0 \text{ (by above)}
\end{aligned}$$

and

$$\begin{aligned}
(u_T)(+_T)(u)\pi_1 &= \langle T(p_T), p_{T^2} \rangle (+_T) \langle T(p), p_T \rangle \pi_1 \\
&= \langle T(p_T), p_{T^2} \rangle (+_T) p_T \\
&= \langle T(p_T), p_{T^2} \rangle (\pi_2 p_T) \text{ by coherence of } +, \\
&= p_{T^2} p_T \\
&= (u_3)(+_l)\pi_1 \text{ (by above)}
\end{aligned}$$

as required. Thus the left region commutes.

For the middle region, we first calculate

$$\begin{aligned}
T(u)w &= T(u) \langle T(\pi_0)c, T(\pi_1)c \rangle \\
&= \langle T(u\pi_0)c, T(u\pi_1)c \rangle \\
&= \langle T(\langle T(p), p_T \rangle \pi_0)c, T(\langle T(p), p_T \rangle \pi_1)c \rangle \\
&= \langle T^2(p)c, T(p_T)c \rangle
\end{aligned}$$

So that the top composite is

$$\langle T^2(p)c, T(p_T)c \rangle (+_T)(c) \langle T(p), p_T \rangle$$

while the middle composite is

$$\begin{aligned}
(u_3)(+_r) &= \langle T(p_T)T(p), p_{T^2}T(p), p_{T^2}p_T \rangle \langle \pi_0, \langle \pi_1, \pi_2 \rangle + \rangle \\
&= \langle \langle T(p_T)T(p), \langle p_{T^2}T(p), p_{T^2}p_T \rangle + \rangle
\end{aligned}$$

Again, since these are two maps into T_2 , to show they are equal, it suffices to show they are equal when post-composed by π_0 and π_1 :

$$\begin{aligned}
&\langle T^2(p)c, T(p_T)c \rangle (+_T)(c) \langle T(p), p_T \rangle \pi_0 \\
&= \langle T^2(p)c, T(p_T)c \rangle (+_T)(c) T(p) \\
&= \langle T^2(p)c, T(p_T)c \rangle (+_T) p_T \text{ by coherence of } c, \\
&= \langle T^2(p)c, T(p_T)c \rangle (\pi_1)(p_T) \text{ by coherence of } +_T, \\
&= T(p_T) c p_T \\
&= T(p_T) T(p) \text{ by coherence of } c, \\
&= (u_3)(+_r)\pi_0 \text{ (by above)}
\end{aligned}$$

and

$$\begin{aligned}
& \langle T^2(p)c, T(p_T)c \rangle (+_T)(c) \langle T(p), p_T \rangle \pi_1 \\
&= \langle T^2(p)c, T(p_T)c \rangle (+_T)(c) p_T \\
&= \langle T^2(p)c, T(p_T)c \rangle (+_T) T(p) \text{ by coherence of } c, \\
&= \langle T^2(p)c, T(p_T)c \rangle T_2(p)(+) \text{ by naturality of } +, \\
&= \langle T^2(p)c, T(p_T)c \rangle \langle \pi_0 T(p), \pi_1 T(p) \rangle (+) \text{ by definition of } T_2(f), \\
&= \langle T^2(p)c T(p), T(p_T)c T(p) \rangle (+) \\
&= \langle T^2(p) p_T, T(p_T) p_T \rangle (+) \text{ by coherence of } c, \\
&= \langle p_{T^2} T(p), p_{T^2} p_T \rangle (+) \text{ by naturality of } p \text{ (twice)}, \\
&= (u_3)(+_r) \pi_1.
\end{aligned}$$

Thus, the diagram commutes, and T is a monad. \square

The fact that T is a monad seems to have been largely overlooked in the differential geometry literature.³ This is perhaps surprising as the Kleisli category of this monad gives a natural generalization of the addition of vector fields:

Proposition 3.5 *If $v, w : M \rightarrow TM$ are vector fields on M , then the Kleisli composite of v and w (denoted by $v;_{\mu_1} w$) is precisely $v + w$.*

PROOF: For arbitrary maps in the Kleisli category $f : A \rightarrow TB$, $g : B \rightarrow TC$, $f;_{\mu_1} g$ is given by the formula

$$fT(g)\mu_1 = fT(g)\langle T(p), p_T \rangle + = \langle fT(gp), fp_g \rangle +.$$

But if f and g are vector fields, then $fp = gp = 1$, so the above simplifies to $\langle f, g \rangle +$, as required. \square

This shows that the maps of the Kleisli category \mathbb{X}_T are generalized vector fields, while the composition of these vector fields generalizes vector field addition. These generalized vector fields do appear in differential geometry literature as “push forwards” of vector fields, however, the Kleisli composition seems to have largely escaped notice.

We turn now to investigating how μ_1 interacts with ℓ and c . We start with:

Lemma 3.6 *If (\mathbb{X}, T) is tangent structure, then $c\mu_1 = \mu_1$, and $\ell\mu_1 = p0$.*

PROOF: For the first claim:

$$\begin{aligned}
c\mu_1 &= c\langle T(p), p_T \rangle + \\
&= \langle cT(p), cp_T \rangle + \\
&= \langle p_T, T(p) \rangle + \text{ (by Proposition 2.4)} \\
&= \langle T(p), p_T \rangle + \text{ (by commutativity of } +) \\
&= \mu_1
\end{aligned}$$

³The fact that T is a monad for Cartesian differential categories was independently noted in [Manzyuk 2012].

For the second claim,

$$\ell\mu_1 = \ell\langle T(p), p_T \rangle_+ = \langle p_0, p_0 \rangle_+ = p_0$$

as p_0 is the unit of $+$. □

Next we observe that c is a self-distributive law which shows that T^2 is also a monad:

Proposition 3.7 *If (\mathbb{X}, T) is tangent structure, then $c : T^2 \rightarrow T^2$ is a distributive law of (T, η_1, μ_1) (where $\eta_1 = 0$) over itself.*

PROOF: The two equations relating the units are $T(0)c = 0_T$ and $0_Tc = T(0)$ are the same equation (since $c^2 = 1$), and are true since c is an additive bundle morphism.

The equations relating the multiplications are also equivalent, again since $c^2 = 1$. Thus, it suffices to only prove one of them. We shall prove the equation $cT(c)\mu_1 = T(\mu_1)c$. We have

$$\begin{aligned} c_T T(c) T(p_T) &= c_T T(cp_T) \\ &= c_T T(T(p)) \text{ (} c \text{ is a bundle morphism)} \\ &= c_T T^2(p) \\ &= T^2(p)c \text{ (naturality of } c\text{).} \\ c_T T(c) p_{T^2} &= c_T p_T c \text{ (naturality of } c\text{)} \\ &= T(p_T)c \text{ (} c \text{ is a bundle morphism).} \end{aligned}$$

So the left side reduces to

$$\begin{aligned} \langle T^2(p)c, T(p_T)c \rangle_{+T} &= T(\langle T(p), p_T \rangle) \langle T(\pi_0)c, T(\pi_1)c \rangle_{+T} \\ &= T(\langle T(p), p_T \rangle) T(+)c \text{ (} c \text{ is an additive bundle morphism)} \\ &= T(\mu_1)c. \end{aligned}$$

□

In particular, note that this gives two monads which we may style as (T, η_1, μ_1) , where $\eta_1 := 0$ and $\mu_1 := \langle p, T(p) \rangle_+ = \langle T(p), p \rangle_+ = u_+$, and (T^2, η_2, μ_2) where $\eta_2 := 0_0$ and $\mu_2 := T(c)\mu_1 T(\mu_1)$.

Proposition 3.8 *$\ell, 0, T(0) : T \rightarrow T^2$ are morphisms of monads from (T, η_1, μ_1) to (T^2, η_2, μ_2) , and $p, T(p) : T^2(A) \rightarrow T(A)$ are morphisms of monads from (T^2, η_2, μ_2) to (T, η_1, μ_1) .*

PROOF: We shall leave the proof that $0, T(0), p$, and $T(p)$ are morphisms to the reader and concentrate on ℓ . It is useful to unwind the definition of μ_2 before starting this proof:

$$\begin{aligned} \mu_2 &= T(c)\mu_1 T(\mu_1) = T(c)\langle T(p), p \rangle_+ + T(\mu_1) = \langle T^2(p), pc \rangle_+ + T(\mu_1) \\ &= \langle T(T(p)\mu_1), pcT(\mu_1) \rangle_+ = \langle T(T(p)\langle T(p), p \rangle_+), pc\langle T^2(p), T(p) \rangle T(+) \rangle_+ \\ &= \langle \langle T^2(pp), T(pp) \rangle T(+), \langle pT^2(p)c, pp \rangle T(+) \rangle_+ \end{aligned}$$

We need to prove $\eta_1\ell = \eta_2$ and $\mu\ell = \ell T^2(\ell)\mu_2$ which follows by:

$$\begin{aligned}
\eta_1\ell &= 0\ell = 0T(0) = \eta_2 \\
\ell T^2(\ell)\mu_2 &= \ell T^2(\ell)\langle\langle T^2(pp), T(pp)\rangle T(+), \langle pT^2(p)c, pp\rangle T(+)\rangle + \\
&= \langle\langle \ell T^2(\ell pp), \ell T^2(\ell)T(pp)\rangle T(+), \langle \ell T^2(\ell)pT^2(p)c, \ell pp\rangle T(+)\rangle + \\
&= \langle\langle \ell T^2(p0p), \ell T(p)T(\ell p)\rangle T(+), \langle \ell pT^2(\ell T(p))c, p0p\rangle T(+)\rangle + \\
&= \langle\langle \ell T^2(p), p0T(p0)\rangle T(+), \langle p0T^2(p0)c, p\rangle T(+)\rangle + \\
&= \langle\langle T(p)\ell, p0T(p0)\rangle T(+), \langle p0T^2(p0)c, p\rangle T(+)\rangle + \\
&= \langle T(p)\ell, p\rangle + \langle T(p), p\rangle + \ell \\
&= \mu_1\ell.
\end{aligned}$$

□

While the Kleisli category for T is easily seen as generalizing vector field addition, its Eilenberg-Moore category seems to be harder to understand. An algebra $\alpha : TM \rightarrow M$ for T describes a way to associate each tangent vector on the space to another point on the space, in a way that is compatible with addition. In particular, sending the tangent vector to its base point is always an algebra:

Proposition 3.9 *If (\mathbb{X}, \mathbb{T}) is a tangent category, then for any object M , the projection map $p_M : TM \rightarrow M$ is an algebra for the monad T .*

PROOF: We begin by showing that p is a morphism of monads from $(T, 0, \mu)$ to the identity monad. For this, we need to show the diagrams

$$\begin{array}{ccc}
T^2 & \xrightarrow{\mu_1} & T \\
T(p)\downarrow & & \downarrow p \\
T & \xrightarrow{p} & I
\end{array}
\qquad
\begin{array}{ccc}
I & & \\
0\downarrow & \searrow 1 & \\
T & \xrightarrow{p} & I
\end{array}$$

commute. For the first diagram, we have

$$\mu_1 p = \langle T(p), p_T \rangle + p = \langle T(p), p_T \rangle \pi_0 p = T(p)p$$

by the properties of additive bundles. The second diagram, $0p = 1$, is also an axiom of additive bundles.

If $\alpha : T \rightarrow S$ is a morphism of monads, then α sends S -algebras to T -algebras via pre-composition. Since any object M is canonically an algebra for the identity functor, and p is a morphism of monads to the identity functor, $p : TM \rightarrow M$ is a T -algebra. □

Of course, any object of the form TM also has an associated free algebra $\mu : T^2M \rightarrow TM$. For example, on $\mathbb{R}^2 = T(\mathbb{R})$, the free algebra structure sends

$$(a, b, c, d) \mapsto (b + c, d).$$

It would be interesting to know if algebras of this monad are used in the literature on differential geometry: we have not been able to find such a use.

3.3 Group tangent structure

As noted earlier, the definition of tangent structure we give here only asks that the bundles be additive (i.e. commutative monoids). In Rosický's original definition, the bundles were assumed to be commutative groups. Here we shall refer to a tangent category in which the bundles are groups as a **group tangent category** or simply a **tangent category with negatives**. Rosický's definition, however, was also different in another respect: he had an alternate form for the universality of the vertical lift. Rosický's simply asked that the following be an equalizer diagram:

$$TM \xrightarrow{\ell} T^2M \begin{array}{c} \xrightarrow{T(p)} \\ \xrightarrow{p} \\ \xrightarrow{pp0} \end{array} TM$$

Recall that our definition asks that the following be an equalizer diagram:

$$T_2(M) \xrightarrow{v} T^2M \begin{array}{c} \xrightarrow{T(p)} \\ \xrightarrow{pp0} \end{array} TM$$

(where $v := \langle \pi_0 \ell, \pi_1 0_T \rangle T(+)$). We have already seen in Lemma 2.13 that our definition implies Rosický's condition. We now show that under the assumption that the bundles are groups the reverse implication holds:

Lemma 3.10 *In any category with group tangent structure the requirement that ℓ is an equalizer of $T(p)$, p , and $pp0$ is, in the presence of the other axioms, equivalent to the universality of vertical lift.*

Thus, in the presence of negatives, our definition is exactly equivalent to Rosický's, but, for mere additive bundles in our sense, Definition 2.3 remains the appropriate one.

PROOF: Suppose all the other axioms for tangent structure hold except the universality of vertical lift, and that ℓ is the equalizer of the three maps. As the additive bundles have negatives, there is a natural transformation $- : T \rightarrow T$ making each TM into an abelian group object over M .

We need to establish that v is the equalizer of $T(p)$ and $pp0$, so suppose we have a map f which equalizes $T(p)$ and $pp0$. Then the map

$$X \xrightarrow{\langle f, fp0T(-) \rangle T(+)} T^2M$$

equalizes $T(p), p, T(p)p0$, as

$$\begin{aligned} \langle f, fp0T(-) \rangle T(+T(p)) &= fT(p) = fpp0 \\ \langle f, fp0T(-) \rangle T(+pp0) &= \langle f, fp0T(-) \rangle T(+T(p)p0) = fT(p)p0 = fpp0 \\ \langle f, fp0T(-) \rangle T(+p) &= \langle fp, fp0p- \rangle + = \langle fp, fp- \rangle + = fpp0 \end{aligned}$$

and hence, by the universality of ℓ , there is a unique map $\{f\} : X \rightarrow TM$ with $\{f\}\ell = \langle f, fp0T(-) \rangle T(+)$. However, this gives the required map for the universality of v , as

$$\begin{aligned} \langle \{f\}\ell, fp0 \rangle T(+) &= \langle \langle f, fp0T(-) \rangle T(+), fpT0 \rangle T(+) \\ &= \langle f, \langle fp0T(-), fpT0 \rangle T(+) \rangle T(+) \\ &= \langle f, fp0 \rangle T(+) = f. \end{aligned}$$

□

In a group tangent category there are some further useful properties of $\{-\}$ along the lines of Lemma 2.14 which we take the opportunity to record at this stage:

Lemma 3.11 *Suppose (\mathbb{X}, \mathbb{T}) is a tangent category with negatives. If $f, g : X \rightarrow T^2M$, then:*

- (i) $\{f\}\ell = \langle f, fp0T(-) \rangle T(+)$ whenever the left side is defined;
- (ii) $\langle f, g \rangle T(+)- = \langle f-, g- \rangle T(+)$ whenever either side is defined;
- (iii) $\{f-\} = \{f\}-$ whenever either side is defined;
- (iv) $\{fT(-)\} = \{f\}-$ whenever either side is defined;
- (v) $\{f-T(-)\} = \{f\}$ whenever either side is defined.

PROOF:

(i) As $\langle \{f\}\ell, fp0 \rangle T(+)=f$ then $\{f\}\ell = \langle f, fp0T(-) \rangle T(+)$.

(ii) We show $\langle f-, g- \rangle T(+)$ is the negation of $\langle f, g \rangle T(+)$ as:

$$\begin{aligned}
& \langle \langle f, g \rangle T(+), \langle f-, g- \rangle T(+)+ \rangle \\
&= \langle f + f-, g + g- \rangle T(+)\quad (\text{interchange from lemma 2.6}) \\
&= \langle fp0, gp0 \rangle T(+)\quad (\text{negation}) \\
&= \langle flp, glp \rangle T(+)\quad (\ell \text{ is a morphism}) \\
&= \langle fl, gl \rangle T^2(+)\quad (p \text{ is natural}) \\
&= \langle f, g \rangle T(+)\ell p\quad (\ell \text{ is natural}) \\
&= \langle f, g \rangle T(+)\ell p0\quad (\ell \text{ is a morphism})
\end{aligned}$$

(iii) We would like to show that $\{f\}-$ satisfies the same universal property as $\{f-\}$. For this we have:

$$\begin{aligned}
\langle \{f\}-\ell, f-p0 \rangle T(+)&= \langle \{f\}\ell-, f-p0 \rangle T(+)\quad (\ell \text{ is a morphism of bundles}) \\
&= \langle \{f\}\ell-, fp0- \rangle T(+)\quad (\text{as } -p = p \text{ and } 0 = 0-) \\
&= \langle \{f\}\ell, fp0 \rangle T(+)-\quad (\text{by (ii)}) \\
&= f-.
\end{aligned}$$

(iv) We would like to show that $\{f\}-$ satisfies the same universal property as $\{fT(-)\}$. For this we have the calculation

$$\begin{aligned}
\langle \{f\}-\ell, fT(-)\ell p0 \rangle T(+)&= \langle \{f\}\ell T(-), fp0T(-) \rangle T(+)\quad (\ell \text{ is a morphism}) \\
&= \langle \{f\}\ell, fp0 \rangle T(+)\ell p0 = fT(-).
\end{aligned}$$

(v) Immediate from the above, as $\{f-T(-)\} = \{f-\}- = \{f\}-- = \{f\}$.

□

As noted in the section on additive bundles, from any additive bundle one can extract the group of units via an equalizer. Here, we show that under certain conditions, extracting the group of units from a tangent structure bundle gives an instance of group tangent structure (note that to make certain calculations clearer, we extract the units via a pullback rather than an equalizer). This extraction of group tangent structure also illustrates the sense in which tangent structure is “structure” rather than “property”, as a given category may support many different tangent structures.

Proposition 3.12 *Suppose (\mathbb{X}, \mathbb{T}) is a tangent category in which, for each M , the pullback diagram*

$$\begin{array}{ccc} G_r(M) & \xrightarrow{e_r} & T_{2r}(M) \\ p_r^G \downarrow & & \downarrow (+)_r \\ M & \xrightarrow{0_r} & T_r(M) \end{array}$$

exists and is preserved by each T^n . Then (\mathbb{X}, \mathbb{G}) is a tangent substructure in which the additive bundles are groups. Furthermore, if (\mathbb{X}, \mathbb{T}) is Cartesian, then so is (\mathbb{X}, \mathbb{G}) .

PROOF: Note first that G is a subfunctor of T , via the monic map $GM \xrightarrow{e} T_2M \xrightarrow{\pi_0} TM$ and that the symmetry map of T_2 induces the inverse of the group tangent bundles. The majority of tangent structure for T then restricts to the subfunctor G : the only difficulty is to show that G^n preserve the required pullbacks and that the vertical lift is universal.

To show G^n preserves the required pullbacks, note that, as G is defined as a limit of a diagram of natural transformations, G immediately preserves all limits which are preserved by the functors in that diagram. These functors are: the identity functor, T , and T_2 which, by assumption, preserve the defining diagrams of G_n . Now G^{n+1} is defined using a diagram in G^nT , G^nT_2 , and G^n , this inductively shows that G^{n+1} preserves all the limits preserved by G^n, T and T_2 .

For the universality of vertical lift, by Proposition 3.10, it suffices to show that ℓ is the required equalizer. For this, we need to know that if a map $f : X \rightarrow T^2M$ (which equalizes $T(p)$, p_T , and $T(p)p_0$) has a negation, then so does $\{f\} : X \rightarrow TM$. But this follows by lemma 2.14 (v), since we have $\{f\} + \{g\} = \{f + g\}$ and $\{00_T\} = 0$, so that the negative for f gives the negative for $\{f\}$.

Finally, if T is Cartesian, then a map $f : X \rightarrow T_G(X \times Y)$ has an additive inverse if and only if both $fT(\pi_0)$ and $fT(\pi_1)$ do, so that T_G is Cartesian. □

3.4 Lie bracket

In this section, we recall the observations of [Rosický 1984] on the Lie bracket of vector fields in this general setting. It is a very basic result of differential geometry that the vector fields on a differential manifold organise themselves into a Lie algebra: the standard proof (which is straightforward) uses the fact that vector fields correspond to derivations on the continuous functions on the manifold. The observation that Rosický made, however, is fundamentally more sophisticated as it does not rely on a representation of vector fields as derivations. Unfortunately, it is also harder⁴ to prove!

⁴We are very grateful to Rosický for providing us with the notes on his proof of the Jacobi identity [Rosický notes]. The notes were in Czech, handwritten, and some 70 pages long! Furthermore, the proof used the existence of some

Indeed, below it should be noted that we have included a proof of everything except the Jacobi identity itself. As we shall discuss further below, there is every reason to believe that this identity does hold, however, we were unable to produce an elementary proof suitable for this exposition.

In this section, we shall work over a fixed object M , and assume that its additive tangent bundle $p : TM \rightarrow M$ is a group bundle: the existence of negation is fundamental in defining the Lie bracket. To reduce notational overhead and to increase readability, given $f, g : X \rightarrow T(M)$, instead of $\langle f, g \rangle +$ we will write $f + g$, and for $f', g' : X \rightarrow T^2(M)$, instead of $\langle f', g' \rangle T(+)$ we will write $f' \oplus g'$. However, we shall continue to write negatives in postfix.

Lemma 3.13 *In a group tangent category with vector fields $w_1, w_2 \in \chi(M)$, the morphism*

$$w_1T(w_2) + w_2T(w_1)c-$$

is well-defined and equalizes $T(p)$ and $pp0$.

PROOF: To show this is well-defined, we must show that these two maps have the same value when post-composed by p_T :

$$w_1T(w_2)p_T = w_1pw_2 = w_2 = w_2T(w_1p) = w_2T(w_1)T(p) = w_2T(w_1)cp_T = w_2T(w_1)c-p_T.$$

For the second claim:

$$\begin{aligned} & (w_1T(w_2) + w_2T(w_1)c-)T(p) \\ &= w_1T(w_2)T(p) + w_2T(w_1)c-T(p) \text{ (by naturality of } +) \\ &= w_1T(w_2p) + w_2T(w_1)p_T - \\ &= w_1 + w_2pw_1 - \\ &= w_1 + w_1- = 0 \\ & (w_1T(w_2) + w_2T(w_1)c-)pp0 \\ &= w_1T(w_2)pp0 = w_1pw_2p0 = 0 \end{aligned}$$

□

We now use the characterization of the universality of vertical lift in lemma 2.12 (i) to define the Lie bracket of two vector fields.

Definition 3.14 *In a group tangent category with vector fields $w_1, w_2 \in \chi(M)$, the **Lie bracket** of w_1 and w_2 is defined to be the morphism*

$$[w_1, w_2] := \{w_1T(w_2) + w_2T(w_1)c-\} : M \rightarrow TM.$$

Here are some useful identities for the Lie bracket:

Lemma 3.15 *If $x, y \in \chi(M)$ then:*

$$(i) \ xT(0)c = xT(y)c + xT(y-)c;$$

limits which, here, we would prefer not to assume. Nonetheless the notes contained some interesting observations which are more discussed below.

- (ii) $0T(x)c = yT(x)c \oplus y-T(x)c;$
- (iii) $[x, y]\ell = (xT(y) + yT(x-)c) \oplus 0T(y-);$
- (iv) $[x, y]\ell = (xT(y) \oplus y-T(x)c) + x-T(0).$

PROOF:

- (i) $xT(0)c = xT(y + y-)c = (xT(y) \oplus xT(y-))c = xT(y)c + xT(y-)c;$
- (ii) $0T(x)c = (y + y-)T(x)c = (yT(x) + y-T(x))c = yT(x)c \oplus y-T(x)c;$
- (iii) We use the identity $\{f\}\ell = f \oplus fp0T(-)$ of Lemma 3.11 (ii) and the fact that $(xT(y) + yT(x)c-)p = xT(y)p = y$ to obtain:

$$\begin{aligned} [x, y] &= \{xT(y) + yT(x)c-\}\ell = \{xT(y) + yT(x-)c\}\ell \\ &= (xT(y) + yT(x-)c) \oplus y0T(-) = (xT(y) + yT(x-)c) \oplus 0T(y-). \end{aligned}$$

- (iv) Observe that $0T(y-) = y - 0 = y - T(0)c$ so that by (iii):

$$\begin{aligned} [x, y]\ell &= (xT(y) + yT(x-)c) \oplus y-T(0)c \\ &= (xT(y) + yT(x-)c) \oplus (y-T(x)c + y-T(x-)c) \quad (\text{by (i)}) \\ &= (xT(y) \oplus y-T(x)c) + (yT(x-)c \oplus y-T(x-)c) \quad (\text{by Lemma 2.6 (i)}) \\ &= (xT(y) \oplus y-T(x)c) + 0T(x-)c \quad (\text{by (ii)}) \\ &= (xT(y) \oplus y-T(x)c) + x-T(0). \end{aligned}$$

□

In [Rosický notes] an alternative formulation of the Lie bracket is given which relates it to the monad on the second tangent bundle:

Lemma 3.16 *In any group tangent category with vector fields $w_1, w_2 \in \chi(M)$, the following diagram commutes:*

$$\begin{array}{ccccccc} M & \xrightarrow{w_1} & T(M) & \xrightarrow{T(w_2)} & T^2(M) & \xrightarrow{T^2(w_1)} & T^3(M) & \xrightarrow{T^3(w_2)} & T^4M \\ & & \downarrow [w_1, w_2] & & & & & & \downarrow -T(-) \\ & & T(M) & \xrightarrow{\ell} & T^2(M) & & & & T^4(M) \\ & & & & & & & & \downarrow \mu_2 \\ & & & & & & & & T^2(M) \end{array}$$

It is useful in this lemma to use the unwound form of μ_2 given in Lemma 3.8, $\mu_2 = (T^2(pp) \oplus T(pp)) + (pT^2(p)c \oplus pp)$ and to note some of its properties. In particular, by a straightforward calculation, when x and y are vector fields, we have

$$xT(y)T^2(y-T(x-)c)\mu_2 = 0T(0) = y-T(x-)cT^2(xT(y))\mu_2.$$

Thus, this operation on double vector fields of the form $xT(y)$ and their flips has an inverse. The operation μ_2 on vector field pairs, however, is not commutative.

PROOF: We have the following calculation:

$$\begin{aligned}
& w_1T(w_2T(w_1T(w_2))) - T(-)\mu_2 \\
&= w_1 - T(w_2 - T(w_1T(w_2)))\mu_2 \\
&= w_1 - T(w_2 - T(w_1T(w_2)))(T^2(pp) \oplus T(pp)) + (pT^2(p)c \oplus pp) \\
&= (w_1 - T(w_2 - \oplus w_1 - T(w_2))) + (w_2 - T(w_1)c \oplus w_1T(w_2)) \\
&= (w_1 - T(0) + (w_2 - T(w_1)c \oplus w_1T(w_2))) \\
&= \{w_1T(w_2) + w_2T(w_1)c-\}\ell \quad (\text{by Lemma 3.15 (iv)}) \\
&= [w_1, w_2]\ell
\end{aligned}$$

□

We are now in a position to describe the fundamental properties of the Lie bracket. While we do prove the first three identities, recall that we shall not present a proof of the Jacobi identity:

$$[w_1, [w_2, w_3]] + [w_3, [w_1, w_2]] + [w_2, [w_3, w_1]] = 0 \quad (\text{Jacobi identity})$$

Rosický did send to us his proof [Rosický notes] of this identity, however, his proof is altogether too long to be included here, and, in addition, uses some limits which we would prefer to avoid. We hope to return to this identity at a later date to provide not only an elementary proof but also to pursue some of its consequences.

Theorem 3.17 [Rosický 1984] *In any group tangent category with vector fields $w_1, w_2, w_3 \in \chi(M)$:*

- (i) $[w_1, w_2]$ is a vector field;
- (ii) $[w_1 + w_2, w_3] = [w_1, w_3] + [w_2, w_3]$;
- (iii) $[w_1, w_2]- = [w_2, w_1]$;

PROOF:

(i) We have:

$$\begin{aligned}
[w_1, w_2]p &= \{w_1T(w_2) + w_2T(w_1)c-\}p \\
&= (w_1T(w_2) + w_2T(w_1)c-)pp \\
&= w_1T(w_2)pp = w_1pw_2p = 1
\end{aligned}$$

(ii) Consider

$$\begin{aligned}
[w_1 + w_2, w_3] &= \{(w_1 + w_2)T(w_3) + w_3T(w_1 + w_2)c-\} \\
&= \{(w_1T(w_3) + w_2T(w_3)) + (w_3T(w_1)c + w_3T(w_2)c)-\} \\
&= \{(w_1T(w_3) + w_3T(w_1)c-) + (w_2T(w_3) + w_3T(w_1)c-)\} \\
&= \{(w_1T(w_3) + w_3T(w_1)c-)\} + \{(w_2T(w_3) + w_3T(w_1)c-)\} \quad \text{by 2.14 (v)} \\
&= [w_1, w_3] + [w_2, w_3]
\end{aligned}$$

(iii) As ℓ is monic $[w_1, w_2] = [w_2, w_1]-$ if and only if $[w_1, w_2]\ell = [w_2, w_1]-\ell$. We then have the following argument:

$$\begin{aligned}
[w_1, w_2]\ell &= (w_1T(w_2) \oplus w_2-T(w_1)c) + w_1-0c \quad (\text{by lemma 3.15 (iv)}) \\
&= ((w_1T(w_2)c + w_2-T(w_1)) \oplus w_1-0)c \quad (c \text{ flips } + \text{ and } \oplus) \\
&= ((w_1T(w_2)c + w_2-T(w_1)) \oplus w_10T(-))c \\
&= \{w_1T(w_2)c + w_2T(w_1)-\}\ell c \quad (\text{by lemma 3.15 (iii)}) \\
&= \{w_1T(w_2)c + w_2T(w_1)-\}\ell \\
&= \{w_2T(w_1) + w_1T(w_2)c-\}-\ell \quad (\text{by Lemma 3.11 (iii)}) \\
&= [w_2, w_1]-\ell
\end{aligned}$$

□

In [Rosický 1984] it is noted that this abstract definition of the Lie bracket does correspond to the usual notion for categories of smooth manifolds, and to the definition in synthetic differential geometry (SDG): for a discussion of SDG and of representable tangent structure see Section 5 below.

4 Cartesian differential categories and tangent structure

As discussed in the introduction, Cartesian differential categories were defined in [Blute et al. 2008] and capture several different notions of differentiation: ordinary calculus, differentials in linear logic [Ehrhard and Regnier 2003], and differentials in combinatorics [Bergeron et al. 1997]. Our goal is not only to show that each Cartesian differential category gives an example of tangent structure, but also that “differential objects” in a tangent category provide an equivalent way to view Cartesian differential categories. As a result, we will show that Cartesian differential categories are in an adjoint relationship with Cartesian tangent categories. We shall also prove that an alternative way to extract the differential objects of a category with tangent structure is to consider its tangent spaces. These results show that Cartesian differential categories have a significant role to play in the study of tangent structure.

4.1 Cartesian differential categories

One way to view the axiomatization of Cartesian differential categories is as an abstraction of the Jacobian of a smooth map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. One ordinarily thinks of the Jacobian as a smooth map

$$J(f) : \mathbb{R}^n \rightarrow \text{Lin}(\mathbb{R}^n, \mathbb{R}^m).$$

Uncurrying, this means the Jacobian can also be seen as a smooth map

$$J(f) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$$

which is linear in its first variable. A Cartesian differential category asks for a operation of this type, satisfying the axioms described below. To express the axioms, one needs the ability to add maps. However, requiring the category to be enriched in commutative monoids would be too strong,

as the central example (smooth maps between Cartesian spaces) is not. One instead works with Cartesian left additive categories, in which addition is only preserved by composition on the left.⁵ The exact definition can be found in [Blute et al. 2008].

Definition 4.1 A *Cartesian differential category* is a Cartesian left additive category with an operation

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow{D(f)} Y}$$

(called “differentiation”) such that

[CD.1] $D(f + g) = D(f) + D(g)$ and $D(0) = 0$;

[CD.2] $\langle a + b, c \rangle D(f) = \langle a, c \rangle D(f) + \langle b, c \rangle D(f)$ and $\langle 0, a \rangle D(f) = 0$;

[CD.3] $D(\pi_0) = \pi_0 \pi_0$, and $D(\pi_1) = \pi_0 \pi_1$;

[CD.4] $D(\langle f, g \rangle) = \langle D(f), D(g) \rangle$;

[CD.5] $D(fg) = \langle D(f), \pi_1 f \rangle D(g)$;

[CD.6] $\langle \langle a, 0 \rangle, \langle c, d \rangle \rangle D(D(f)) = \langle a, d \rangle D(f)$;

[CD.7] $\langle \langle 0, b \rangle, \langle c, d \rangle \rangle D(D(f)) = \langle \langle 0, c \rangle, \langle b, d \rangle \rangle D(D(f))$;

It may be helpful to give the reader an intuition for these axioms. [CD.1] says that differentiation preserves addition, [CD.2] that the derivative is additive in its first variable. [CD.3] and [CD.4] demand that differentiation is compatible with the product structure of the category. [CD.5] is the chain rule. [CD.6] is a formulation of the fact that the derivative is linear in its first variable, and [CD.7] represents the symmetry of second partial derivatives.

We begin with an alternative version of these axioms which is more appropriate for tangent structure.

Proposition 4.2 The axioms for a Cartesian differential category are equivalently given by replacing [CD.6] and [CD.7] with the following axioms:

- [CD.6'] $\langle \langle a, 0 \rangle, \langle 0, d \rangle \rangle D(D(f)) = \langle a, d \rangle D(f)$;
- [CD.7'] $\langle \langle a, b \rangle, \langle c, d \rangle \rangle D(D(f)) = \langle \langle a, b \rangle, \langle c, d \rangle \rangle D(D(f))$.

PROOF: Assume that D satisfies the usual set of axioms. Clearly, it then satisfies [CD.6'], by setting $c = 0$. For [CD.7'], consider:

$$\begin{aligned} & \langle \langle a, b \rangle, \langle c, d \rangle \rangle D^2 f \\ &= \langle \langle a, 0 \rangle + \langle 0, b \rangle, \langle c, d \rangle \rangle D^2 f \\ &= \langle \langle a, 0 \rangle, \langle c, d \rangle \rangle D^2 f + \langle \langle 0, b \rangle, \langle c, d \rangle \rangle D^2 f \text{ by [CD.2]}, \\ &= \langle a, d \rangle D^2 f + \langle \langle 0, c \rangle, \langle b, d \rangle \rangle D^2 f \text{ by [CD.6] and [CD.7]}, \\ &= \langle a, d \rangle D^2 f + \langle \langle 0, c \rangle, \langle b, d \rangle \rangle D^2 f \text{ by [CD.6] and [CD.7]}, \\ &= \langle \langle a, 0 \rangle, \langle b, d \rangle \rangle D^2 f + \langle \langle 0, c \rangle, \langle b, d \rangle \rangle D^2 f \text{ by [CD.6] again}, \\ &= \langle \langle a, c \rangle, \langle b, d \rangle \rangle D^2 f \text{ by [CD.2]}. \end{aligned}$$

⁵These are, in fact, examples of so called *skew* enriched categories following [Street 2012].

as required.

Now assume that D satisfies the alternate set of axioms, with [CD.6] and [CD.7] replaced with [CD.6'] and [CD.7']. Clearly, it then satisfies [CD.7], by setting $a = 0$. To show that it satisfies [CD.6], consider:

$$\begin{aligned}
& \langle \langle a, 0 \rangle, \langle b, d \rangle \rangle D^2 f \\
&= \langle \langle a, b \rangle, \langle 0, d \rangle \rangle D^2 f \text{ (by [CD.7'])} \\
&= \langle \langle a, 0 \rangle, \langle 0, d \rangle \rangle D^2 f + \langle \langle 0, b \rangle, \langle 0, d \rangle \rangle D^2 f \text{ (by [CD.2])} \\
&= \langle a, d \rangle Df + \langle \langle 0, 0 \rangle, \langle b, d \rangle \rangle D^2 f \text{ (by [CD.6'] and [CD.7'])} \\
&= \langle a, d \rangle Df + 0 \text{ (by [CD.2])} \\
&= \langle a, d \rangle Df
\end{aligned}$$

as required. □

As we shall see, any Cartesian differential category has tangent structure, with [CD.6'] giving the naturality of a vertical lift map, and [CD.7'] giving the naturality of a canonical flip.

We recall a number of examples of Cartesian differential categories. The standard example is of course:

Example 4.3 *Smooth functions defined on the Cartesian spaces \mathcal{R}^n forms a Cartesian differential category.*

From [Blute et al. 2012], we also have:

Example 4.4 *The category of convenient vector spaces and smooth maps between them is a Cartesian differential category.*

In [Cockett and Seely 2011], the authors prove a surprising result: there is a comonad $\mathbf{Fa}\hat{a}$ on Cartesian left additive categories whose coalgebras are Cartesian differential categories. In particular, this means that any Cartesian left additive category has an associated Cartesian differential category:

Example 4.5 *If \mathbb{X} is a Cartesian left additive category, $\mathbf{Fa}\hat{a}(\mathbb{X})$ is a Cartesian differential category.*

One can check that for any Cartesian left additive category, defining the differential of f to be $\pi_0 f$ satisfies all axioms with the exception of [CD.2]. For this, we would need $(a + b)f = af + bf$ and $0f = 0$ for all a, b . Of course, this is true by definition if f is additive. Thus, if all maps in \mathbb{X} are additive (as in the case of the category of commutative monoids or commutative rings), then $D(f) = \pi_0 f$ does define a differential.

Example 4.6 *If \mathbb{X} is an additive Cartesian category, then $D(f) = \pi_0 f$ gives \mathbb{X} the structure of a Cartesian differential category.*

4.2 Tangent structure of a Cartesian differential category

We now turn to showing that each Cartesian differential category has a canonical tangent structure associated to it.

Proposition 4.7 *Any Cartesian differential category has a “differential” Cartesian tangent structure given by:*

$$TM := M \times M, Tf := \langle Df, \pi_1 f \rangle$$

with:

- $p := \pi_1$;
- $T_n(M) := M \times M \dots \times M$ ($n + 1$ times);
- $+\langle x_1, x_2, x_3 \rangle := \langle x_1 + x_2, x_3 \rangle$, $0(x_1) := \langle 0, x_1 \rangle$;
- $\ell(\langle x_1, x_2 \rangle) := \langle \langle x_1, 0 \rangle, \langle 0, x_2 \rangle \rangle$;
- $c(\langle \langle x_1, x_2 \rangle, \langle x_3, x_4 \rangle \rangle) := \langle \langle x_1, x_3 \rangle, \langle x_2, x_4 \rangle \rangle$.

PROOF: That T is a functor follows from **[CD.5]**:

$$T(f)T(g) = \langle Df, \pi_1 f \rangle \langle Dg, \pi_1 g \rangle = \langle D(fg), \pi_1 fg \rangle = T(fg) \text{ and } T(1) = \langle D(1), \pi_1 \rangle = \langle \pi_0, \pi_1 \rangle = 1.$$

For the additive bundle structure, it is clear that $T_n(M) := M \times M \dots \times M$ ($n + 1$ times) is the pullback of n copies of $p : TM \rightarrow M$. It is also clear that $+p = \pi_0 p = \pi_1 p$ and $0p = 1$, and the additive is associative, commutative and unital since addition of maps in a left additive category is associative, commutative and unital.

We also have

$$\langle x_1, x_2, x_3 \rangle v = \langle x_1, 0, x_2, x_3 \rangle$$

and it is then clear that v is an equalizer of $T(p)$ and $T(p)p0$.

As noted in the notes after the definition of tangent structure, asking that each $T(f) = \langle Df, \pi_1 f \rangle$ is an additive bundle morphism is equivalent to asking that each of p , $+$, and 0 be natural.

That π_1 is natural is immediate:

$$T(f)\pi_1 = \langle Df, \pi_1 f \rangle \pi_1 = \pi_1 f.$$

$+$ is natural by **[CD.2]**:

$$\begin{aligned} & \langle x_1, \langle x_2, x_3 \rangle \rangle T_2(f)(+Y) \\ &= \langle \langle x_1, x_3 \rangle Df, \langle \langle x_2, x_3 \rangle Df, \pi_1 \pi_1 x_3 \rangle \rangle (+x) \\ &= \langle \langle x_1, x_3 \rangle Df + \langle x_2, x_3 \rangle Df, \pi_1 \pi_1 x_3 \rangle \\ &= \langle \langle x_1 + x_2, x_3 \rangle Df, \pi_1 \pi_1 x_3 \rangle \text{ by [CD.2]}, \\ &= \langle x_1 + x_2, \pi_1 \pi_1 x_3 \rangle (Tf) \\ &= \langle x_1, \langle x_2, x_3 \rangle \rangle (+X)(Tf) \end{aligned}$$

as required. The naturality of $0 : X \xrightarrow{\langle 0, 1 \rangle} X \times X$ similarly follows by the other part of **[CD.2]**.

Obviously, T preserves products and the pullbacks defining the T_n 's. That it preserves pairings follows from [CD.3]:

$$\begin{aligned}
T(\langle f, g \rangle) &= \langle D(\langle f, g \rangle), \pi_1 \langle f, g \rangle \rangle \\
&= \langle \langle Df, Dg \rangle, \langle \pi_1 f, \pi_1 g \rangle \rangle \text{ by [CD.3],} \\
&= s \langle \langle Df, \pi_1 f \rangle, \langle Dg, \pi_1 g \rangle \rangle \\
&= s \langle Tf, Tg \rangle
\end{aligned}$$

(where s is the map that switches the two interior terms). Similarly, preservation of the projections follows from [CD.4]. The other diagrams for Cartesian tangent structure are automatic, with the exception of the preservation of the addition axioms, which follows by the axioms for a Cartesian left additive category.

The rest of the proof will involve calculations on maps whose domains are either T^2 or T_2 . To make these calculations easier to follow, we will show that they are true when composing with an arbitrary map into T^2 or T_2 , so that rather than dealing with projections of projections, we are dealing with maps in the product spaces.

The vertical lift is natural by [CD.6] and [CD.2]:

$$\begin{aligned}
&\langle x_1, x_2 \rangle (\ell_X)(T^2 f) \\
&= \langle \langle x_1, 0 \rangle, \langle 0, x_2 \rangle \rangle (T^2 f) \\
&= \langle \langle x_1, 0 \rangle, \langle 0, x_2 \rangle \rangle \langle \langle D^2 f, \langle \pi_0 \pi_1, \pi_1 \pi_1 \rangle Df \rangle, \pi_1 \langle Df, \pi_1 f \rangle \rangle \\
&= \langle \langle \langle x_1, x_2 \rangle Df, 0 \rangle, \langle 0, x_2 f \rangle \rangle \text{ by [CD.6] in the first variable, and [CD.2] in the second and third,} \\
&= \langle \langle x_1, x_2 \rangle Df, x_2 f \rangle \ell_Y = \langle x_1, x_2 \rangle \langle Df, \pi_1 f \rangle \ell_Y = \langle x_1, x_2 \rangle T(f)(\ell_Y).
\end{aligned}$$

The canonical flip is natural by [CD.7']:

$$\begin{aligned}
&\langle \langle x_1, x_2 \rangle \langle x_3, x_4 \rangle \rangle (T^2 f)(c_Y) \\
&= \langle \langle x_1, x_2 \rangle \langle x_3, x_4 \rangle \rangle \langle \langle D^2 f, \langle \pi_0 \pi_1, \pi_1 \pi_1 \rangle Df \rangle, \pi_1 \langle Df, \pi_1 f \rangle \rangle (c_Y) \\
&= \langle \langle \langle \langle x_1, x_3 \rangle \langle x_2, x_4 \rangle \rangle D^2 f, \langle x_2, x_4 \rangle Df \rangle, \langle \langle x_3, x_4 \rangle Df, x_4 f \rangle \rangle (c_Y) \text{ by [CD.7']} \\
&= \langle \langle \langle \langle x_1, x_3 \rangle \langle x_2, x_4 \rangle \rangle D^2 f, \langle x_3, x_4 \rangle Df \rangle, \langle \langle x_2, x_4 \rangle Df, x_4 f \rangle \rangle \\
&= \langle \langle x_1, x_3 \rangle, \langle x_2, x_4 \rangle \rangle T^2 f = \langle \langle x_1, x_2 \rangle, \langle x_3, x_4 \rangle \rangle (c_X)(T^2 f).
\end{aligned}$$

To show that these maps are additive bundle morphisms, we need to determine $T(+)$ and $T(0)$. By [CD.1] and [CD.3], $T(+_X) = T(\langle \pi_0 + \pi_1 \pi_0, \pi_1 \pi_1 \rangle)$ is given by

$$\langle \langle \pi_0 \pi_0 + \pi_0 \pi_1 \pi_0, \pi_0 \pi_1 \pi_1 \rangle, \langle \pi_1 \pi_0 + \pi_1 \pi_0 \pi_1, \pi_1 \pi_1 \pi_1 \rangle \rangle$$

while

$$T(0) = T(\langle 0, 1 \rangle) = \langle \langle D0, D1 \rangle, \pi_1 \rangle = \langle \langle 0, 1 \rangle, \pi_1 \rangle$$

Now, the map $\langle \pi_0 c, \pi_1 c \rangle$ sends

$$\langle \langle x_1, \langle x_2, x_3 \rangle \rangle, \langle x_4, \langle x_5, x_6 \rangle \rangle \rangle \mapsto \langle \langle x_1, x_4 \rangle, \langle \langle x_2, x_5 \rangle, \langle x_3, x_6 \rangle \rangle \rangle$$

We can then show that c preserves addition:

$$\begin{aligned}
& \langle \langle x_1, \langle x_2, x_3 \rangle \rangle, \langle x_4, \langle x_5, x_6 \rangle \rangle \rangle \langle \pi_1 x_3, \pi_2 x_3 \rangle (+_{TX}) \\
&= \langle \langle x_1, x_4 \rangle, \langle \langle x_2, x_5 \rangle, \langle x_3, x_6 \rangle \rangle \rangle (+_{TX}) \\
&= \langle \langle x_1 + x_2, x_4 + x_5 \rangle, \langle x_3, x_6 \rangle \rangle \\
&= \langle \langle x_1 + x_2, x_3 \rangle, \langle x_4 + x_5, x_6 \rangle \rangle (c_X) \\
&= \langle \langle x_1, \langle x_2, x_3 \rangle \rangle, \langle x_4, \langle x_5, x_6 \rangle \rangle \rangle T(+_X)(c_X).
\end{aligned}$$

Preservation of 0 similarly uses the equation $T(0) = \langle \langle 0, 1 \rangle, \pi_1 \rangle$, and the calculations to show ℓ preserves addition are similar.

Finally, the coherence axioms for ℓ and c are a simple exercise. \square

Note that example 1 of [Rosický 1984] is an instance of this proposition, applied to example 4.6 of this paper.

4.3 Differential structure in a tangent category

Not only are Cartesian differential categories examples of tangent structure, but one can also identify precisely which examples of tangent structure give Cartesian differential categories. We begin by abstracting the structure that occurs on objects in a Cartesian differential category.

Definition 4.8 *For an object A in a Cartesian tangent category, **differential structure on A** consists of a commutative monoid structure $\sigma : A \times A \rightarrow A$, $\zeta : 1 \rightarrow A$ on A together with a map $\hat{p} : TA \rightarrow A$ such that*

- $A \xleftarrow{\hat{p}} TA \xrightarrow{p} A$ is a product diagram;
- The diagrams

$$\begin{array}{ccc}
A & \xrightarrow{0_A} & TA \\
\downarrow ! & & \downarrow \hat{p} \\
1 & \xrightarrow{\zeta} & A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
T_2 A & \xrightarrow{+_A} & TA \\
\downarrow \langle \pi_0 \hat{p}, \pi_1 \hat{p} \rangle & & \downarrow \hat{p} \\
A \times A & \xrightarrow{\sigma} & A
\end{array}$$

commute;

- $+$ is “linear”, in the sense that

$$\begin{array}{ccccc}
T(A \times A) & \xrightarrow{\langle T(\pi_0), T(\pi_1) \rangle} & TA \times TA & \xrightarrow{\hat{p} \times \hat{p}} & A \times A \\
\downarrow T(\sigma) & & & & \downarrow \sigma \\
TA & \xrightarrow{\hat{p}} & & & A
\end{array}$$

commutes.

An object with a specified differential structure is a **differential object**.

It may seem odd to not include a linearity axiom for ζ , but the uniqueness of maps to a terminal object allows one to show such an axiom is automatic:

Lemma 4.9 *If $(A, \widehat{p}_A, \sigma_A, \zeta_A)$ is differential structure on A , then*

$$\begin{array}{ccc} T(1) & \xrightarrow{T(\zeta)} & TM \\ \downarrow !_{T(1)} & & \downarrow \hat{p} \\ 1 & \xrightarrow{\zeta} & M \end{array}$$

commutes.

PROOF: Consider the calculation:

$$\begin{aligned} !_{T(1)}\zeta &= p_1\zeta!_A\zeta \text{ (uniqueness of terminal objects)} \\ &= p_1\zeta 0_A \hat{p} \text{ (differential structure axiom)} \\ &= p_1 0_1 T(\zeta) \hat{p} \text{ (naturality of 0)} \\ &= T(\zeta) \hat{p} \text{ (} T(1) \text{ is terminal)}. \end{aligned}$$

□

Note that in the tangent structure of a Cartesian differential category, each object canonically has the structure of a differential object, with $\hat{p} = \pi_0$, $\sigma = \pi_0 + \pi_1$, and $\zeta = 0$. Conversely, we wish to show if we start with an arbitrary Cartesian tangent category, then the full subcategory category of differential objects is a Cartesian differential category.

The first thing we need to show is that the product of two differential object is a differential object.

Proposition 4.10 *If $(A, \widehat{p}_A, \sigma_A, \zeta_A)$ and $(B, \widehat{p}_B, \sigma_B, \zeta_B)$ are differential objects, then $A \times B$ is also, with:*

- $\widehat{p}_{A \times B} := \langle T(\pi_0)\widehat{p}_A, T(\pi_1)\widehat{p}_B \rangle$,
- $\sigma_{A \times B} := \langle \langle \pi_0\pi_0, \pi_1\pi_0 \rangle \sigma_A, \langle \pi_0\pi_1, \pi_1\pi_1 \rangle \sigma_B \rangle$,
- $\zeta_{A \times B} := \langle \zeta_A, \zeta_B \rangle$.

PROOF: First, note that since T preserves products and the pairs (\widehat{p}_A, p_A) , (\widehat{p}_B, p_B) are product diagrams, then

$$A \times B \xleftarrow{\langle T(\pi_0)\widehat{p}_A, T(\pi_1)\widehat{p}_B \rangle} T(A \times B) \xrightarrow{\langle T(\pi_1)p_A, T(\pi_2)p_B \rangle} A \times B$$

is also a product diagram. By naturality,

$$\langle T(\pi_0)p_A, T(\pi_1)p_B \rangle = \langle p_{A \times B}\pi_0, p_{A \times B}\pi_1 \rangle = p_{A \times B},$$

so the pair $(\widehat{p}_{A \times B}, p_{A \times B})$ is a product diagram.

Next, we need to check that the defined addition and zero are compatible with this structure. For the zero, we need the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{0_{A \times B}} & T(A \times B) \\ \downarrow ! & & \downarrow \widehat{p_{A \times B}} \\ \mathbf{1} & \xrightarrow{\langle \zeta_A, \zeta_B \rangle} & A \times B \end{array}$$

to commute, for which we have:

$$\begin{aligned} 0_{A \times B} \widehat{p_{A \times B}} &= 0_{A \times B} \langle T(\pi_0) \widehat{p_A}, T(\pi_1) \widehat{p_B} \rangle \\ &= \langle 0_{A \times B} T(\pi_0) \widehat{p_A}, 0_{A \times B} T(\pi_1) \widehat{p_B} \rangle \\ &= \langle \pi_0 0_A \widehat{p_A}, \pi_1 0_B \widehat{p_B} \rangle \quad (T \text{ is Cartesian}) \\ &= \langle \pi_0 ! \zeta_A, \pi_1 ! \zeta_B \rangle \quad (\widehat{p_A}, \widehat{p_B} \text{ compatible with zero}) \\ &= ! \langle \zeta_A, \zeta_B \rangle \end{aligned}$$

For the addition, we need

$$\begin{array}{ccc} T_2(A \times B) & \xrightarrow{+_{A \times B}} & T(A \times B) \\ \downarrow \langle \pi_0 \widehat{p_{A \times B}}, \pi_1 \widehat{p_{A \times B}} \rangle & & \downarrow \widehat{p_{A \times B}} \\ A \times B \times A \times B & \xrightarrow{\sigma_{A \times B}} & A \times B \end{array}$$

to commute, for which we have:

$$\begin{aligned} &\langle \pi_0 \widehat{p_{A \times B}}, \pi_1 \widehat{p_{A \times B}} \rangle \sigma_{A \times B} \\ &= \langle \pi_0 \langle T(\pi_A) \widehat{p_A}, T(\pi_2) \widehat{p_B} \rangle, \pi_1 \langle T(\pi_0) \widehat{p_A}, T(\pi_1) \widehat{p_B} \rangle \rangle \langle \langle \pi_0 \pi_0, \pi_1 \pi_0 \rangle \sigma_A, \langle \pi_0 \pi_1, \pi_1 \pi_1 \rangle \sigma_B \rangle \\ &= \langle \langle \pi_0 T(\pi_0) \widehat{p_A}, \pi_1 T(\pi_0) \widehat{p_A} \rangle \sigma_A, \langle \pi_0 T(\pi_B) \widehat{p_B}, \pi_1 T(\pi_1) \widehat{p_B} \rangle \sigma_B \rangle \\ &= \langle \langle \pi_0 T(\pi_0), \pi_1 T(\pi_0) \rangle +_A \widehat{p_A}, \langle \pi_0 T(\pi_1), \pi_1 T(\pi_2) \rangle +_B \widehat{p_B} \rangle \quad (\widehat{p_A}, \widehat{p_B} \text{ compatible with addition}) \\ &= \langle T_2(\pi_0) +_A \widehat{p_A}, T_2(\pi_1) +_B \widehat{p_B} \rangle = \langle +_{A \times B} T(\pi_0) \widehat{p_A}, +_{A \times B} T(\pi_1) \widehat{p_B} \rangle \quad (T \text{ is Cartesian}) \\ &= +_{A \times B} \langle T(\pi_0) \widehat{p_A}, T(\pi_1) \widehat{p_B} \rangle = +_{A \times B} \widehat{p_{A \times B}}. \end{aligned}$$

Checking the linearity of $+$ similarly uses the fact that T is Cartesian. □

We can now give one of the main results of this section.

Theorem 4.11 *Suppose (\mathbb{X}, \mathbb{T}) is a Cartesian tangent category. Let $\text{Diff}(\mathbb{X}, \mathbb{T})$ denote the category whose objects are differential objects, with a map from $(A, \widehat{p_A}, \sigma_A, \zeta_A)$ to $(B, \widehat{p_B}, \sigma_B, \zeta_B)$ simply consisting of a map $f : A \rightarrow B$. Then:*

- (i) $\text{Diff}(\mathbb{X}, \mathbb{T})$ is a Cartesian left additive category;
- (ii) $\text{Diff}(\mathbb{X}, \mathbb{T})$ is a Cartesian differential category, with $D(f)$ given by

$$A \times A \xrightarrow{\langle \pi_0, \pi_1 \rangle} TA \xrightarrow{T(f)} TB \xrightarrow{\widehat{p_B}} B.$$

PROOF:

- (i) We first need the category to have specified products. We define the product of $(A, \widehat{p}_A, \sigma_A, \zeta_A)$ and $(B, \widehat{p}_B, \sigma_B, \zeta_B)$ to be as in proposition 4.10. Then, by definition, each object has a monoidal structure, and this structure is compatible with products by construction. Thus, by proposition 1.2.2 of [Blute et al. 2008], \mathbb{X} has the structure of a Cartesian left additive category, with

$$f + g := \langle f, g \rangle \sigma \text{ and } 0 := !\zeta.$$

- (ii) We will begin by establishing [CD.1]. We first show that the linearity axiom for σ implies that D is linear, in the sense that $D(\sigma) = \pi_0 \sigma$. Indeed,

$$\begin{aligned} D(\sigma) &= \langle \pi_0, \pi_1 \rangle T(\sigma) \hat{p} \\ &= \langle \pi_0, \pi_1 \rangle \langle T(\pi_0) \hat{p}, T(\pi_1) \hat{p} \rangle \sigma \text{ (linearity axiom for } \sigma) \\ &= \langle \pi_0, \pi_1 \rangle \widehat{p_{A \times A}} \sigma \\ &= \pi_0 \sigma. \end{aligned}$$

Then

$$\begin{aligned} D(f + g) &= D(\langle f, g \rangle +) = \langle D(\langle f, g \rangle), \pi_1 \langle f, g \rangle \rangle D(+) \\ &= \langle \langle D(f), D(g) \rangle, \pi_1 \langle f, g \rangle \rangle \pi_0 + = \langle D(f), D(g) \rangle + = D(f) + D(g), \end{aligned}$$

as required. Similarly, by lemma 4.9, we have

$$D(!\zeta) = \langle \pi_0, \pi_1 \rangle T(!\zeta) \hat{p} = \langle \pi_0, \pi_1 \rangle !\zeta = !\zeta$$

and so $D(0) = 0$.

From now on, to reduce the repeated use of isomorphisms, we will assume we are working with objects such that $TA = A \times A$, $\widehat{p}_A = \pi_0$, and $p_A = \pi_1$.

We begin by determining the form of the natural transformations $+$, 0 , ℓ , and c . Since $p = \pi_1$, $\langle x_1, x_2, x_3, x_4 \rangle p_T = \langle x_3, x_4 \rangle$, and since T is Cartesian, we then have

$$\langle x_1, x_2, x_3, x_4 \rangle T(p) = \langle x_1, x_2, x_3, x_4 \rangle T(\pi_1) = \langle x_1, x_2, x_3, x_4 \rangle \pi_0 \pi_1 = \langle x_2, x_4 \rangle.$$

By the equations for differential structure, we then get $0_A(x) = \langle 0, x \rangle$ and $+_A \langle x_1, x_2, x_3 \rangle = \langle x_1 + x_2, x_3 \rangle$. We then get $\langle \langle x_1, x_2 \rangle \rangle 0_T = \langle 0, 0, x_1, x_2 \rangle$ and $\langle x_1, x_2 \rangle T(0) = \langle 0, x_1, 0, x_2 \rangle$ since T is Cartesian.

Then since $\ell : TM \rightarrow T^2M$ is the equalizer of $T(p)$, p_T , and $T(p)p_0$ (Proposition 3.10), we have $\langle x_1, x_4 \rangle \ell = \langle x_1, 0, 0, x_4 \rangle$.

Now write $\langle x_1, x_2, x_3, x_4 \rangle c$ as $\langle c_1, c_2, c_3, c_4 \rangle$. Then the axioms $cT(p) = p_T$ and $c^2 = 1$ tell us that $c_2 = x_3$ and $c_3 = x_2$. Since $lc = l$, we have $c_1(x_1, 0, 0, x_4) = x_1$. But then we have

$$\begin{aligned} c_1(x_1, x_2, x_3, x_4) &= c_1((x_1, 0, x_3, x_4) + (0, x_2, x_3, x_4)) \\ &= c_1(x_1, 0, x_3, x_4) + c_1(0, x_2, x_3, x_4) \text{ (since } c \text{ preserves addition)} \\ &= c_1(x_1, 0, x_3, x_4) \text{ (since } c \text{ preserves } 0) \\ &= c_1((x_1, 0, 0, x_4) + (0, 0, x_3, x_4)) \\ &= c_1(x_1, 0, 0, x_4) + c_1(0, 0, x_3, x_4) \text{ (since } c \text{ preserves addition)} \\ &= x_1 \text{ (by above and since } c \text{ preserves } 0). \end{aligned}$$

Thus we have $\langle x_1, x_2, x_3, x_4 \rangle c = \langle x_1, x_3, x_2, x_4 \rangle$.

We now turn to showing that $D(f) = T(f)\pi_0$ satisfies the required axioms for a Cartesian differential category.

By naturality of p , we can determine that:

$$T(f) = \langle T(f)\pi_0, T(f)\pi_1 \rangle = \langle Df, \pi_1 f \rangle$$

Similarly,

$$T_2(f) = \langle \langle \pi_0, \pi_1 \pi_0 \rangle D(f), \langle \pi_1 D(f), \pi_1 \pi_1 f \rangle \rangle$$

and

$$D^2(f) = T(D(f))\pi_0 = T(T(f)\pi_0)\pi_0 = T^2(f)T(\pi_0)\pi_0 = T^2(f)\langle \pi_0\pi_0, \pi_1\pi_0 \rangle\pi_0 = T^2(f)\pi_0\pi_0.$$

We begin with [CD.5]:

$$\begin{aligned} D(fg) &= T(fg)\pi_0 \\ &= T(f)T(g)\pi_0 \\ &= \langle T(f)\pi_0, T(f)\pi_1 \rangle D(g) \\ &= \langle Df, \pi_1 f \rangle D(g) \text{ by naturality of } p = \pi_1. \end{aligned}$$

Since the functor T is Cartesian, with isomorphism $s : T(M \times N) \longrightarrow TM \times TN$ given by

$$s(m_1, n_1, m_2, n_2) = (m_1, m_2, n_1, n_2),$$

we have $T(\langle f, g \rangle) = \langle Tf, Tg \rangle s$, $T(\pi_0) = s\pi_0 = \langle \pi_0\pi_0, \pi_1\pi_0 \rangle$, and $T(\pi_1) = \langle \pi_1\pi_0, \pi_1\pi_1 \rangle$. Then we get [CD.4]:

$$\begin{aligned} D(\langle f, g \rangle) &= T(\langle f, g \rangle)\pi_0 \\ &= \langle Tf, Tg \rangle s\pi_0 \\ &= \langle Tf, Tg \rangle \langle \pi_0\pi_0, \pi_1\pi_0 \rangle \\ &= \langle T(f)\pi_0, T(g)\pi_0 \rangle \\ &= \langle Df, Dg \rangle \end{aligned}$$

and for [CD.3]:

$$\begin{aligned} D(\pi_0) &= T(\pi_0)\pi_0 = \langle \pi_0\pi_0, \pi_1\pi_0 \rangle\pi_0 = \pi_0\pi_0, \\ D(\pi_1) &= T(\pi_1)\pi_0 = \langle \pi_0\pi_1, \pi_1\pi_1 \rangle\pi_0 = \pi_0\pi_1, \\ D(1) &= T(1)\pi_0 = \pi_0 \end{aligned}$$

For [CD.2], we have

$$\begin{aligned} \langle x_1, x_3 \rangle D(f) + \langle x_2, x_3 \rangle D(f) &= \langle \langle x_1, x_3 \rangle D(f) + \langle x_2, x_3 \rangle D(f), x_3 f \rangle \pi_0 \\ &= \langle \langle x_1, x_3 \rangle D(f), \langle x_2, x_3 \rangle D(f), x_3 f \rangle (+)\pi_0 \\ &= \langle x_1, x_2, x_3 \rangle T_2(f)(+)\pi_0 \\ &= \langle x_1, x_2, x_3 \rangle (+)T(f)\pi_0 \\ &= \langle x_1 + x_2, x_3 \rangle D(f) \end{aligned}$$

and the 0 axiom is similar. For [CD.6'] we use naturality of ℓ in the following calculation:

$$\begin{aligned}
\langle x_1, 0, 0, x_2 \rangle D^2(f) &= \langle x, 0, 0x_2 \rangle T(T(f)\pi_0)\pi_0 \\
&= \langle x_1, x_2 \rangle \ell T^2(f)T(\pi_0)\pi_0 \\
&= \langle x_1, x_2 \rangle T(f)\ell T(\pi_0)\pi_0 \\
&= \langle x_1, x_2 \rangle T(f)\pi_0 = \langle x_1, x_2 \rangle D(f).
\end{aligned}$$

For [CD.7'], using the naturality of c , we have:

$$\begin{aligned}
\langle \langle x_1, x_2 \rangle \langle x_3, x_4 \rangle \rangle D^2 f &= \langle \langle x_1, x_2 \rangle \langle x_3, x_4 \rangle \rangle T(T(f)\pi_0)\pi_0 \\
&= \langle \langle x_1, x_2 \rangle \langle x_3, x_4 \rangle \rangle T(T(f))cT(\pi_0)\pi_0 \\
&= \langle \langle x_1, x_2 \rangle \langle x_3, x_4 \rangle \rangle cT(T(f))T(\pi_0)\pi_0 \\
&= \langle \langle x_1, x_3 \rangle \langle x_2, x_4 \rangle \rangle T(T(f))T(\pi_0)\pi_0 \\
&= \langle \langle x_1, x_3 \rangle \langle x_2, x_4 \rangle \rangle D^2(f).
\end{aligned}$$

□

It may seem slightly odd to consider all maps between differential objects, but restricting to the maps which preserve addition and/or those which preserve \hat{p} would be altogether too strong. In the first case, we would only be looking at maps which preserve addition; in the second, we would only be considering maps which are linear (in the differential sense of [Blute et al. 2008]).

Combining the results of the previous two sections, one can express the relationship between tangent structure and Cartesian differential categories by an adjoint. Let **cartTanStrong** denote the category of Cartesian tangent categories (definition 2.8) and strong morphisms (definition 2.7). It is straightforward to check that any strong morphism lifts differential structure; hence the above result gives a functor $\text{Diff} : \mathbf{cartTanStrong} \rightarrow \mathbf{cartDiff}$. Similarly, if we let **cartDiff** denote the category whose objects are cartesian differential categories, and whose maps are functors which preserve products, the additive structure, and the differential (exactly), then proposition 4.7 gives us a functor $\text{Tan} : \mathbf{cartDiff} \rightarrow \mathbf{cartTanStrong}$. We then have:

Theorem 4.12 $\text{Tan} \dashv \text{Diff}$; that is, if we have $(\mathbb{X}, D) \in \mathbf{cartDiff}$ and $(\mathbb{X}', T) \in \mathbf{cartTanStrong}$, then there is a natural isomorphism

$$\frac{\text{Tan}(\mathbb{X}, D) \rightarrow (\mathbb{X}', T)}{(\mathbb{X}, D) \rightarrow \text{Diff}(\mathbb{X}', T)}.$$

PROOF: The unit of the adjunction

$$F : (\mathbb{X}, D) \rightarrow \text{Diff}(\text{Tan}(\mathbb{X}, D))$$

was already described after the definition of differential structure: F sends the object \mathbb{X} to its canonical differential structure $(X, \pi_0, +, 0)$ (and does nothing to the maps). This functor preserves the differential exactly since the differential in $\text{Diff}(\text{Tan}(\mathbb{X}, D))$ is given by $\langle D(f), \pi_1 f \rangle \pi_0 = D(f)$.

For the co-unit of the adjunction

$$G : \text{Tan}(\text{Diff}(\mathbb{X}', T)) \rightarrow (\mathbb{X}', T),$$

we simply send $(A, \pi_0, \zeta, \sigma)$ to A (and again do nothing to maps). Denoting the tangent functor in $\text{Tan}(\text{Diff}(\mathbb{X}', T))$ by T_D , we then need a natural isomorphism $\alpha : G(T_D(A)) \rightarrow T(GA)$; that is, a natural isomorphism $A \times A \rightarrow TA$. By the definition of a differential structure, TA is a product, with projections $\langle \widehat{p}_A, p_A \rangle$, so we take $\alpha = \langle \pi_0, \pi_1 \rangle$. For naturality of α , for a map $f : A \rightarrow B$, we need

$$\begin{array}{ccc} A \times A & \xrightarrow{\langle \pi_0, \pi_1 \rangle} & TA \\ T_D(f) \downarrow & & \downarrow T(f) \\ B \times B & \xrightarrow{\langle \pi_0, \pi_1 \rangle} & TB \end{array}$$

to commute. Since the target TB is a product, it suffices to show the maps are equal when post-composing by its projections:

$$\begin{aligned} & T_D(f) \langle \pi_0, \pi_1 \rangle \langle \widehat{p}, p \rangle \\ &= \langle Df, \pi_1 f \rangle \langle \widehat{p}, p \rangle \\ &= \langle \langle \pi_0, \pi_1 \rangle T(f) \widehat{p}, \pi_1 f \rangle \langle \widehat{p}, p \rangle \text{ (by definition of } D) \\ &= \langle \langle \pi_0, \pi_1 \rangle T(f) \widehat{p}, \langle \pi_0, \pi_1 \rangle p f \rangle \\ &= \langle \langle \pi_0, \pi_1 \rangle T(f) \widehat{p}, \langle \pi_0, \pi_1 \rangle T(f) p \rangle \text{ (by naturality)} \end{aligned}$$

Checking that α preserves the other elements of the tangent structure is similar. The triangle equalities are straightforward. \square

In the next section, we give an alternate description of differential objects, showing that they always arise as tangent spaces.

4.4 Tangent spaces and differential structure

In most standard descriptions of the tangent bundle of a smooth manifold, tangent spaces are described before the tangent bundle itself. In these descriptions, one begins by describing the tangent space of tangent vectors at each point of the space, and then glues these tangent spaces together to form the tangent bundle. In this development, however, we have begun with the tangent bundle, and will recover the tangent spaces from it. The purpose of this section is to prove that each tangent space is canonically a differential object, and, in fact, all differential objects arise as tangent spaces.

Definition 4.13 *If \mathbb{X} is a Cartesian tangent category, and M is an object with a point $a : 1 \rightarrow M$, then, if the pullback of a along p exists*

$$\begin{array}{ccc} T_a M & \xrightarrow{i_a} & TM \\ \downarrow ! & & \downarrow p \\ 1 & \xrightarrow{a} & M \end{array}$$

*and is preserved by T^n , then we will call this pullback, $T_a(M)$, the **tangent space of M at a** .*

Notice that we require not just that the pullback exists but also that it is preserved by the tangent functor. Because $T(M)$ is a commutative monoid in the slice category over M , it now

immediately follows that each tangent space $T_a M$ is a commutative monoid in \mathbb{X} itself, with unit $a0|_{i_a} = 0_1 T(a)|_{i_a} = !T(a)|_{i_a} : 1 \rightarrow T_a M$ (recall 0_1 is an isomorphism) and addition $(i_a \times i_a) + |_{i_a} : T_a(M) \times T_a(M) \rightarrow T_a(M)$.

The following is a useful preliminary to our main result.

Lemma 4.14 *If the tangent space $T_a(M)$ exists then*

$$T(T_a(M)) \xrightarrow{T(i_a)} T^2(M) \begin{array}{c} \xrightarrow{T(p)} \\ \xrightarrow{p_T p_0} \end{array} T(M)$$

commutes, giving a unique map $\{T(i_a)\} : T(T_a(M)) \rightarrow T(M)$.

PROOF: First, observe that for any point $a : 1 \rightarrow M$, by naturality of 0 , $0_1 T(a) = a0_M$, so that since T is Cartesian, $!T(a) = !a0_M$. Then we have:

$$\begin{aligned} T(i_a)p_T p_0 &= T(i_a p)p_0 \text{ (naturality of } p) \\ &= T(!a)p_0 \text{ (pullback diagram)} \\ &= !T(a)p_0 \text{ (} T \text{ is Cartesian)} \\ &= !a0 p_0 = !a0 \text{ (as above and as } 0p \text{ is the identity)} \\ &= !T(a) = T(!a) \text{ (} T \text{ is Cartesian)} \\ &= T(i_a p) \text{ (pullback diagram)} \\ &= T(i_a)T(p). \end{aligned}$$

□

We can now demonstrate that tangent spaces satisfy a form of the Kock-Lawvere axiom and thus have differential structure. As noted earlier, the proof of the result uses the universality of the vertical lift in a fundamental way.

Theorem 4.15 *In a Cartesian tangent category every tangent space, $T_a(M)$, is a differential object.*

PROOF: First observe that $T_a(M) \times T_a(M)$ is isomorphic to $T(T_a(M))$ as by assumption:

$$\begin{array}{ccc} & T(T_a(M)) & \xrightarrow{T(i_a)} & T^2(M) \\ & \downarrow T(!) & & \downarrow T(p) \\ ! & T(1) & \xrightarrow{T(a)} & T(M) \\ \downarrow & \parallel & \nearrow a_0 & \\ 1 & & & \end{array}$$

is a pullback. However we then have an isomorphism γ_a :

$$\begin{array}{ccccc}
T(T_a(M)) & & & & \\
\downarrow \gamma_a & \searrow T(i) & & & \\
T_a(M) \times T_a(M) & \xrightarrow{\langle \pi_0 i_a, \pi_1 i_a \rangle} & T_2(M) & \xrightarrow{v} & T^2(M) \\
\downarrow \pi_0 & & \downarrow \pi_0 & & \downarrow T(p) \\
T_a(M) & \xrightarrow{i_a} & T(M) & & \\
\downarrow ! & & \downarrow p & & \\
1 & \xrightarrow{a} & M & \xrightarrow{0} & T(M)
\end{array}$$

because the large square is a pullback as all of its inner regions are pullbacks: the lower left by definition of being a tangent space, the upper left because it is a projection, and the right by lemma 2.12(ii). This in turn provides a unique map $\{T(i)\}_{i_a} = \gamma_a \pi_0$:

$$\begin{array}{ccc}
T(T_a(M)) & \xrightarrow{\{T(i_a)\}} & T(M) \\
\downarrow \{T(i_a)\}_{i_a} & \searrow i_a & \downarrow p \\
1 & \xrightarrow{a} & M
\end{array}$$

Next observe $\gamma_a \pi_1 = p_{T_a(M)}$. Since these are maps into the tangent space $T_a(M)$, it suffices to check they are equal when post-composed by i_a :

$$\begin{aligned}
p_{T_a(M)} i_a &= T(i_a) p_M \quad (\text{naturality}) \\
&= \gamma_a \langle \pi_0 i_a, \pi_1 i_a \rangle v p_M \\
&= \gamma_a \langle \pi_0 i_a, \pi_1 i_a \rangle \pi_1 \quad (\text{lemma 2.11}) \\
&= \gamma_a \pi_1 i_a.
\end{aligned}$$

So, our choice of \hat{p} is $\{T(i_a)\}_{i_a}$. Now, we need to show that the zero and addition on $T_a(M)$ are compatible with the addition and zero of the tangent structure. For the zero, we need the following diagram to commute:

$$\begin{array}{ccc}
T_a(M) & \xrightarrow{0} & T(T_a M) \\
! \downarrow & & \downarrow \{T(i_a)\}_{i_a} \\
1 & \xrightarrow{0_a} & T_a M
\end{array}$$

It suffices to check they are equal when post-composed by i_a , so we need $0\{T(i_a)\} = !a0$. Thus, it suffices to show $!a0$ has the same universal property of lemma 2.12(i) as $\{0T(i_a)\}$. Thus, consider:

$$\begin{aligned}
& \langle !a0\ell, 0T(i_a)p_T0_T \rangle T(+) \\
&= \langle i_a p 0\ell, 0p i_a 0_T \rangle T(+) \text{ (pullback diagram and naturality of } p) \\
&= \langle i_a p 0T(0), i_a 0_T \rangle T(+) \text{ (} \ell \text{ is an additive bundle morphism and } 0p = 1) \\
&= \langle i_a 0_T T(p)T(0), i_a 0_T \rangle T(+) \text{ (naturality of } 0) \\
&= i_a 0_T \langle T(p)T(0), 1 \rangle T(+) \\
&= i_a 0_T \text{ (adding a zero term)} \\
&= 0T(i_a) \text{ (naturality of } 0)
\end{aligned}$$

For addition, we need the following diagram to commute:

$$\begin{array}{ccc}
T_2(T_a(M)) & \xrightarrow{+} & T(T_a(M)) \\
\langle \pi_0\{T(i_a)\}|_{i_a}, \pi_1\{T(i_a)\}|_{i_a} \rangle \downarrow & & \downarrow \{T(i_a)\}|_{i_a} \\
T_a(M) \times T_a(M) & \xrightarrow{\langle \pi_0 i_a, \pi_1 i_a \rangle + |_{i_a}} & T_a(M)
\end{array}$$

Again, it suffices to check the results are equal when post-composed by i_a , so we need

$$+\{T(i_a)\} = \langle \pi_0\{T(i_a)\}, \pi_1\{T(i_a)\} \rangle +$$

Thus, we need $\langle \pi_0\{T(i_a)\}, \pi_1\{T(i_a)\} \rangle +$ to satisfy the same universal property as $\{+T(i_a)\}$. For this we have:

$$\begin{aligned}
+T(i_a) &= T_2(i_a) +_T \text{ (naturality)} \\
&= \langle \pi_0 T(i_a), \pi_1 T(i_a) \rangle +_T \\
&= \langle \langle \pi_0\{T(i_a)\}\ell, \pi_0 T(i_a)p_T0_T \rangle T(+), \langle \pi_1\{T(i_a)\}\ell, \pi_1 T(i_a)p_T0_T \rangle T(+), \rangle +_T \text{ (defn of } \{-\}) \\
&= \langle \langle \pi_0\{T(i_a)\}\ell, \pi_1\{T(i_a)\}\ell \rangle +_T, \langle \pi_0 T(i_a)p_T0_T, \pi_1 T(i_a)p_T0_T \rangle +_T \rangle T(+) \text{ (by lem 2.6 (i))} \\
&= \langle \langle \pi_0\{T(i_a)\}\ell c, \pi_1\{T(i_a)\}\ell \rangle T(+), \pi_0 T(i_a)p_T0_T \rangle T(+) \text{ (lem 2.6 (ii), and adding zero)} \\
&= \langle \langle \pi_0\{T(i_a)\}\ell, \pi_1\{T(i_a)\}\ell \rangle T(+), \pi_0 p i_a 0_T \rangle T(+) \\
&= \langle \langle \pi_0\{T(i_a)\}, \pi_1\{T(i_a)\} \rangle + \ell, +p i_a 0_T \rangle T(+) \\
&= \langle \langle \pi_0\{T(i_a)\}, \pi_1\{T(i_a)\} \rangle + \ell, +T(i_a)p_T0_T \rangle T(+)
\end{aligned}$$

Finally, we need to show that our addition is linear, so we need

$$\begin{array}{ccc}
T(T_a(M) \times T_a(M)) & \xrightarrow{\langle T(\pi_0)\{T(i_a)\}|_{i_a}, T(\pi_1)\{T(i_a)\}|_{i_a} \rangle} & T_a(M) \times T_a(M) \\
T((i_a \times i_a) + |_{i_a}) \downarrow & & \downarrow (i_a \times i_a) + |_{i_a} \\
T(T_a(M)) & \xrightarrow{\{T(i_a)\}|_{i_a}} & T_a(M)
\end{array}$$

to commute. It suffices to check the two maps are equal when post-composed by i , so we need

$$\langle T(\pi_0)\{T(i_a)\}|_{i_a}, T(\pi_1)\{T(i_a)\}|_{i_a} \rangle (i_a \times i_a) + = T((i_a \times i_a) + |_{i_a})\{T(i_a)\}$$

which follows as:

$$\begin{aligned}
& \langle T(\pi_0)\{T(i_a)\}|_{i_a}, T(\pi_1)\{T(i_a)\}|_{i_a} \rangle (i_a \times i_a) + \\
&= \langle T(\pi_0)\{T(i_a)\}, T(\pi_1)\{T(i_a)\} \rangle + \\
&= \langle \{T(\pi_0)T(i_a)\}, \{T(\pi_1)T(i_a)\} \rangle + \\
&= \langle \{T(\pi_0 i_a)\}, \{T(\pi_1 i_a)\} \rangle + \\
&= \{ \langle T(\pi_0 i_a), T(\pi_1 i_a) \rangle T(+)\} \text{ by lemma 2.14 (iv)} \\
&= \{T(\langle \pi_0 i_a, \pi_1 i_a \rangle +)\} \text{ since } T \text{ is Cartesian} \\
&= \{T((i_a \times i_a) +)\} \\
&= \{T((i_a \times i_a) + |_{i_a} i_a)\} \\
&= \{T((i_a \times i_a) + |_{i_a})T(i_a)\} \\
&= T((i_a \times i_a) + |_{i_a})\{T(i_a)\}.
\end{aligned}$$

□

Corollary 4.16 *If (\mathbb{X}, \mathbb{T}) is a Cartesian tangent category, then the category of tangent spaces, with their associated differential structures, is a Cartesian differential category equivalent to $\text{Diff}(\mathbb{X}, \mathbb{T})$.*

PROOF: We have just seen that every tangent space has a differential structure. But, in fact every object with differential structure $(A, \hat{p}, \sigma, \zeta)$ is the tangent space to itself at ζ , as

$$\begin{array}{ccc}
A & \xrightarrow{\langle 1, !\zeta \rangle} & TA \\
! \downarrow & & \downarrow p \\
1 & \xrightarrow{\zeta} & A
\end{array}$$

is a pullback diagram. So the category of tangent spaces with their differential structure is the category of all differential structures, and hence by Theorem 4.11 forms a Cartesian differential category. □

Finally, we make some remarks about linear maps between tangent spaces. By the universal property of tangent spaces, any smooth map $f : M \rightarrow N$ induces a map between tangent spaces

$$T_a(M) \xrightarrow{T_a(f) := i_a T(f) |_{i_f a}} T_{af}(N).$$

In the category of finite-dimensional smooth manifolds, these maps are linear (in the vector space sense). While Cartesian differential categories assumes no vector space structure, there is still a notion of linear map between two objects: one says a map $f : X \rightarrow Y$ in a Cartesian differential category is linear if $D(f) = \pi_0 f$. In the standard example of smooth maps between Cartesian spaces, this notion agrees with the usual one. Here, we show that the maps $T_a(f)$ are linear in this differential sense.

Proposition 4.17 *Suppose (\mathbb{X}, \mathbb{T}) is a Cartesian tangent category, with a map $f : M \rightarrow N$ and a point $a : 1 \rightarrow M$ whose tangent space $T_a(M)$ exists. Then the map $T_a(f) : T_a(M) \rightarrow T_{af}(N)$ is linear in the differential sense; ie.,*

$$D(T_a(f)) = \hat{p}T_a(f).$$

PROOF: Recalling that $D(T_a(f)) := T(T_a(f))\hat{p}$, we need to show the following diagram commutes:

$$\begin{array}{ccc} T(T_a(M)) & \xrightarrow{T(i_a T(f)|_{i_{af}})} & T(T_{af}(N)) \\ \{T(i_a)\}_{i_a} \downarrow & & \downarrow \{T(i_{af})\}_{i_{af}} \\ T_a(M) & \xrightarrow{i_a T(f)|_{i_{af}}} & T_{af}(N) \end{array}$$

It suffices to show they are equal when post-composed by i_{af} , so we need to show

$$T(i_a T(f)|_{i_{af}})\{T(i_{af})\} = \{T(i_a)\}T(f).$$

But

$$\begin{aligned} T(i_a T(f)|_{i_{af}})\{T(i_{af})\} &= \{T(i_a T(f)|_{i_{af}})T(i_{af})\} \\ &= \{T(i_a T(f))\} \\ &= \{T(i_a)T^2(f)\} \\ &= \{T(i_a)\}T(f) \text{ (by lemma 2.14)} \end{aligned}$$

as required. □

This shows that the differential definition of linear map is an appropriate replacement for the vector space definition in this abstract context.

5 Representable tangent structure and infinitesimal objects

The previous section demonstrated a fundamental link between differentials and tangent structure. We now turn our attention to another important relationship. Tangent structure in which the tangent functor is of the form $(-)^D$ for some object D – which we shall characterize as being an *infinitesimal* object – is said to be *representable*. We shall show that such tangent structure places one in the domain of synthetic differential geometry.

We begin this section by recalling – and generalizing to our additive context – the basic ideas of synthetic differential geometry. In particular, we recall how a model of synthetic differential geometry (which here we take to be an infinitesimally linear rig of line type in a Cartesian closed category with finite limits) produces a setting with representable tangent structure. For the reverse direction we show how representable tangent structure corresponds to having an infinitesimal object, and how having an infinitesimal object produces a system of infinitesimals. We then extend the ideas of [Rosický 1984] and show how (given sufficient limits and exponent objects) the central notion of synthetic differential geometry, the rig of line type, can be recovered from the presence of an infinitesimal object. As synthetic models are more sophisticated than their standard cousins from differential geometry, we end the section by discussing a basic (additive) synthetic model. The model is based on finitely presented commutative rigs and exposes one further key structural aspect of tangent structure: whenever tangent structure is representable, the opposite category has (dual) tangent structure.

5.1 Basic synthetic differential geometry

Synthetic differential geometry was introduced by Lawvere as a way to give formal mathematical content to the “synthetic” reasoning employed by Sophus Lie. Lie would often reason about differential geometry by using “infinitesimally small” vectors: he called this “synthetic” reasoning. However, Lie was unable to provide a rigorous framework for these infinitesimals, and as a result, fell back on analytical techniques which he found less intuitive. Lawvere showed how one could make Lie’s synthetic reasoning precise by providing topos models in which synthetic arguments using infinitesimals could be interpreted.

The purpose of this section is to introduce these ideas in the additive context. The starting point for Lawvere’s ideas, as described in [Kock 2006], is a “ring of line type” usually situated in a topos. We shall generalize these ideas in stages: in this section we will take the first step by showing how these ideas can be translated into the additive and Cartesian closed context. This means, in particular, that we shall use rig of line type rather than ring of line type:

Definition 5.1 *If \mathbb{X} is a finitely-complete Cartesian-closed category, a **rig of line type** consists of an internal commutative rig R such that the canonical map from*

$$\alpha : R \times R \longrightarrow R^D; (a, b) \mapsto \lambda d.a + b \cdot d$$

where $D := \{d \in R : d^2 = 0\}$, is an isomorphism.

The requirement that α is an isomorphism is the Kock-Lawvere axiom. It says that every map from D (the “object of infinitesimals”) to R (the “real line”) is linear: that is all infinitesimal curves are straight. It should be noted that often additional assumptions are made about the relationship of R to other infinitesimal objects (such as $D_n = \{d \in R : d^{n+1} = 0\}$), which allows one to consider other jet bundles for manifolds. Since our concern is only with the tangent bundle itself, we will not consider these additional assumptions here.

It is important to appreciate that a category with a rig of line type is necessarily rather special. In particular, there are no non-zero rigs of line type in the category of sets (see exercise 1.1 in [Kock 2006]). In fact, it takes some quite sophisticated mathematical work to show that such categories even exist! For example, the book [Moerdijk and Reyes 1991] is almost entirely devoted to building such categories. Nonetheless, there are models of SDG in which the classical category of smooth manifolds fully and faithfully embeds. This shows that one can indeed reason synthetically about classical differential geometry and the results will be classically valid. Thus, reasoning in SDG is analogous to reasoning in real analysis by passing to the richer setting of complex analysis.

In a model of SDG, because one thinks of a curve $f : D \longrightarrow M$ as an “infinitesimal curve” on M , one wishes to think of the object M^D as the tangent bundle of M . However, in a model of SDG simply setting $TM := M^D$ does not directly give tangent structure. This is because, while the map $0 : 1 \longrightarrow D$ gives the projection $p : M^D \longrightarrow M$, the second tangent bundle $T^2M = (M^D)^D = M^{D \times D}$ and the pullback bundle T_2M may not behave in the correct manner. Thus, we need some additional requirements on M to secure the correct behavior. These, following [Kock 2006], are given by the following definitions:

Definition 5.2 *In a finitely-complete Cartesian-closed category with a rig of line type, an object*

M is **vertically linear**⁶ if it perceives the diagram:

$$D \begin{array}{c} \xrightarrow{\langle 0, 1 \rangle} \\ \xrightarrow{\langle 0, 0 \rangle} \end{array} D \times D \xrightarrow{\langle \cdot, \pi_1 \rangle} D(2)$$

as a coequalizer, where $D(n) := \{(d_1, d_2 \cdots d_n) \in D^n : \forall i, j \leq n, d_i \cdot d_j = 0\}$.

To say an object, M , **perceives** a diagram as being a colimit is to say that using the diagram as an exponent of M turns the diagram into a limit cone. The objects which perceive a certain collection of cones as being limits are always closed under both limits and exponentiation. The definition means that the universality of the vertical lift holds for vertically linear objects.

Definition 5.3 Say that an object M is **infinitesimally linear** if M perceives

$$\begin{array}{ccc} 1 & \xrightarrow{0} & D \\ 0 \downarrow & & \downarrow \langle 0, 1 \rangle \\ D & \xrightarrow{\langle 1, 0 \rangle} & D(2) \end{array}$$

as a pushout and, similarly, perceives the analogous diagram for each $D(n)$ as a pushout power.

This tells us that on an infinitesimally linear object M , the tangent bundle functor $T_n(M)$ is $M^{D(n)}$, and allows us to define the addition map $+: T_2 \rightarrow T$.

Proposition 5.4 (Kock’s construction for rigs of line type) *If \mathbb{X} is a finitely-complete Cartesian closed category with a rig of line type, R , which is infinitesimally linear, then \mathbb{X}_{iv} , the full subcategory of objects which are infinitesimally and vertically linear, includes R , is a finitely complete Cartesian closed subcategory, and has Cartesian tangent structure, with the tangent functor given by $(-)^D$.*

PROOF: The objects which perceive certain cocones as being colimits are always closed to finite limits and exponentiation with any object. Thus the subcategory \mathbb{X}_{iv} is certainly a finitely complete Cartesian closed subcategory. Furthermore, as noted above, infinitesimal linearity allows us to represent $T_n(M)$ as $M^{D(n)}$. The various natural transformations are then given by applying the functor $M^{(-)}$ to particular maps between these infinitesimal objects:

- addition is induced by the diagonal map $\Delta : D \rightarrow D(2)$, while the 0 map is induced by the unique map $D \rightarrow 1$;
- the multiplication map $(x_1, x_2) \mapsto (x_1 \cdot x_2)$ gives the vertical lift $\ell : TM \rightarrow T^2M$;
- the canonical flip $c : T^2M \rightarrow T^2M$ is induced by the “twist” map $t : D^2 \rightarrow D^2$.

⁶This is a strong version of property W in [Kock 2006]: there it is so named as it was due to Gavin Wraith.

The coherence axioms are then straightforward to check. For example, the assumption that (D, \cdot) is commutative gives $\ell c = \ell$, while c is an additive bundle morphism follows since the diagram

$$\begin{array}{ccc} D \times D & \xrightarrow{\Delta \times 1} & D(2) \times D \\ \downarrow t & & \downarrow t \\ D \times D & \xrightarrow{1 \times \Delta} & D \times D(2) \end{array}$$

commutes. Finally, the fact that every object is vertically linear provides precisely that $v : T_2M \rightarrow T^2M$ (represented by the map $(d_1, d_2) \mapsto (d_1 \cdot d_2, d_2)$) has the required equalizer property.

Finally, we need to check that R is vertically linear. By the Kock-Lawvere axiom and the infinitesimal linearity of R , the diagram

$$R^{D(2)} \longrightarrow R^{D \times D} \begin{array}{c} \xrightarrow{R^{(0,1)}} \\ \xrightarrow{R^{(0,0)}} \end{array} R^D$$

becomes

$$R \times R \times R \xrightarrow{\langle \pi_0, \pi_1, 0, \pi_2 \rangle} R \times R \times R \times R \begin{array}{c} \xrightarrow{\langle \pi_0, \pi_2 \rangle} \\ \xrightarrow{\langle \pi_0, 0 \rangle} \end{array} R \times R,$$

which is an equalizer, as required. \square

The Kock-Lawvere axiom also tells us that R is the tangent space of D at 0; in other words, the following diagram is a pullback:

$$\begin{array}{ccc} R & \longrightarrow & D^D \\ \downarrow ! & & \downarrow p \\ 1 & \xrightarrow{0} & D \end{array}$$

Hence, by Theorem 4.15, we have the following:

Corollary 5.5 *In the tangent structure associated to a model of SDG, R and all of its finite powers have differential structure.*

A typical synthetic model constructed in this manner may contain many more differential objects than simply the powers of R . For example, it was shown in [Kock 1986] and [Kock and Reyes 1986] that the convenient vector spaces fully and faithfully embed into the Cahiers topos, one of the first models of SDG. Moreover, one can show that this embedding is a strong morphism of tangent structure. Since the category of convenient vector spaces forms a Cartesian differential category ([Blute et al. 2012]), this shows that each convenient vector space (as an object of the Cahiers topos) also has differential structure. It would be interesting to know whether there are other differential objects in the Cahiers topos besides the convenient vector spaces.

5.2 Representable tangent structure and synthetic models

By the result of the previous section, each additive model of synthetic differential geometry provides an example of tangent structure. The tangent structure, however, has the key property that the functors T and T_n are representable. Our objective now is to show that demanding tangent structure be representable is sufficient to place one in a setting which is morally synthetic. The general ideas follow the last section of [Rosický 1984]; however, we go into greater detail. In particular, we show how one can extract a *commutative* rig of line type from an example of representable tangent structure; the ring constructed in [Rosický 1984] is not necessarily commutative.

The starting point for the development is to characterize the representing objects for the functors used in tangent structure. We view the object representing the tangent functor to be an “infinitesimal” object. The main observations connecting this to synthetic geometry come in the next subsection, where we will show that from such infinitesimal object, one can (given certain limits) construct various rig objects which satisfy the Kock-Lawvere axiom, and this places one squarely in the synthetic territory.

Recall that a functor on a Cartesian category is **representable** if it is equivalent to an exponential functor, $(-)^A = A \Rightarrow _$. The object A is then the representing object. Notice that the object A must be an **exponent object** in the sense that for each object X the exponential object $X^A = A \Rightarrow X$ must exist. Composition of representable functors is, up to equivalence, given by taking the product of the representing objects and natural transformation between such functors are completely determined by ordinary maps, in a contravariant direction, between the representing objects. In this way functorial (macroscopic) structure on representable functors is mirrored by (microscopic) structure on the representing objects.

A category has **representable tangent structure** in case it is Cartesian and has tangent structure in which all the functors T^n and T_n are representable. Notice that this necessarily means that the tangent structure is Cartesian. Note that we only require certain objects to be exponentiable; thus, we do not need to demand that the category be Cartesian closed.

Our first objective is to translate the macroscopic functorial behavior of tangent structure into the microscopic behavior on the representing objects:

Definition 5.6 *A Cartesian category \mathbb{X} has an infinitesimal object D in case:*

[Infsml.1] *D is a commutative semigroup with multiplication $_ \odot _ : D \times D \longrightarrow D$ and a zero $\varphi : 1 \longrightarrow D$ such that the following diagrams commute:*

$$\begin{array}{ccc}
 D \times D & \xrightarrow{c_\times} & D \times D \\
 \searrow \odot & & \swarrow \odot \\
 & & D
 \end{array}
 \quad
 \begin{array}{ccc}
 D \times D \times D & \xrightarrow{1 \times \odot} & D \times D \\
 \odot \times 1 \downarrow & & \downarrow \odot \\
 D \times D & \xrightarrow{\quad} & D \\
 & \odot &
 \end{array}
 \quad
 \begin{array}{ccc}
 D & \xrightarrow{\langle \varphi, 1 \rangle} & D \times D & \xleftarrow{\langle 1, \varphi \rangle} & D \\
 \searrow \varphi & & \downarrow \odot & & \swarrow \varphi \\
 & & D & &
 \end{array}$$

[Infsml.2] *Pushout powers of $1 \xrightarrow{\varphi} D$ exist:*

$$\begin{array}{ccc}
 1 & \xrightarrow{\varphi} & D \\
 \varphi \downarrow & & \downarrow i_0 \\
 D & \xrightarrow{i_1} & D \star D
 \end{array}$$

[Infsml.3] *There is a map $\delta : D \rightarrow D \star D$ which makes the object φ , in the pointed category $1/\mathbb{X}$, a commutative comonoid with respect to the coproduct, $-\star-$ (the unit is necessarily the unique map to the final object). Explicitly this means that the following diagrams commute:*

$$\begin{array}{ccc}
D & \xrightarrow{\delta} & D \star D \\
\delta \downarrow & & \downarrow 1 \star \delta \\
D \star D & \xrightarrow{\delta \star 1} & D \star D \star D
\end{array}
\quad
\begin{array}{ccc}
& D & \\
\delta \swarrow & & \searrow \delta \\
D \star D & \xrightarrow{c_\star} & D \star D
\end{array}
\quad
\begin{array}{ccc}
& D & \\
\delta \swarrow & & \searrow \delta \\
D & \xleftarrow{! \star 1} & D \star D \xrightarrow{! \star !} D
\end{array}$$

[Infsml.4] *The following diagram commutes*

$$\begin{array}{ccc}
D \times D & \xrightarrow{\odot} & D \\
1 \times \delta \downarrow & & \downarrow \delta \\
D \times (D \star D) & \xrightarrow{\langle \odot_{t_0} | \odot_{t_1} \rangle} & D \star D
\end{array}$$

[Infsml.5] *The following is a coequalizer:*

$$D \begin{array}{c} \xrightarrow{\langle \varphi, \varphi \rangle} \\ \xrightarrow{\langle \varphi, 1 \rangle} \end{array} D \times D \xrightarrow{(\delta \times 1) \langle \odot | \star \pi_0 \rangle} D \star D$$

[Infsml.6] *The objects D^n and D_n are exponent objects.*

We shall, as usual, write $D^0 := 1$, $D^1 := D$, $D^2 = D \times D$, etc. and write, suggestively, $D(0) := 1$, $D(1) := D$, $D(2) := D \star D$, $D(3) := D \star D \star D$, etc. We observe:

Proposition 5.7 *A Cartesian category has representable tangent structure if and only if it has an infinitesimal object in the above sense.*

PROOF: This is a matter of translating between the macroscopic functorial structure and the microscopic structure. First, consider the map $0 : A \rightarrow TA = A^D$. This must be induced by a map $D \rightarrow 1$; however, there is only one such map $! : D \rightarrow 1$. Now consider $c : T^2A = A^{D \times D} \rightarrow A^{D \times D}$. This corresponds to a map $c_\times : D \times D \rightarrow D \times D$: we also know $T(0)cp = 0$ and $0c = T(c)$, on the microscopic level this implies $c_\times \pi_1 = \pi_0$ and $c_\times \pi_0 = \pi_1$. Thus, c_\times is the symmetry of the product.

The transformation $p : TA = A^D \rightarrow A$ must correspond to a point in D . This is $\varphi : 1 \rightarrow D$. The canonical lift must correspond to a map $\odot : A \times A \rightarrow A$: as $\ell c = \ell$, this is a commutative operation and the first coherence diagram for ℓ asserts that it is associative. The bundle homomorphism for ℓ forces φ to be a zero for this multiplication.

Clearly $T_2(A) = A^{D \star D}$ as a pushout in the exponent becomes a pullback. Then the bundle addition is given by δ ; the additive properties of the bundle addition give a commutative comonoid structure to the copowers of φ in the coslice $1/\mathbb{X}$.

It is clear that the symmetry map will induce the desired morphism of bundles. However, the behavior of the lift is less clear. A little scrutiny reveals that **[Infsml.4]** is the required diagram, and it is asserting the ‘‘distributivity’’ of multiplication over addition.

Finally [**Infsm1.5**] is the translation of the universality of the vertical lift. \square

Here is a synopsis of the correspondence between the two levels:

Microscopic level (Infinitesimal object operations)	Macroscopic level (Functorial properties)
$! : D \rightarrow 1$ final map	$0 : A \rightarrow T(A)$ bundle zero
$c_\times : D \times D \rightarrow D \times D$ symmetry of product	$c : T^2 A \rightarrow T^2 A$ canonical flip
$\wp : 1 \rightarrow D$ zero of infinitesimal object	$p : TA \rightarrow A$ projection of tangent bundle
$\odot : D \times D \rightarrow D$ multiplication of infinitesimals	$\ell : TA \rightarrow T^2 A$ vertical lift
$\delta : D \rightarrow D \star D$ comultiplication of infinitesimals	$+$: $T_2 A \rightarrow TA$ bundle addition.

An infinitesimal object in this sense has the property that every element under the multiplication has square zero:

Lemma 5.8 *If D is an infinitesimal object, then the following diagram commutes:*

$$\begin{array}{ccc} D & & \\ \Delta \downarrow & \searrow \wp & \\ D \times D & \xrightarrow{\quad \odot \quad} & D \end{array}$$

PROOF: First, we claim that if we define

$$D \star D \xrightarrow{\chi := \langle 1, \wp \rangle \star \langle \wp, 1 \rangle} D \times D,$$

then $\Delta = \delta\chi$. Indeed, by [**Infsm1.3**],

$$\delta\chi\pi_0 = \delta(1 \star \wp) = 1$$

and similarly $\delta\chi\pi_1 = 1$, so that $\Delta = \langle 1, 1 \rangle = \delta\chi$.

So $\Delta\odot = \delta\chi\odot$. But

$$\iota_0\chi\odot = \langle 1, \wp \rangle\odot = \wp$$

by [**Infsm1.1**]; similarly $\iota_1\chi\odot = \wp$, so that $\chi\odot = \wp$. Thus

$$\Delta\odot = \delta\chi\odot = \delta\wp = \wp,$$

as required. \square

An alternate axiomatization of what it means for an object to be infinitesimal is discussed in [Lawvere 2011]: the paper also serves to promote the idea that infinitesimal objects should be regarded as being primary in developing synthetic models. Toward this end, below, we reformulate the notion of a synthetic model so that it no longer relies on the presence of a rig of line type.

By a **tiny object** in a Cartesian category we shall mean a 2-nilpotent commutative semigroup. This means, explicitly, a tiny object $\mathcal{A} = (A, m, 0)$, where $m : A \times A \rightarrow A$ is the multiplication and $0 : 1 \rightarrow M$ is the zero, must satisfy:

$$\begin{array}{ccccc} A \times A \times A & \xrightarrow{m \times 1} & A \times A & & A \times A & \xrightarrow{c_\times} & A \times A & & A & \xrightarrow{\langle 0, 1 \rangle} & A \times A & \xleftarrow{\langle 1, 0 \rangle} & A & & A & \xrightarrow{\Delta} & A \times A \\ 1 \times m \downarrow & & \downarrow m & & \downarrow m & & \downarrow m & & \downarrow m & & \downarrow m & & \downarrow m & & \downarrow m & & \downarrow m \\ A \times A & \xrightarrow{m} & A & & A & & A & & A & & A & & A & & A & & A \end{array}$$

Notice that each $D(n)$, of the previous section, is a tiny object with the multiplication defined pointwise.

Clearly for any Cartesian category, \mathbb{X} , we may form a category of tiny objects, $\text{Tiny}(\mathbb{X})$. This category has a zero: that is an object which is both initial and final which induces unique zero maps between any two objects. We shall call a zero preserving functor from the opposite of the Lawvere theory of commutative monoids, \mathbb{T}_{cm} , into the category of tiny objects of a category, $\mathcal{D} : \mathbb{T}_{\text{cm}}^{\text{op}} \rightarrow \text{Tiny}(\mathbb{X})$ a **system of infinitesimals** provided the tiny multiplication on $\mathcal{D}(2)$ satisfies:

$$\begin{array}{ccc} \mathcal{D}(1) \times \mathcal{D}(1) & \xrightarrow{0} & \mathcal{D}(2) \\ \mathcal{D}(i_0) \times \mathcal{D}(i_1) \downarrow & \nearrow m_{\mathcal{D}(2)} & \\ \mathcal{D}(2) \times \mathcal{D}(2) & & \end{array}$$

We do not demand that the functor preserves products/coproducts.

Because \mathbb{T}_{cm} has biproducts there is, for each system of infinitesimals, a unique comonoidal transformation $\chi : \mathcal{D}(n+m) \rightarrow \mathcal{D}(n) \times \mathcal{D}(m)$ such that

$$\mathcal{D}(n_i) \xrightarrow{\delta_{ij}} \mathcal{D}(n_j) = \mathcal{D}(n_i) \xrightarrow{\mathcal{D}(i_i)} \mathcal{D}(n_0 + n_1) \xrightarrow{\chi} \mathcal{D}(n_0) \times \mathcal{D}(n_1) \xrightarrow{\pi_j} \mathcal{D}(n_j)$$

where δ_{ij} is Kronecker's delta; that is, when $i = j$ the map is the identity, otherwise it is zero.

Clearly, the objects $D(n)$ (with $D(1) = D$ and $D(0) = 1$) of the previous section provide a system of infinitesimals with comultiplication given by the diagonal map, $\delta : D \rightarrow D(2); d \mapsto (d, d)$, and the counit given by the final map $! : D \rightarrow 1$. On the other hand, when a category has an infinitesimal object, the objects $D_{(0)} = 1, D_{(1)} = D, D_{(2)} = D \star D, D_{(3)} = D \star D \star D, \dots$ also form a system of infinitesimals. In particular, each $D_{(n)}$ is a tiny object: the multiplication for $D_{(2)}$ is:

$$\begin{aligned} (D \star D) \times (D \star D) &\xrightarrow{\odot_{A \star A}} D \star D \\ &= (D \star D) \times (D \star D) \xrightarrow{\theta(\theta' \star \theta')} (D \times D) \star (D \times D) \star (D \times D) \star (D \times D) \xrightarrow{\odot_{\star ! \star \odot}} D \star D \end{aligned}$$

where we use strength to express the distribution of the product inside the $_ \star _$. Notice that the ‘‘cross’’ term multiplications are zero so that the requirement on the tiny multiplication is met.

A useful observation which we will have occasion to use shortly concerns a slight re-expression of [Infsml.4]:

Lemma 5.9 *In a category with an infinitesimal object, $(\delta \times \delta) \odot_{A \star A} = (1 \times \delta) \langle \odot i_0 | \odot i_1 \rangle$ so that [Infsml.4] holds if and only if*

$$\begin{array}{ccc} D \times D & \xrightarrow{\odot} & D \\ \delta \times \delta \downarrow & & \downarrow \delta \\ (D \star D) \times (D \star D) & \xrightarrow{\odot_{D \star D}} & D \star D \end{array}$$

so that δ is a homomorphism of tiny objects.

PROOF: First note that [Infsml.4] also implies δ preserves the tiny zero; simply precompose the diagram for [Infsml.4] with the map $\langle \wp, \wp \rangle : 1 \longrightarrow D \times D$.

In any category there is always a natural map $(X \times A) \star (X \times A) \longrightarrow X \times (A \star A)$; when X is an exponent object this is a natural isomorphism. The inverse of this transformation provides a strength $\theta : X \times (A \star A) \longrightarrow (X \times A) \star (X \times A)$ (and its symmetric dual $\theta' : (A \star A) \times X \longrightarrow (A \times X) \star (A \times X)$) which we shall use explicitly in the calculation below.

$$\begin{aligned}
(\delta \times \delta) \odot_{A \star A} &= (1 \times \delta)(\delta \times 1) \odot_{A \star A} \\
&= (1 \times \delta)(\delta \times 1)\theta(\theta' \star \theta')(\odot \star ! \star ! \star \odot) \\
&= (1 \times \delta)\theta((\delta \times 1) \star (\delta \times 1))(\theta' \star \theta')((\odot \star !) \star (! \star \odot)) \\
&= (1 \times \delta)\theta((\delta \times 1) \star (\delta \times 1))(((1 \star !) \times 1) \star (!(\star 1) \times 1))(\odot \star \odot) \\
&= (1 \times \delta)\theta(\odot \star \odot) \\
&= (1 \times \delta)\langle \odot \iota_0 | \odot \iota_1 \rangle.
\end{aligned}$$

□

A **synthetic model** is a system of infinitesimals in a finitely complete Cartesian closed category in which every object $\mathcal{D}(n)$ is both infinitesimally linear and infinitesimally vertical. To say that $\mathcal{D}(i)$ is infinitesimally linear is to say that $\mathcal{D}(i)$ perceives pointed copower diagrams of the form:

$$\begin{array}{ccc}
\mathcal{D}(0) & \xrightarrow{\wp} & \mathcal{D}(1) \\
\wp \downarrow & & \downarrow \iota_0 \\
\mathcal{D}(1) & \xrightarrow[\iota_1]{} & \mathcal{D}(2)
\end{array}$$

to be pushout diagrams. To say that $\mathcal{D}(i)$ is vertically linear is to say that it perceives

$$\mathcal{D}(1) \begin{array}{c} \xrightarrow{\langle \wp, \wp \rangle} \\ \xrightarrow[\langle \wp, 1 \rangle]{} \end{array} \mathcal{D}(1) \times \mathcal{D}(1) \xrightarrow{(\delta \times 1)\langle \odot | \star \pi_0 \rangle} \mathcal{D}(2)$$

to be a coequalizer. The map $\langle \odot | \star \pi_0 \rangle$ can be formed, of course, because $\mathcal{D}(2)$ is infinitesimally linear.

The system of infinitesimals in an additive model of synthetic geometry, given by the objects $\mathcal{D}(n)$, satisfies these properties if and only if R is infinitesimally linear (as $\mathcal{D}(n) \subseteq R^n$ and R is then vertically linear from the Kock-Lawvere axiom). Thus, in the formulation of synthetic geometry using rigs of line type, the testing for linearity is conveniently transferred onto the rig itself. In a category with an infinitesimal object, of course, such a transference may not be possible, however, these requirements are immediately satisfied! We now obtain a generalization of Kock's construction which avoids talking about a rig of line type altogether:

Proposition 5.10 (Kock's construction for infinitesimals) *In any synthetic model, the infinitesimally and vertically linear objects form a complete Cartesian closed subcategory which has representable tangent structure.*

PROOF: It suffices to show that there is an infinitesimal object; the obvious candidate is $\mathcal{D}(1)$. This certainly satisfies **[Infsml.1]**. The requirement of infinitesimal linearity ensures the $\mathcal{D}(n)$ is the pushout power required by **[Infsml.1]**. The fact that one is picking out a commutative comonoid in the tiny objects (which are forced to be copowers) guarantees **[Infsml.3]**. Lemma 5.9, together with the requirement of how the tiny multiplication on $\mathcal{D}(2)$ is defined, ensures **[Infsml.4]** holds. Vertical linearity ensures **[Infsml.5]**, while cartesian closure ensures **[Infsml.6]**. \square

An important corollary of this is that any category with an infinitesimal object can be fully and faithfully embedded in a complete Cartesian closed category which has tangent structure. The trick is to embed the category in its presheaf category: it is well-known that the Yoneda embedding preserves both finite limits and exponentials; thus the embedding turns the presheaf category into a synthetic model from which a complete Cartesian closed subcategory with representable tangent structure can be extracted using Proposition 5.10.

Corollary 5.11 *Any category with representable tangent structure can be embedded in a synthetic model, and can therefore be embedded in a complete Cartesian closed category with representable tangent structure.*

It is perhaps worth highlighting the differences between a standard model of synthetic differential geometry and a synthetic model as introduced above. The lack of reference to a rig of line type is clear. However as we shall see shortly such rigs can be constructed. More seriously in the standard model $\mathcal{D}(n) \subseteq \mathcal{D}(1)^n$ and this subobject relationship (and the particular subobjects they must be) is something we have not demanded in our more general notion of a synthetic model.

5.3 Infinitesimals and line objects

Given an infinitesimal object D , under mild assumptions, we will show how one can build various associated **line objects**: these are rigs, R , of endomorphisms of D into which D embeds in a structure preserving manner and satisfy the Kock-Lawvere axiom.

The discussion after proposition 5.4 already hints at how we can do this: we begin by assuming the tangent space of D at 0 exists; that is, the following pullback exists:

$$\begin{array}{ccc} (D \rightrightarrows_{\varphi} D) & \longrightarrow & D^D \\ \downarrow ! & & \downarrow p \\ 1 & \xrightarrow{0} & D \end{array}$$

(this can be thought of as the 0-preserving maps from D to D). Since $T = (-)^D$ is a right adjoint, it automatically preserves all limits, and hence Theorem 4.15 tells us that $(D \rightrightarrows_{\varphi} D)$ is a differential object; moreover, part (i) of the the theorem tells us that the map

$$(D \rightrightarrows_{\varphi} D) \times (D \rightrightarrows_{\varphi} D) \xrightarrow{\langle \pi_0^i, \pi_1^i \rangle v} (D \rightrightarrows_{\varphi} D)^D$$

is an isomorphism. In the term logic, this map can be viewed as sending

$$(r_1, r_0) \mapsto \lambda d. r_0 + r_1 \cdot j(d)$$

where j is the natural embedding $D \rightarrow (D \Rightarrow_{\varphi} D)$, and multiplication is composition. Thus, we already have an object with additive structure that satisfies the Kock-Lawvere axiom.

Unfortunately, for this object, composition will not necessarily distribute over the addition (see lemma 5.15 for when it does). In general, to get a rig object, we shall need to further restrict the object $(D \Rightarrow_{\varphi} D)$ to those maps which “preserve addition” in a sense we define below.

The addition on $(D \Rightarrow_{\varphi} D)$ can be most easily expressed in the term logic by:

$$+ : (D \Rightarrow_{\varphi} D) \times (D \Rightarrow_{\varphi} D) \rightarrow (D \Rightarrow_{\varphi} D); (f, g) \mapsto \lambda d. \left\{ \begin{array}{l} \iota_0(x) \mapsto f \bullet x \\ \iota_1(x) \mapsto g \bullet x \end{array} \right\} \delta(d)$$

(where \bullet is application). The pattern matched “case” expresses a map from $D \star D$: for this case construct to make sense, the two maps in the branches of the case must agree on the point – in this case, of course, they both actually preserve the point.

To say a pointed map $f : D \rightarrow D$ **preserves addition** is the requirement that the following diagram commutes:

$$\begin{array}{ccc} D & \xrightarrow{\delta} & D \star D \\ f \downarrow & & \downarrow f \star f \\ D & \xrightarrow{\delta} & D \star D \end{array}$$

This is expressed in the term logic as an equality:

$$\delta(f(x)) = \left\{ \begin{array}{l} \iota_0(x) \mapsto \iota_0(f(x)) \\ \iota_1(x) \mapsto \iota_1(f(x)) \end{array} \right\} \delta(x)$$

We then define the **large line object** to be the equalizer:

$$\mathfrak{R} \rightarrow (D \Rightarrow_{\varphi} D) \begin{array}{c} \xrightarrow{\gamma} \\ \xrightarrow{D \Rightarrow_{\varphi} \delta} \end{array} D \Rightarrow (D \star D)$$

where

$$\begin{aligned} \gamma(f) &= \lambda d. \left\{ \begin{array}{l} \iota_0(x) \mapsto \iota_0(f \bullet x) \\ \iota_1(x) \mapsto \iota_1(f \bullet x) \end{array} \right\} \delta(d) \\ D \Rightarrow_{\varphi} \delta(f) &= \lambda d. \delta(f \bullet d) \end{aligned}$$

We must assume that this (joint) equalizer can be constructed.

It is routine – but involves some lengthy calculations – to check that \mathfrak{R} with multiplication given by composition,

$$\cdot \cdot : R \times R \rightarrow R; (f, g) \mapsto \lambda d. g \bullet (f \bullet d)$$

is a rig with addition as above. To give a flavor of the calculations involved we verify the distribu-

tivity laws:

$$\begin{aligned}
f \cdot (g + h) &= \lambda d. (g + h) \bullet (f \bullet d) \\
&= \lambda d. \left(\lambda d. \left\{ \begin{array}{l} \iota_0(x) \mapsto g \bullet x \\ \iota_1(x) \mapsto h \bullet x \end{array} \right\} \delta(d) \right) \bullet (f \bullet d) \\
&= \lambda d. \left\{ \begin{array}{l} \iota_0(x) \mapsto g \bullet x \\ \iota_1(x) \mapsto h \bullet x \end{array} \right\} \delta(f \bullet d) \\
&= \lambda d. \left\{ \begin{array}{l} \iota_0(x) \mapsto g \bullet x \\ \iota_1(x) \mapsto h \bullet x \end{array} \right\} \left(\left\{ \begin{array}{l} \iota_0(x) \mapsto \iota_0(f \bullet x) \\ \iota_1(x) \mapsto \iota_1(f \bullet x) \end{array} \right\} \delta(d) \right) \\
&= \lambda d. \left\{ \begin{array}{l} \iota_0(x) \mapsto \left\{ \begin{array}{l} \iota_0(x) \mapsto g \bullet x \\ \iota_1(x) \mapsto h \bullet x \end{array} \right\} \iota_0(f \bullet x) \\ \iota_1(x) \mapsto \left\{ \begin{array}{l} \iota_0(x) \mapsto g \bullet x \\ \iota_1(x) \mapsto h \bullet x \end{array} \right\} \iota_1(f \bullet x) \end{array} \right\} \delta(d) \\
&= \lambda d. \left\{ \begin{array}{l} \iota_0(x) \mapsto g \bullet (f \bullet x) \\ \iota_1(x) \mapsto h \bullet (f \bullet x) \end{array} \right\} \delta(d) \\
&= \lambda d. \left\{ \begin{array}{l} \iota_0(x) \mapsto (f \cdot g) \bullet x \\ \iota_1(x) \mapsto (f \cdot h) \bullet x \end{array} \right\} \delta(d) \\
&= (f \cdot g) + (g \cdot h)
\end{aligned}$$

and

$$\begin{aligned}
(g + h) \cdot h &= \lambda d. f \bullet ((g + h) \bullet d) \\
&= \lambda d. f \bullet \left(\left\{ \begin{array}{l} \iota_0(x) \mapsto g \bullet x \\ \iota_1(x) \mapsto h \bullet x \end{array} \right\} \delta(d) \right) \\
&= \lambda d. \left\{ \begin{array}{l} \iota_0(x) \mapsto f \bullet (g \bullet x) \\ \iota_1(x) \mapsto f \bullet (h \bullet x) \end{array} \right\} \delta(d) \\
&= \lambda d. \left\{ \begin{array}{l} \iota_0(x) \mapsto (g \cdot f) \bullet x \\ \iota_1(x) \mapsto (h \cdot f) \bullet x \end{array} \right\} \delta(d) \\
&= (g \cdot f) + (h \cdot f)
\end{aligned}$$

Notice that in the first calculation we use the fact that $f : \mathfrak{R}$ preserves the addition, while the second calculation uses the distribution associated with the case map.

We are now ready to state:

Theorem 5.12 *In any category with an infinitesimal object, D , which admits the construction of its large line object, \mathfrak{R} , the Kock-Lawvere axiom holds; that is the map*

$$\alpha : \mathfrak{R} \times \mathfrak{R} \longrightarrow \mathfrak{R}^D; (r_1, r_0) \mapsto \lambda d. r_0 + r_1 \cdot j(d)$$

is an isomorphism.

PROOF: That the map j is well-defined requires that the multiplication of D preserves addition; but this is exactly the condition **[Infsml.4]**.

From Theorem 4.15, on $(D \Rightarrow_{\varphi} D)$, the inverse to α is given by

$$f \mapsto (\{f\}, f \bullet \varphi),$$

where the term $\{f\}$ is the unique one given by the universality of the canonical lift. In the term logic, its universal property can be expressed as saying that the map $(D \Rightarrow_{\varphi} D)^D \times D \times D \rightarrow D : (f, x, y) \mapsto (f \bullet x) \bullet y$ can be written as

$$(f, x, y) \mapsto \left\{ \begin{array}{l} \iota_0(y) \mapsto \{f\} \bullet (x \odot y) \\ \iota_1(y) \mapsto (f \bullet \varphi) \bullet y \end{array} \right\} \delta(y). \quad (\dagger)$$

Thus, to show that \mathfrak{R} satisfies the Kock-Lawvere axiom, it suffices to show that if f preserves addition, so does $\{f\}$. In order to show this we show that both $\{f\}\delta$ and $\delta(\{f\} \star \{f\})$ serve as $\{f\}\delta$. Clearly the former is immediately true so we shall focus on the latter:

$$\begin{aligned} & \left\{ \begin{array}{l} \iota_0(y) \mapsto \left\{ \begin{array}{l} \iota_0(y) \mapsto \iota_0(\{f\} \bullet y) \\ \iota_1(y) \mapsto \iota_1(\{f\} \bullet y) \end{array} \right\} \delta(x \odot y) \\ \iota_1(y) \mapsto \delta((f \bullet \varphi) \bullet y) \end{array} \right\} \delta(y) \\ &= \left\{ \begin{array}{l} \iota_0(y) \mapsto \left\{ \begin{array}{l} \iota_0(y) \mapsto \iota_0(\{f\} \bullet (x \odot y)) \\ \iota_1(y) \mapsto \iota_1(\{f\} \bullet (x \odot y)) \end{array} \right\} \delta(y) \\ \iota_1(y) \mapsto \delta((f \bullet \varphi) \bullet y) \end{array} \right\} \delta(y) \\ &= \left\{ \begin{array}{l} \iota_0(y) \mapsto \left\{ \begin{array}{l} \iota_0(y) \mapsto \iota_0(\{f\} \bullet (x \odot y)) \\ \iota_1(y) \mapsto \iota_1(\{f\} \bullet (x \odot y)) \end{array} \right\} \delta(y) \\ \iota_1(y) \mapsto \left\{ \begin{array}{l} \iota_0(y) \mapsto \iota_0((f \bullet \varphi) \bullet y) \\ \iota_1(y) \mapsto \iota_1((f \bullet \varphi) \bullet y) \end{array} \right\} \delta(y) \end{array} \right\} \delta(y) \\ &= \left\{ \begin{array}{l} \iota_0(y) \mapsto \left\{ \begin{array}{l} \iota_0(y) \mapsto \iota_0(\{f\} \bullet (x \odot y)) \\ \iota_0(y) \mapsto \iota_0((f \bullet \varphi) \bullet y) \end{array} \right\} \delta(y) \\ \iota_1(y) \mapsto \left\{ \begin{array}{l} \iota_1(y) \mapsto \iota_1(\{f\} \bullet (x \odot y)) \\ \iota_1(y) \mapsto \iota_1((f \bullet \varphi) \bullet y) \end{array} \right\} \delta(y) \end{array} \right\} \delta(y) \\ &= \left\{ \begin{array}{l} \iota_0(y) \mapsto \iota_0 \left(\left\{ \begin{array}{l} \iota_0(y) \mapsto \{f\} \bullet (x \odot y) \\ \iota_0(y) \mapsto (f \bullet \varphi) \bullet y \end{array} \right\} \delta(y) \right) \\ \iota_1(y) \mapsto \iota_1 \left(\left\{ \begin{array}{l} \iota_1(y) \mapsto \{f\} \bullet (x \odot y) \\ \iota_1(y) \mapsto (f \bullet \varphi) \bullet y \end{array} \right\} \delta(y) \right) \end{array} \right\} \delta(y) \\ &= \left\{ \begin{array}{l} \iota_0(y) \mapsto \iota_0((f \bullet x) \bullet y) \\ \iota_1(y) \mapsto \iota_1((f \bullet x) \bullet y) \end{array} \right\} \delta(y) \\ &= \delta((f \bullet x) \bullet y) \end{aligned}$$

Notice the calculation uses [**Infsm1.4**], the fact that $f \bullet x$ preserves addition, and that δ is associative and commutative. \square

We cannot see any reason why \mathfrak{R} should be commutative. In fact, as far as we can see, the infinitesimals need not even sit in the center of \mathfrak{R} , which is something one might expect. However, this is a defect that we can correct by restricting to the commutator of D in \mathfrak{R} . To form this subrig requires the following equalizer to be present:

$$\mathfrak{R}_{\varepsilon} \longrightarrow \mathfrak{R} \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} \mathfrak{R}^D$$

where $\phi(r) = \lambda d.(j(d) \cdot r)$ and $\psi(r) = \lambda d.(r \cdot j(d))$. It is standard that this does define a subrig. What is not so obvious, but was noted in [Rosický 1984], is that this subrig of \mathfrak{R} also satisfies the Kock-Lawvere axiom. We shall refer to this rig as the **Rosický line object**.

Proposition 5.13 *In any category with an infinitesimal object, in which the Rosický line object, \mathfrak{R}_ε , can be constructed, \mathfrak{R}_ε satisfies the Kock-Lawvere axiom.*

PROOF: We must check that when $f : \mathfrak{R}_\varepsilon^D$ (so that $d' \odot ((f \bullet d) \bullet y) = (f \bullet d) \bullet (d' \odot y)$) then $\{f\} : \mathfrak{R}_\varepsilon$. That is if $f \bullet d$ commutes with the action of D then so will $\{f\}$. As above we use the universal property of $\{f\}$ to establish that $d' \odot (\{f\} \bullet y) = (\{f\} \bullet (d' \odot y))$:

$$\begin{aligned}
\left\{ \begin{array}{l} \iota_0(y) \mapsto \{f\} \bullet (d' \odot x \odot y) \\ \iota_1(y) \mapsto (f \bullet \varphi) \bullet (d' \odot y) \end{array} \right\} \delta(y) &= \left\{ \begin{array}{l} \iota_0(y) \mapsto \{f\} \bullet (x \odot y) \\ \iota_1(y) \mapsto (f \bullet \varphi) \bullet y \end{array} \right\} \delta(d' \odot y) \\
&= (f \bullet x) \bullet (d' \odot y) \\
&= d' \odot ((f \bullet x) \bullet y) \\
&= d' \odot \left(\left\{ \begin{array}{l} \iota_0(y) \mapsto \{f\} \bullet (x \odot y) \\ \iota_1(y) \mapsto (f \bullet \varphi) \bullet y \end{array} \right\} \delta(y) \right) \\
&= \left\{ \begin{array}{l} \iota_0(y) \mapsto d' \odot (\{f\} \bullet y) \\ \iota_1(y) \mapsto d' \odot ((f \bullet \varphi) \bullet y) \end{array} \right\} \delta(y) \\
&= \left\{ \begin{array}{l} \iota_0(y) \mapsto d' \odot (\{f\} \bullet y) \\ \iota_1(y) \mapsto (f \bullet \varphi) \bullet (d' \odot y) \end{array} \right\} \delta(y)
\end{aligned}$$

□

We are supposing that neither \mathfrak{R} or \mathfrak{R}_ε are *commutative* rigs, in general, yet it is usual to start with a commutative rig of line type in order to construct a model of synthetic geometry. This suggests that we should consider the center of \mathfrak{R} (which lies inside \mathfrak{R}_ε): we shall denote it by \mathbb{R} and call it the **Lawvere line object**. Notice that to form this rig we need \mathfrak{R} to be an exponent object, as it is defined by the equalizer

$$\mathbb{R} \longrightarrow \mathfrak{R} \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} \mathfrak{R}^{\mathfrak{R}}$$

where, much as above, $\phi(r) = \lambda r'.(r' \cdot r)$ and $\psi(r) = \lambda r'.(r \cdot r')$. Again, it is standard that this defines a subrig. What is less obvious is that it still satisfies the Kock-Lawvere axiom, which we now check:

Proposition 5.14 *In any category with an infinitesimal object, if the Lawvere line object \mathbb{R} can be constructed, then \mathbb{R} satisfies the Kock-Lawvere axiom.*

PROOF: Again we must check that when $f : \mathbb{R}^D$ (so that $r' \cdot (f \bullet d) = (f \bullet d) \cdot r'$) then $\{f\} : \mathbb{R}$. That is, if $f \bullet d$ is in the center of \mathfrak{R} , then $\{f\}$ is also. Again we use the universal property of $\{f\}$

to establish that $r' \cdot \{f\} = \{f\} \cdot r'$:

$$\begin{aligned}
\left\{ \begin{array}{l} \iota_0(y) \mapsto (\{f\} \cdot r') \bullet (x \odot y) \\ \iota_1(y) \mapsto ((f \bullet \varphi) \cdot r') \bullet y \end{array} \right\} \delta(y) &= \left\{ \begin{array}{l} \iota_0(y) \mapsto r' \bullet (\{f\} \bullet (x \odot y)) \\ \iota_1(y) \mapsto r' \bullet ((f \bullet \varphi) \bullet y) \end{array} \right\} \delta(y) \\
&= r' \bullet \left(\left\{ \begin{array}{l} \iota_0(y) \mapsto \{f\} \bullet (x \odot y) \\ \iota_1(y) \mapsto (f \bullet \varphi) \bullet y \end{array} \right\} \delta(y) \right) \\
&= r' \bullet ((f \bullet x) \bullet y) \\
&= ((f \bullet x) \cdot r') \bullet y \\
&= (r' \cdot (f \bullet x)) \bullet y \\
&= (f \bullet x) \bullet (r' \bullet y) \\
&= \left(\left\{ \begin{array}{l} \iota_0(y) \mapsto \{f\} \bullet (x \odot y) \\ \iota_1(y) \mapsto (f \bullet \varphi) \bullet y \end{array} \right\} \delta(r' \bullet y) \right) \\
&= \left\{ \begin{array}{l} \iota_0(y) \mapsto \{f\} \bullet (r' \bullet (x \odot y)) \\ \iota_1(y) \mapsto (f \bullet \varphi) \bullet (r' \bullet y) \end{array} \right\} \delta(y) \\
&= \left\{ \begin{array}{l} \iota_0(y) \mapsto (r' \cdot \{f\}) \bullet (x \odot y) \\ \iota_1(y) \mapsto (r' \cdot (f \bullet \varphi)) \bullet y \end{array} \right\} \delta(y) \\
&= \left\{ \begin{array}{l} \iota_0(y) \mapsto (r' \cdot \{f\}) \bullet (x \odot y) \\ \iota_1(y) \mapsto ((f \bullet \varphi) \cdot r') \bullet y \end{array} \right\} \delta(y)
\end{aligned}$$

□

Thus, from an infinitesimal object one can obtain at least three line objects. The last – which we referred to as the Lawvere line object – is a commutative line object and allows one to reconceive of the setting as arising through a commutative line object. In fact, by moving to the presheaf category (recalling that the Yoneda embedding preserves exponentials) in this manner one can always construct a commutative line object. However, what one cannot guarantee is that the infinitesimal object is exactly $\{d|d^2 = 0\} \subseteq \mathbb{R}$, indeed nor can one guarantee the form of $D(n)$ as mandated in synthetic differential geometry.

As discussed in proposition 5.8, in any category with an infinitesimal object there is a map

$$\chi : D \star D \longrightarrow D \times D; z \mapsto \left\{ \begin{array}{l} \iota_0(z) \mapsto (z, \varphi) \\ \iota_1(z) \mapsto (\varphi, z) \end{array} \right\} z$$

if pulling back along χ preserves the pushout diagram for $A \star A$ (as happens in any topos) then it is easy to see that χ must be monic. However, if χ is monic we can show:

Lemma 5.15 *In a category with an infinitesimal object, D , if the map $\chi : D \star D \longrightarrow D \times D$ is monic then every zero preserving endomorphism of D preserves addition.*

PROOF: Note that $\delta\chi = \Delta$ and χ is natural so $\chi(f \times f) = (f \star f)\chi$ but then

$$f\delta\chi = f\Delta = \Delta(f \times f) = \delta\chi(f \times f) = \delta(f \star f)\chi$$

so that $f\delta = \delta(f \star f)$. □

This means that [Infsml.4] is unnecessary, and the large line object \mathfrak{R} can be constructed as $D \Rightarrow_{\varphi} D$; that is, the tangent space of D at 0. However, it also means that one may regard

$D(n)$ to be a subobject of D^n . We have, thus, almost returned to the perspective of (additive) synthetic differential geometry: the remaining difference is that we have not required D to be all of $\{d|d^2 = 0\}$, or $D(2)$ to be all of $\{(d_1, d_2) \in D \times D \mid d_1 d_2 = 0\}$.

5.4 Models of infinitesimal objects

Our objective now is to provide a basic example of a category with an infinitesimal object. The example we shall consider is the opposite of the category of finitely presented commutative rigs, $\mathbf{crig}_{fp}^{\text{op}}$. We shall show that in this category, the object

$$\mathbb{N}[\epsilon] := \mathbb{N}[x]/(x^2 = 0)$$

is an infinitesimal object. As the category has all finite limits, we can build the large line object and we shall show that in this case it coincides with the Lawvere line object. This provides an example of a rig of line type; to obtain a synthetic model, one passes to its presheaf category $\mathbf{Set}^{\mathbf{crig}_{fp}}$. One can then use Kock's construction, Proposition 5.10, to extract a finitely-complete Cartesian closed category with a representable infinitesimal object.

While the construction of a complete Cartesian closed category with an infinitesimal object from a category with an infinitesimal object can be carried out quite generally by passing to the presheaf category and using Kock's construction, there is another way to obtain this model using some observations from [Rosický 1984]. As we shall show, if \mathbb{X} has representable tangent structure, then \mathbb{X}^{op} also has tangent structure, called the dual tangent structure. Applying this to the above example shows that \mathbf{crig}_{fp} has tangent structure, a fact that Rosický himself was aware of. Finally, Rosický has a further construction: he observed that the full subcategory of functors in $\mathbf{Set}^{\mathbf{crig}_{fp}}$ which preserve the tangent structure limits itself has tangent structure. As we shall see, Rosický's construction produces the same tangent structure as Kock's construction.

We start with the following basic observation:

Proposition 5.16 $\mathbb{N}[\epsilon] := \mathbb{N}[x]/(x^2 = 0)$ is an infinitesimal object in $\mathbf{crig}_{fp}^{\text{op}}$.

PROOF: In \mathbf{crig}_{fp} the multiplication on the infinitesimal object is given by:

$$\odot : \mathbb{N}[\epsilon] \longrightarrow \mathbb{N}[\epsilon] \otimes \mathbb{N}[\epsilon]; x \mapsto x \otimes x$$

the zero map is given by

$$\wp : \mathbb{N}[\epsilon] \longrightarrow \mathbb{N}; x \longrightarrow 0.$$

The pushout powers are here pullback powers and they are the objects $\mathbb{N}_{(n)} = \mathbb{N}[x_1, \dots, x_n]/(x_i x_j = 0)$ and the comonoid map is

$$\delta : \mathbb{N}(2) \longrightarrow \mathbb{N}[\epsilon]; \quad \begin{array}{l} x_1 \mapsto x \\ x_2 \mapsto x \end{array}$$

The reader is left to check [Infsml.4] and [Infsml.5], which one can do quite concretely.

It remains to argue that $\mathbb{N}[\epsilon]$ is an exponent object. This is a standard result from commutative algebra, and uses a series of equivalences:

$$\begin{array}{c} S(\Omega_A) \longrightarrow B \text{ a rig map} \\ \hline \Omega_A \longrightarrow B \text{ an } A\text{-module map} \\ \hline A \longrightarrow B \text{ an } A\text{-derivation} \\ \hline A \xrightarrow{f} \mathbb{N}[\epsilon] \otimes B \text{ a rig map} \end{array}$$

We explain the series of equivalences starting at the top: $S(\Omega_A)$ is the symmetric rig construction on the A -module Ω_A . A map from this algebra corresponds to an A -module map $\Omega_A \rightarrow B$ where B is viewed as an A -module using the rig map $A \rightarrow S(\Omega_A) \rightarrow B$. Now $A \rightarrow \Omega_A$ is the Kähler module of differential forms, and corresponds to derivations from A to B (still being regarded as an A -module). But derivations exactly correspond to maps $f : A \rightarrow \mathbb{N}[\epsilon] \otimes B$ as $\mathbb{N}[\epsilon] \otimes B$ is the rig $B \times B$ with addition pointwise and the “dual” multiplication $(b_0, b'_0) \cdot (b_1, b'_1) = (b_0 \cdot b_1, b_0 \cdot b'_1 + b_1 \cdot b'_0)$: the second component of the map to this algebra gives a derivation (the first is a rig map from $A \rightarrow B$ which allows B to be regarded as an A -module). This movement can be done within finitely presented rigs.

As this category has finite limits, the exponential of the pushout objects are given by pullbacks.

□

This already gives us an example of a category with an infinitesimal object. To get a synthetic model, we must identify a rig of line type. We only have sufficient structure to build the large line object. However, a straightforward calculation shows that $\mathbb{N}[x]$ is this line object and, as this is a commutative rig, it is also the Lawvere line object.

To check that the presheaf category is a synthetic model it is necessary to know that the rig of line type is infinitesimal. However, as this calculation is entirely within the Yoneda objects, the result is guaranteed. Using Kock’s construction, we can extract from the presheaf category a Cartesian closed category with finite limits which contains the infinitesimal object and its Lawvere line object.

There is an alternative approach to the construction which brings to light another important aspect of tangent structure:

Proposition 5.17 (Dual tangent structure) ⁷ *If \mathbb{X} is a category with tangent structure such that the tangent functor T has a left adjoint S , and each T_n has a left adjoint S_n , then \mathbb{X}^{op} has “dual” tangent structure on S and the functors S_n .*

PROOF: The argument uses (Australian) mates and is quite general: a natural transformation between right adjoints induces a natural transformation in the opposite direction between the left adjoints. This immediately means we obtain a dual structure satisfying all the *equational* coherences expected of cotangent structure on the left adjoints. What is less obvious is that the universal requirements also transfer. To show this it suffices to show that mating in this manner also carries limit cones into colimit cones and to show this it suffices to demonstrate just that cones transfer bijectively!

⁷Note that dual tangent structure is not giving a cotangent bundle; it is instead giving another tangent bundle, but on the opposite category.

The definition of mates is as follows:

$$\begin{array}{ccc}
F_0(A) & \xrightarrow{f_A^\sharp = g} & F_1(A) \\
F_0(\eta_1) \downarrow & & \uparrow \epsilon_0 \\
F_0(G_1(F_1(A))) & \xrightarrow{F_0(f_{F_1(A)})} & F_0(G_0(F_1(A))) \\
\hline
G_1 A & \xrightarrow{f = g^\flat} & G_0(A) \\
\eta_0 \downarrow & & \uparrow G_0(\epsilon_1) \\
G_0(F_0(G_1(A))) & \xrightarrow{G_0(g_{G_1(A)})} & G_0(F_1(G_1(A)))
\end{array}$$

where $(\eta_i, \epsilon_i) : F_i \vdash G_i : \mathbb{X} \rightarrow \mathbb{X}$ are the adjoints.

To show that cones transfer bijectively it suffices to show that the commuting triangles determined by the arrows in the diagram transfer. Recall that the arrows in the diagram are natural transformations and so the transfer need not fix the objects. Here is the correspondence:

$$\begin{array}{ccc}
F_0(A) & \xrightarrow{f^\sharp = g} & F_1(A) \\
& \searrow F_0(p_0)\epsilon_0 & \swarrow q_1 = F_1(p_1)\epsilon_1 \\
& & X \\
\hline
& & A \\
& \swarrow p_1 = \eta_1 G_1(q_1) & \searrow p_0 = \eta_0 G_0(q) \\
G_1(X) & \xrightarrow{f = g^\flat} & G_0(X)
\end{array}$$

□

Corollary 5.18 *If \mathbb{X} has an infinitesimal object D , then \mathbb{X}^{op} has tangent structure.*

PROOF: As \mathbb{X} has an infinitesimal object, it has representable tangent structure, with tangent functor $T = (-)^D$ and pullbacks $T_n = (-)^{D(n)}$. Each of these have a left adjoint $D(n) \times _$, so that there is an induced dual tangent structure on \mathbb{X}^{op} . □

Applying this to $\mathbf{crig}_{fp}^{\text{op}}$ shows that $\mathbb{N}[\epsilon] \otimes _$ is a tangent functor on \mathbf{crig}_{fp} . This structure (on commutative rigs) was a basic example in [Rosický 1984]. Explicitly, the category of finitely presented commutative rigs has tangent structure, with

$$TA := A[x]/(x^2 = 0),$$

$$T_2A := A[x, y]/(x^2 = 0, y^2 = 0, xy = 0),$$

$$T^2A = A[x, y]/(x^2 = 0, y^2 = 0),$$

and

$$+ : T_2A \rightarrow TA; a_0 + a_1x + a_2y \mapsto a_0 + (a_1 + a_2)x,$$

$$\begin{aligned}\ell : TA &\longrightarrow T^2A; a_0 + a_1x \mapsto a + a_1xy, \\ c : T^2A &\longrightarrow T^2A; a_0 + a_1x + a_2y + a_3xy \mapsto a_0 + a_2x + a_1y + a_3xy.\end{aligned}$$

The tangent functor TA of this example may, of course, less explicitly be written as $\mathbb{N}[\epsilon] \otimes A$.

Now, a general way of constructing a category with tangent structure was provided in [Rosický 1984]:

Proposition 5.19 (Rosický’s construction) *If \mathbb{X} is a small category with tangent structure T , then the functors from \mathbb{X} to \mathbf{set} which preserve the pullbacks and equalizer conditions of the tangent structure also have tangent structure, with*

$$T^*(F) = TF, T^*(\alpha) = \alpha_T.$$

Furthermore, this is always Cartesian tangent structure. Denote this tangent structure by $\mathbf{ROS}(\mathbb{X}, T)$.

The proposition, which is broadly applicable, is immediate. The only somewhat surprising aspect is that the resulting category always has Cartesian tangent structure even if \mathbb{X} does not. This is because:

$$\begin{aligned}(T^*(F \times G))(X) &= (F \times G)(T(X)) = F(T(X)) \times G(T(X)) \\ &= T^*(F)(X) \times T^*(G)(X) = (T^*(F) \times T^*(G))(X)\end{aligned}$$

In particular, we may apply Rosický’s constructions to \mathbf{crig}_{fp} with its dual tangent structure. This gives us a subcategory of $\mathbf{ROS}[\mathbf{crig}_{fp}] \subseteq \mathbf{set}^{\mathbf{crig}_{fp}}$, and it is reasonable to wonder how it compares to the subcategory determined by Kock’s construction. In fact, they are the same category, and we can show this quite generally.

Proposition 5.20 *Suppose that \mathbb{X} is a finitely complete small category with an infinitesimal object D . Then the tangent structure*

$$([\mathbb{X}^{\text{op}}, \mathbf{set}]_{iv}, (-)^{\mathcal{Y}(D)})$$

of proposition 5.10 is the same as the tangent structure

$$\mathbf{ROS}(\mathbb{X}^{\text{op}}, (D \times -))$$

of proposition 5.19.

PROOF: For an object F and object $S \in \mathbb{X}$, consider the definition of the tangent functor T^* in $\mathbf{ROS}([\mathbb{X}^{\text{op}}, (D \times -)])$:

$$T^*F(S) = F(D \times S) = [\mathcal{Y}(D \times S), F] = [\mathcal{Y}(D) \times \mathcal{Y}(S), F] = [\mathcal{Y}(S), F^{\mathcal{Y}(D)}] = F^{\mathcal{Y}(D)}(S),$$

which is the same as the definition of the tangent functor in $([\mathbb{X}^{\text{op}}, \mathbf{set}]_{iv}, (-)^{\mathcal{Y}(D)})$. A similar argument shows that the action on morphisms and the forms of the natural transformations also agree.

The only thing left to show is that the categories have the same set of objects. That is, we wish to show that F preserves the pullbacks and equalizers of the tangent structure $(\mathbb{X}^{\text{op}}, (D \times -))$

if and only if F is vertically and infinitesimally linear. Now F is infinitesimally linear if and only if the diagram

$$\begin{array}{ccc} F^{\mathcal{Y}(D(2))} & \longrightarrow & F^{\mathcal{Y}(D)} \\ \downarrow & & \downarrow \\ F^{\mathcal{Y}(D)} & \longrightarrow & F \end{array}$$

is a pullback. Since limits in $[\mathbb{X}^{\text{op}}, \mathbf{set}]$ are pointwise, this is a pullback if and only if

$$\begin{array}{ccc} F^{\mathcal{Y}(D(2))}(S) & \longrightarrow & F^{\mathcal{Y}(D)}(S) \\ \downarrow & & \downarrow \\ F^{\mathcal{Y}(D)}(S) & \longrightarrow & F(S) \end{array}$$

is a pullback for each $S \in \mathbb{X}$. But by the above argument, this is the same as the diagram

$$\begin{array}{ccc} F(D(2) \times S) & \longrightarrow & F(D \times S) \\ \downarrow & & \downarrow \\ F(D \times S) & \longrightarrow & F(S) \end{array}$$

which is F applied to the pullback diagram for the tangent structure $([\mathbb{X}^{\text{op}}, (D \times _)])$. Thus F is infinitesimally linear if and only if it preserves the pullback diagrams of the tangent structure $(\mathbb{X}^{\text{op}}, (D \times _))$; similarly F is vertically linear if and only if it preserves the equalizer diagrams of the tangent structure $(\mathbb{X}^{\text{op}}, (D \times _))$. Thus the categories have the same objects, as required. \square

Note that this implies that $\text{ROS}(\mathbb{X}^{\text{op}}, (D \times _))$ is Cartesian closed, a fact which is not immediately obvious from the definition. Finally, we note that the tangent structure

$$\text{ROS}(\mathbb{X}^{\text{op}}, (D \times _)) = ([\mathbb{X}^{\text{op}}, \mathbf{set}]_{iv}, (-)^{\mathcal{Y}(D)})$$

is, of course, itself representable. But this implies that the tangent bundle functor has a left adjoint, so that by proposition 5.17, the category $[\mathbb{X}^{\text{op}}, \mathbf{set}]_{iv}^{\text{op}}$ also has a “dual” tangent structure. For example, with $\mathbb{X} = \mathbf{crig}_{fp}^{\text{op}}$, we get a new example of tangent structure, on the category $[\mathbf{crig}_{fp}, \mathbf{set}]_{iv}^{\text{op}}$.

6 Restriction tangent structure and manifolds

Our goal in this section is to show the category of manifolds built from a category with tangent structure also has tangent structure. In particular, we shall apply the result to differential restriction categories to show that manifolds built from categories from “smooth partial maps” have tangent structure. This will show not only that the usual category of smooth manifolds is a tangent category, but that less standard models for analysis, such as the category of smooth convenient manifolds of [Kriegl and Michor 1997], are tangent categories.

Restriction categories [Cockett and Lack 2002] were defined so as to give an equational presentation of categories of partial maps. Moreover, restriction categories in which compatible partial maps can be joined is an appropriate abstract setting in which to describe manifolds, as done in

[Grandis 1989]. In [Cockett et al. 2011], the authors combined restriction structure with Cartesian differential structure so as to give an equational presentation of categories of smooth partial maps, with the resulting structure being called differential restriction categories.

An abstract category of manifolds built out a differential restriction category will not again be a differential restriction category as there is no longer a differential operation of the required type. However, the reader might expect that such an abstract “category of smooth manifolds” would have an associated tangent bundle and that the tangent structure of differential restriction category would lift to tangent structure on the manifold completion. This is precisely what we show in this section and it provides an abstract proof that the various categories of smooth manifolds constructed from differential restriction categories have tangent structure.

6.1 Differential restriction categories

To begin, we first recall the definition of a restriction category from [Cockett and Lack 2002]:

Definition 6.1 *Given a category, \mathbb{X} , a restriction structure on \mathbb{X} gives for each map $A \xrightarrow{f} B$, a map, $A \xrightarrow{\bar{f}} A$, satisfying four axioms:*

[R.1] $\bar{f}f = f$;

[R.2] *If $\text{dom}(f) = \text{dom}(g)$ then $\bar{g}\bar{f} = \bar{f}\bar{g}$;*

[R.3] *If $\text{dom}(f) = \text{dom}(g)$ then $\bar{g}\bar{f} = \bar{g}\bar{f}$;*

[R.4] *If $\text{dom}(h) = \text{cod}(f)$ then $f\bar{h} = \bar{h}f$.*

*A category with a specified restriction structure is a **restriction category**.*

The canonical example is that of partial functions between sets, where, given a partial function $f : X \rightarrow Y$, \bar{f} is essentially the identity on the domain of definition of f :

$$\bar{f}(x) = \begin{cases} x & \text{if } f(x) \text{ defined} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Recall also that map in a restriction category is said to be **total** if $\bar{f} = 1$.

We now recall the definition of a differential restriction category from [Cockett et al. 2011] – the reader may usefully compare it with the definition 4.1 of Cartesian differential categories.

Definition 6.2 *A **differential restriction category** is a Cartesian left additive restriction category with an operation*

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow{D(f)} Y}$$

(“differentiation”) such that

[DR.1] $D(f + g) = D(f) + D(g)$ and $D(0) = 0$;

[DR.2] $\langle a + b, c \rangle D(f) = \langle a, c \rangle D(f) + \langle b, c \rangle D(f)$ and $\langle 0, a \rangle D(f) = \overline{af}0$;

[DR.3] $D(\pi_0) = \pi_0\pi_0$, and $D(\pi_1) = \pi_0\pi_1$;

[DR.4] $D(\langle f, g \rangle) = \langle D(f), D(g) \rangle$;

[DR.5] $D(fg) = \langle D(f), \pi_1 f \rangle D(g)$;

[DR.6] $\langle \langle a, 0 \rangle, \langle c, d \rangle \rangle D(D(f)) = \bar{c} \langle a, d \rangle D(f)$;

[DR.7] $\langle \langle 0, b \rangle, \langle c, d \rangle \rangle D(D(f)) = \langle \langle 0, c \rangle, \langle b, d \rangle \rangle D(D(f))$;

[DR.8] $D(\bar{f}) = (1 \times \bar{f})\pi_0 = \overline{\pi_1 f} \pi_0$;

[DR.9] $\overline{D(f)} = 1 \times \bar{f} = \overline{\pi_1 f}$.

Note the addition of a restriction in axioms [DR.2] and [DR.6]: this is necessary since we must keep track of the partiality of the maps that are lost across the equality. The 8th and 9th axioms demand that the differential operator be total in its first variable.

Just as for Cartesian differential categories, an alternate version of the 6th and 7th axioms is more relevant for tangent structure.

Proposition 6.3 *The axioms for a differential restriction category are equivalently given by replacing [DR.6] and [DR.7] with the following axioms:*

- [DR.6'] $\langle \langle a, 0 \rangle, \langle 0, d \rangle \rangle D(D(f)) = \langle a, d \rangle D(f)$;
- [DR.7'] $\langle \langle a, b \rangle, \langle c, d \rangle \rangle D(D(f)) = \langle \langle a, b \rangle, \langle c, d \rangle \rangle D(D(f))$.

PROOF: Assume that D satisfies the usual set of axioms. Clearly, it then satisfies [DR.6'], by setting $c = 0$. For [DR.7'], consider:

$$\begin{aligned}
& \langle \langle a, b \rangle, \langle c, d \rangle \rangle D^2 f \\
&= \langle \langle a, 0 \rangle + \langle 0, b \rangle, \langle c, d \rangle \rangle D^2 f \\
&= \langle \langle a, 0 \rangle, \langle c, d \rangle \rangle D^2 f + \langle \langle 0, b \rangle, \langle c, d \rangle \rangle D^2 f \text{ by [DR.2]}, \\
&= \bar{c} \langle a, d \rangle D^2 f + \bar{b} \langle \langle 0, c \rangle, \langle b, d \rangle \rangle D^2 f \text{ by [DR.6] and [DR.7]}, \\
&= \bar{b} \langle a, d \rangle D^2 f + \bar{c} \langle \langle 0, c \rangle, \langle b, d \rangle \rangle D^2 f \text{ by [DR.6] and [DR.7]}, \\
&= \langle \langle a, 0 \rangle, \langle b, d \rangle \rangle D^2 f + \langle \langle 0, c \rangle, \langle b, d \rangle \rangle D^2 f \text{ by [DR.6] again}, \\
&= \langle \langle a, c \rangle, \langle b, d \rangle \rangle D^2 f \text{ by [DR.2]}.
\end{aligned}$$

as required.

Now assume that D satisfies the alternate set of axioms, with [DR.6] and [DR.7] replaced with [DR.6'] and [DR.7']. Clearly, it then satisfies [DR.7], by setting $a = 0$. To show that it satisfies [DR.6], we begin with a short calculation:

$$\begin{aligned}
\overline{\langle a, d \rangle Df} &= \overline{\langle a, d \rangle \overline{Df}} \\
&= \overline{\langle a, d \rangle \overline{\pi_1 f}} \\
&= \overline{\langle a, d \rangle \pi_1 f} \\
&= \overline{\bar{a} df} = \bar{a} \overline{df}
\end{aligned}$$

Then to show [DR.6], consider:

$$\begin{aligned}
& \langle \langle a, 0 \rangle, \langle b, d \rangle \rangle D^2 f \\
&= \langle \langle a, b \rangle, \langle 0, d \rangle \rangle D^2 f \text{ (by [DR.7'])} \\
&= \langle \langle a, 0 \rangle, \langle 0, d \rangle \rangle D^2 f + \langle \langle 0, b \rangle, \langle 0, d \rangle \rangle D^2 f \text{ (by [DR.2])} \\
&= \langle a, d \rangle Df + \langle \langle 0, 0 \rangle, \langle b, d \rangle \rangle D^2 f \text{ (by [DR.6'] and [DR.7'])} \\
&= \langle a, d \rangle Df + \overline{\langle b, d \rangle} \overline{Df} 0 \text{ (by [DR.2])} \\
&= \overline{\langle b, d \rangle} Df \langle a, d \rangle Df \\
&= \overline{\langle b, d \rangle} Df \overline{\langle a, d \rangle} Df \langle a, d \rangle Df \\
&= \overline{b} \overline{d} \overline{a} \overline{d} \langle a, d \rangle Df \text{ (by the calculation above)} \\
&= \overline{b} \overline{\langle a, d \rangle} Df \langle a, d \rangle Df \\
&= \overline{b} \langle a, d \rangle Df.
\end{aligned}$$

□

We recall a number of examples.

Example 6.4 *Any Cartesian differential category is a differential restriction category, when equipped with the trivial restriction structure ($\overline{f} = 1$ for all f).*

The standard non-trivial example is:

Example 6.5 *The categories whose objects are the natural numbers, with a map $f : n \rightarrow m$ consisting of a smooth partial function defined on an open subset of \mathbb{R}^n to \mathbb{R}^m .*

From [Cockett et al. 2011], we also have the following more complicated example:

Example 6.6 *If R is a commutative ring, then the restriction category of rational functions over R , RAT_R , is a differential restriction category.*

Extending the result of [Blute et al. 2012], we also have:

Example 6.7 *The category of convenient vector spaces and smooth maps defined on smooth open subsets is a differential restriction category.*

In any restriction category, one can make the following definitions:

Definition 6.8 *Two parallel maps f, g in a restriction category are compatible, written $f \smile g$, if $\overline{f} g = \overline{g} f$.*

Compatibility captures the idea of f and g being equal wherever they are both defined. Note that compatibility is not transitive relation. Recall also the notion of when a map f is less than or equal to a map g :

Definition 6.9 *For parallel maps f, g in a restriction category, write $f \leq g$ if $\overline{f} g = f$.*

This captures the notion of f having the same values as g , but with a smaller domain of definition. This is a partial order.

As a helpful tool for certain calculations, we note the following results from [Cockett et al. 2011]:

Proposition 6.10 *In a differential restriction category:*

- (i) $D(\bar{f}g) = (1 \times \bar{f})D(g) = \overline{\pi_1 f} D(g)$;
- (ii) *If $f \leq g$ then $D(f) \leq D(g)$;*
- (iii) *If $f \smile g$ then $D(f) \smile D(g)$.*

6.2 Restriction tangent structure

Just as Cartesian differential categories are the tangent categories in which all objects are differential, so too should differential restriction categories be the tangent restriction categories in which all objects are differential. To make this precise, we must define tangent restriction structure.

We first give the restriction version of additive bundles (compare with definition 2.1).

Definition 6.11 *If A is an object in a restriction category \mathbb{X} then an **additive bundle over A** consists of the following:*

- *a total map $X \xrightarrow{p} A$ such that the restriction pullback of n copies of $X \xrightarrow{p} A$ exists; denote these by X_n , with structure maps $\pi_i : X_n \rightarrow X$;*
- *total maps $+ : X_2 \rightarrow X$ and $0 : M \rightarrow X$, with $+p = \pi_0 p = \pi_1 p$ and $0p = 1$ such that this operation is associative, commutative, and unital; that is, each of the following diagrams commute:*

$$\begin{array}{ccc}
 X_3 \xrightarrow{\langle \langle \pi_0, \pi_1 \rangle +, \pi_2 \rangle} X_2 & & X_2 \xrightarrow{+} X \\
 \langle \pi_0, \langle \pi_1, \pi_2 \rangle + \rangle \downarrow & & \langle \pi_1, \pi_0 \rangle \downarrow \\
 X_2 \xrightarrow{+} X & & X_2 \xrightarrow{+} X
 \end{array}
 \qquad
 \begin{array}{ccc}
 X_2 \xrightarrow{+} X & & X \xrightarrow{1} X \\
 \langle \pi_1, \pi_0 \rangle \downarrow & & \langle p0, 1 \rangle \downarrow \\
 X_2 \xrightarrow{+} X & & X_2 \xrightarrow{+} X
 \end{array}$$

A restriction pullback is a lax pullback, with commutativity replaced by inequalities. For restriction categories restriction limits are the “correct” definition of limit. For further details and the exact definition, see [Cockett and Lack 2007].

When restricted to the total maps of \mathbb{X} , an additive bundle over A is the same as asking for a commutative monoid in the slice category $\text{Total}(\mathbb{X})/A$. There appears to be no notion of slice category for restriction categories which corresponds to the requirement that the pullbacks be restriction pullbacks, while giving the notion of morphism that we wish:

Definition 6.12 *Suppose that $p : X \rightarrow A$ and $q : Y \rightarrow B$ are additive bundles. An **additive bundle morphism** consists of a pair of maps $f : X \rightarrow Y$, $g : A \rightarrow B$ so that the following diagrams commute:*

$$\begin{array}{ccc}
 X \xrightarrow{f} Y & & X_2 \xrightarrow{\langle \pi_0 f, \pi_1 f \rangle} Y_2 \\
 p \downarrow & & + \downarrow \\
 A \xrightarrow{g} B & & X \xrightarrow{f} Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 A \xrightarrow{g} B & & A \xrightarrow{g} B \\
 0 \downarrow & & \downarrow 0 \\
 X \xrightarrow{f} Y & & X \xrightarrow{f} Y
 \end{array}$$

It is important to note that even though these maps may be partial, we still ask that the diagrams commute on the nose (rather than with an inequality).

Proposition 6.13 *If \mathbb{X} is a restriction category, then with the obvious composition, restriction, and identities, additive bundles in \mathbb{X} and their morphisms form a restriction category. If \mathbb{X} has joins, so does this category.*

PROOF: For composites, we define $(f_1, g_1) \circ (f_2, g_2) := (f_1 f_2, g_1 g_2)$. For such a map, the first and third diagrams for an additive bundle morphism obviously commute, while the second diagram commutes since

$$\begin{aligned} & \langle \pi_0 f_1 f_2, \pi_1 f_1 f_2 \rangle + \\ &= \langle \pi_0 f_1, \pi_1 f_2 \langle \pi_1 f_2, \pi_2 f_2 \rangle \rangle + \text{(by the universal property of } \langle, \rangle \text{)} \\ &= \langle \pi_0 f_1, \pi_1 f_2 \rangle + f_2 \text{ (since } f_2 \text{ is an additive bundle morphism)} \\ &= f_1 f_2 + \text{(since } f_1 \text{ is an additive bundle morphism)} \end{aligned}$$

as required.

For restrictions, we define $\overline{(f, g)} := (\overline{f}, \overline{g})$. This satisfies the first diagram:

$$p\overline{g} = \overline{p}g = \overline{f}q = \overline{f}p$$

since q is total. For the second diagram,

$$\begin{aligned} +\overline{f} &= \overline{+f} + \\ &= \overline{\langle \pi_0 f, \pi_1 f \rangle} + \\ &= \overline{\pi_0 f \pi_1 f} + \text{(since } + \text{ is total)} \\ &= \overline{\pi_0 f \pi_0, \pi_1 f \pi_1} + \\ &= \langle \pi_0 \overline{f}, \pi_1 \overline{f} \rangle + \end{aligned}$$

as required. For the last diagram,

$$0\overline{f} = \overline{0f} = \overline{g}0 = \overline{g}0$$

since 0 is total.

If \mathbb{X} has joins, we define $\bigvee_i (f_i, g_i) := (\bigvee_i f_i, \bigvee_i g_i)$. The first and third diagrams commute since joins preserve composition. The second diagram requires slightly more care. We first show that for any map (f, g) from $p : X \rightarrow A$ to $q : Y \rightarrow B$, $\overline{\pi_0 f} = \overline{\pi_1 f}$:

$$\begin{aligned} \overline{\pi_0 f} &= \overline{\pi_0 f p} \text{ (since } p \text{ is total)} \\ &= \overline{\pi_0 p g} \text{ (by the first diagram for } (f, g)) \\ &= \overline{\pi_1 p g} \\ &= \overline{\pi_1 f p} \text{ (by the first diagram for } ((f, g)) \\ &= \overline{\pi_1 f} \text{ (since } p \text{ is total)} \end{aligned}$$

Now, we need to show that the pair $(\bigvee_i f_i, \bigvee_j f_j)$ satisfies the second diagram to be an additive

bundle morphism:

$$\begin{aligned}
\langle \pi_0 \bigvee_i f_i, \pi_1 \bigvee_j f_j \rangle + &= \bigvee_{i,j} \langle \pi_0 f_i, \pi_1 f_j \rangle + \\
&= \bigvee_{i,j} \langle \pi_0 f_i, \overline{\pi_1 f_i} \pi_1 f_j \rangle + \\
&= \bigvee_{i,j} \langle \pi_0 f_i, \overline{\pi_1 f_i} \pi_1 f_j \rangle + \text{ (by the result above)} \\
&= \bigvee_{i,j} \langle \pi_0 f_i, \overline{\pi_1 f_j} \pi_1 f_i \rangle + \text{ (since } f_i \smile f_j \text{)} \\
&= \bigvee_{i,j} \overline{\pi_2 f_j} \langle \pi_0 f_i, \pi_1 f_i \rangle + \\
&= \bigvee_i \langle \pi_0 f_i, \pi_1 f_i \rangle + \\
&= \bigvee_i +f_i = + \bigvee_i f_i.
\end{aligned}$$

□

With the restriction version of additive bundles defined, we turn to the restriction version of tangent structure. There are three main differences: (i) the tangent functor should preserve restrictions, (ii) the pullbacks and equalizers must be restriction pullbacks and equalizers, and (iii) all natural transformations must be total.

Definition 6.14 *Tangent restriction structure* for a restriction category \mathbb{X} consists of a restriction preserving functor $T : \mathbb{X} \rightarrow \mathbb{X}$ and associated total transformations such that:

- (**tangent bundle is additive**) for each $M \in \mathbb{X}$, TM has the structure of an additive bundle over M ; so we have total maps $p_M : TM \rightarrow M$, restriction pullbacks $T_n(M)$, and total maps $+_M : T_2M \rightarrow TM$, $0_M : M \rightarrow TM$; we also ask that for each $f : M \rightarrow N$, the pair (Tf, f) is an additive bundle morphism;
- (**preservation of restriction pullbacks**) for each $n, k \in \mathbb{N}$, T^n preserves the restriction pullbacks of k copies of p ;
- (**vertical lift**) there is a total natural transformation $T \xrightarrow{\ell} T^2$ such that for each M , the pair $(\ell_M, 0_M)$ is an additive bundle morphism from $(p : TM \rightarrow M)$ to $(Tp : T^2M \rightarrow TM)$;
- (**canonical flip**) there is a total natural transformation $T^2 \xrightarrow{c} T^2$ such that for each M , the pair $(c_M, 1)$ is an additive bundle morphism from $(Tp : T^2M \rightarrow TM)$ to $(p_T : T^2M \rightarrow TM)$;
- (**coherence of ℓ and c**) we have $c^2 = 1$, $\ell c = \ell$, and the following diagrams commute:

$$\begin{array}{ccccc}
T & \xrightarrow{\ell} & T^2 & & T^3 & \xrightarrow{T(c)} & T^3 & \xrightarrow{c_T} & T^3 & & T^2 & \xrightarrow{\ell_T} & T^3 & \xrightarrow{T(c)} & T^3 \\
\ell \downarrow & & \downarrow T(\ell) & & c_T \downarrow & & \downarrow T(c) & & \downarrow c_T & & c \downarrow & & \downarrow c_T \\
T^2 & \xrightarrow{\ell_T} & T^3 & & T^3 & \xrightarrow{T(c)} & T^3 & \xrightarrow{c_T} & T^3 & & T^2 & \xrightarrow{T(\ell)} & T^3 & & T^3
\end{array}$$

- (**universality of vertical lift**) the map

$$T_2M \xrightarrow{v:=\langle\pi_0\ell,\pi_10_T\rangle T(+)} T^2M$$

is the restriction equalizer of

$$T^2M \begin{array}{c} \xrightarrow{T(p)} \\ \xrightarrow{T(p)p0} \end{array} TM$$

A restriction category with tangent structure is a **tangent restriction category**.

If \mathbb{X} has restriction products and T preserves them, then (\mathbb{X}, \mathbb{T}) is a **Cartesian tangent restriction category**.

Note that the total maps of any tangent restriction category is always an (ordinary) tangent category. Thus, this definition strictly subsumes the definition of a tangent category.

Due to the naturality of p , restriction tangent structure automatically preserves joins, if they exist.

Proposition 6.15 *If (\mathbb{X}, \mathbb{T}) is restriction tangent category, then:*

- (i) $\overline{Tf} = \overline{pf}$.
- (ii) *If \mathbb{X} has joins, then T preserves them.*

PROOF:

(i) since p is total, $\overline{T(f)} = \overline{T(f)p} = \overline{pf}$ by naturality of p .

(ii) Consider:

$$\begin{aligned} \bigvee_{i \in I} T(f_i) &= \bigvee_{i \in I} T \left(\bigvee_{j \in I} f_j \right) \quad (\text{since } f_i \leq \bigvee_{j \in I} f_j) \\ &= \bigvee_{i \in I} \overline{T(f_i)} T \left(\bigvee_{j \in I} f_j \right) \quad (\text{since } T \text{ is a restriction functor}) \\ &= \bigvee_{i \in I} \overline{pf_i} T \left(\bigvee_{j \in I} f_j \right) \quad (\text{by (i)}) \\ &= \overline{\bigvee_{i \in I} pf_i} T \left(\bigvee_{j \in I} f_j \right) \\ &= \overline{p \bigvee_{i \in I} f_i} T \left(\bigvee_{j \in I} f_j \right) \\ &= T \left(\bigvee_{i \in I} f_i \right) T \left(\bigvee_{j \in I} f_j \right) \quad (\text{by (i)}) \\ &= T \left(\bigvee_{i \in I} f_i \right) \end{aligned}$$

as required. □

We now turn to the promised connection between differential restriction categories and tangent restriction structure.

Proposition 6.16 *Any differential restriction category has a tangent restriction structure given by:*

$$TM := M \times M, Tf := \langle Df, \pi_1 f \rangle$$

with:

- $p := \pi_1$;
- $T_n(M) := M \times M \dots \times M$ ($n + 1$ times);
- $+\langle a, b, c \rangle := \langle a + b, c \rangle$, $0(a) := \langle 0, a \rangle$;
- $l(\langle a, b \rangle) := \langle \langle a, 0 \rangle, \langle 0, b \rangle \rangle$;
- $c(\langle \langle a, b \rangle, \langle c, d \rangle \rangle) := \langle \langle a, c \rangle, \langle b, d \rangle \rangle$.

PROOF: The proof is almost identical to that of proposition 4.7, as for most axioms the restrictions stay hidden. For example, consider the proof of **[DR.5]**:

$$T(f)T(g) = \langle Df, \pi_1 f \rangle \langle Dg, \pi_1 g \rangle = \langle D(fg), \pi_1 fg \rangle = T(fg)$$

When we consider the term $\langle Df, \pi_1 f \rangle \pi_1$, the result is $\overline{Df} \pi_1 f$; but since $\overline{Df} = \overline{\pi_1 f}$, the \overline{Df} term vanishes. A similar phenomenon occurs with the other axioms.

Since **[DR.2]** is different, we demonstrate the proof of naturality of 0:

$$0_M T(f) = \langle 0, 1 \rangle \langle Df, \pi_1 f \rangle = \langle \overline{f} 0, f \rangle = f \langle 0, 1 \rangle = f 0_N.$$

Finally, T preserves restrictions since:

$$\begin{aligned} T(\overline{f}) &= \langle D(\overline{f}), \pi_1 \overline{f} \rangle = \langle \overline{\pi_1 f} \pi_0, \overline{\pi_1 f} \pi_1 \rangle = \overline{\pi_1 f} \\ \overline{T(f)} &= \overline{\langle D(f), \pi_1 f \rangle} = \overline{D(f)} \overline{\pi_1 f} = \overline{\pi_1 f}. \end{aligned}$$

□

Just as in the total case, restriction tangent structure on differential structure gives a differential restriction category.

Definition 6.17 *An object M in a Cartesian restriction tangent category has **differential structure** if it has a total map $\hat{p} : TM \rightarrow M$ such that*

$$M \xleftarrow{\hat{p}} TM \xrightarrow{p} M$$

is a restriction product diagram, and M has the structure of a (total) commutative monoid $\sigma : M \times M \rightarrow M$, $\zeta : 1 \rightarrow M$ that is compatible with the addition and zero of the tangent structure; that is, such that

$$\begin{array}{ccc} M & \xrightarrow{0_M} & TM \\ \downarrow ! & & \downarrow \hat{p} \\ 1 & \xrightarrow{\zeta} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} T_2M & \xrightarrow{+_M} & TM \\ \downarrow \langle \pi_0 \hat{p}, \pi_1 \hat{p} \rangle & & \downarrow \hat{p} \\ M \times M & \xrightarrow{\sigma} & M \end{array}$$

commute.

Theorem 6.18 *Suppose (\mathbb{X}, \mathbb{T}) is a Cartesian restriction tangent category. Let $\text{Diff}(\mathbb{X}, \mathbb{T})$ denote the restriction category whose objects are differential structures, with a map from $(A, \widehat{p}_A, \sigma_A, \zeta_A)$ to $(B, \widehat{p}_B, \sigma_B, \zeta_B)$ simply consisting of a map $f : A \rightarrow B$. Then:*

- (i) $\text{Diff}(\mathbb{X}, \mathbb{T})$ is a Cartesian left additive restriction category;
- (ii) $\text{Diff}(\mathbb{X}, \mathbb{T})$ is a differential restriction category, with $D(f)$ given by

$$A \times A \xrightarrow{\langle \pi_0, \pi_1 \rangle} TA \xrightarrow{T(f)} TB \xrightarrow{\widehat{p}_B} B.$$

PROOF: Again, most of the theorem is similar to the total case. By naturality of p , we can determine that:

$$T(f) = \langle T(f)\pi_0, T(f)\pi_1 \rangle = \langle Df, \pi_1 f \rangle$$

Similarly,

$$T_2(f) = \langle \langle \pi_0, \pi_1 \pi_0 \rangle D(f), \langle \pi_1 D(f), \pi_1 \pi_1 f \rangle \rangle$$

and

$$D^2(f) = T(D(f))\pi_0 = T(T(f)\pi_0)\pi_0 = T^2(f)T(\pi_0)\pi_0 = T^2(f)\langle \pi_0\pi_0, \pi_1\pi_0 \rangle\pi_0 = T^2(f)\pi_0\pi_0.$$

For [DR.8] we have $D(\bar{f}) = T(\bar{f})\pi_0 = \overline{\pi_1 f} \pi_0$ and for [DR.9] we have $\overline{D(f)} = \overline{T(f)\pi_0} = \overline{T(f)} = \overline{\pi_1 f}$. For [DR.2], by naturality of 0 , we have:

$$\begin{aligned} \langle 0, a \rangle D(f) &= \langle 0, a \rangle T(f)\pi_0 = a \langle 0, 1 \rangle T(f)\pi_0 = a 0_N T(f)\pi_0 \\ &= a f 0_M \pi_0 = a f \langle 0, 1 \rangle \pi_0 = \overline{a f} 0. \end{aligned}$$

□

6.3 Manifolds and tangent Structure

In the previous section, we showed that a differential restriction category is canonically a restriction tangent category. Our goal now is to show that the manifold completion of a category with restriction tangent structure also has restriction tangent structure. In particular, this shows that the total category of such a category of manifolds has (total) tangent structure.

We begin by briefly recalling the notion of the manifold completion of a join restriction category as introduced in [Grandis 1989].

Definition 6.19 Let \mathbb{X} be a join restriction category. An **atlas** in \mathbb{X} consists of a family of objects $(X_i)_{i \in I}$ of \mathbb{X} , together with, for each $i, j \in I$, a map $\phi_{ij} : X_i \rightarrow X_j$ such that for each $i, j, k \in I$,

[Atl. 1] $\phi_{ii}\phi_{ij} = \phi_{i,j}$ (partial charts);

[Atl. 2] $\phi_{ij}\phi_{jk} \leq \phi_{ik}$ (cocycle condition);

[Atl. 3] ϕ_{ij} is the partial inverse of ϕ_{ji} .

Definition 6.20 Suppose (X_i, ϕ_{ij}) and (Y_k, ψ_{kh}) are atlases in \mathbb{X} . An **atlas map**

$$A : (X_i, \phi_{ij}) \rightarrow (Y_k, \psi_{kh})$$

is a family of maps $X_i \xrightarrow{A_{ik}} Y_k$ such that

[AtlM. 1] $\phi_{ii}A_{ik} = A_{ik}$;

[AtlM. 2] $\phi_{ij}A_{jk} \leq A_{ik}$,

[AtlM. 3] $A_{ik}\psi_{kh} = \overline{A_{ik}} A_{ih}$.

Morphisms of atlases are composed by matrix composition. Given atlas maps

$$U \xrightarrow{A} V \xrightarrow{B} W$$

we define $(AB)_{im} = \bigvee_h A_{ih}B_{hm}$. The identity map for an atlas is the atlas itself. There is a restriction given by

$$\overline{A}_{ij} = \left(\bigvee_h \overline{A}_{ih} \right) \phi_{ij}.$$

Theorem 6.21 (Grandis) If \mathbb{X} is a join restriction category, then $\mathbf{Mf}(\mathbb{X})$, with objects atlases, morphisms atlas maps, and composition, identities, and restriction as described above, is a join restriction category.

The following is easily checked:

Proposition 6.22 \mathbf{Mf} is an endofunctor on join restriction categories and join preserving restriction functors, where

$$\mathbf{Mf}(F)(U_i, \phi_{ij}) := (F(U_i), F(\phi_{ij})),$$

and

$$\mathbf{Mf}(F)(A_{ik}) = (F(A_{ik})).$$

Moreover, if $F \xrightarrow{\alpha} G$ is natural, then we get a natural transformation from $\mathbf{Mf}(F)$ to $\mathbf{Mf}(G)$ by

$$(F(U_i), F(\phi_{ij})) \xrightarrow{F(\phi_{ij})\alpha_j = \alpha_i G(\phi_{ij})} (G(U_i), G(\phi_{ij}))$$

so that \mathbf{Mf} is a 2-functor. If α is total, then $\mathbf{Mf}(\alpha)$ is as well.

Thus, since we have a 2-functor, applying \mathbf{Mf} to a tangent functor $T : \mathbb{X} \rightarrow \mathbb{X}$ gives all of the equational properties of tangent structure. The only thing left to check, then, is that $(\mathbf{Mf}(\mathbb{X}), \mathbf{Mf}(T))$ has the required restriction pullbacks and equalizers. For this, recall from lemma 2.12 that $v : T_2M \rightarrow T^2M$ being the equalizer of $T(p)$ and $T(p)p0$ is equivalent to asking that

$$\begin{array}{ccc} T_2M & \xrightarrow{v} & T^2M \\ \pi_0 p \downarrow & & \downarrow T(p) \\ M & \xrightarrow{0} & TM \end{array}$$

is a pullback (and the proof generalizes to the restriction case). Thus, we only need to check that \mathbf{Mf} lifts certain functorial restriction pullbacks.

We begin by recording some useful results regarding these types of restriction pullbacks.

Proposition 6.23 *Suppose that we have functors $F, G, H : \mathbb{X} \rightarrow \mathbb{Y}$ between restriction categories, natural transformations $\alpha : F \rightarrow H, \beta : G \rightarrow H$, and for each $X \in \mathbb{X}$, there is an object $PX \in \mathbb{Y}$ and maps $l_X : PX \rightarrow FX, r_X : PX \rightarrow GX$ so that*

$$\begin{array}{ccc} PX & \xrightarrow{r_X} & GX \\ l_X \downarrow & & \downarrow \beta_X \\ FX & \xrightarrow{\alpha_X} & HX \end{array}$$

is a restriction pullback. Then:

- (i) P is a restriction functor, with $P(f) := \langle l_X F(f), r_X G(f) \rangle$;
- (ii) if F and G preserve joins, then so does P ;
- (iii) if both α and β are total, then l and r are natural.

PROOF:

- (i) First, we need to check $P(f)$ is well-defined; that is, we need $l_X F(f) \alpha_Y \smile r_X G(f) \beta_Y$. In fact, they are equal as $l_X F(f) \alpha_Y = l_X \alpha_X f = r_X \beta_X f = r_X G(f) \beta_Y$. Clearly P is functorial, as

$$\begin{aligned} P(f)P(g) &= \langle l_X F(f), r_X G(f) \rangle \langle l_Y F(g), r_Y G(g) \rangle \\ &= \langle l_X F(f)F(g), r_X G(f)G(g) \rangle \\ &= \langle l_X F(fg), r_X G(fg) \rangle \\ &= P(fg) \end{aligned}$$

and

$$P(1) = \langle l_X F(1), r_X G(1) \rangle = \langle l_X, r_X \rangle = 1.$$

As F and G preserve restrictions, then

$$\begin{aligned} P(\bar{f}) &= \langle l_X F(\bar{f}), r_X G(\bar{f}) \rangle \\ &= \langle \overline{l_X F(f)}, \overline{r_X G(f)} \rangle \\ &= \overline{l_X F(f) r_X G(f)} \langle l_X, r_X \rangle \\ &= \overline{l_X F(f) r_X G(f)} \\ &= \overline{\langle l_X F(f), r_X G(f) \rangle} \\ &= \overline{P(f)} \end{aligned}$$

so P is a retraction functor.

(ii) If F and G preserves joins, then so does P , as restriction pullbacks are easily seen to preserve joins.

(iii) If l_X and r_X are total, we first show that $\overline{r_X F(f)} = \overline{l_X G(f)}$:

$$\begin{aligned} \overline{r_X G(f)} &= \overline{r_X G(f) \beta_Y} \text{ since } \beta \text{ is total,} \\ &= \overline{r_X \beta_X H(f)} \text{ by naturality of } \beta, \\ &= \overline{l_X \alpha_X H(f)} \\ &= \overline{r_X F(f) \alpha_Y} \text{ by naturality of } \alpha, \\ &= \overline{r_X F(f)} \text{ since } \alpha \text{ is total.} \end{aligned}$$

Then l is natural since

$$P(f)l_Y = \langle l_X F(f), r_X G(f) \rangle l_Y = \overline{r_X G(f)} l_X F(f) = l_X F(f),$$

and similarly for r .

□

We now check that \mathbf{Mf} lifts the required pullbacks.

Proposition 6.24 *Suppose we have all the conditions of the previous proposition, and \mathbb{X} has joins. Then for any object $M = (U_i, \phi_{ij})$ in $\mathbf{Mf}(\mathbb{X})$, the diagram*

$$\begin{array}{ccc} \mathbf{Mf}(\mathbb{X})(P)(M) & \longrightarrow & \mathbf{Mf}(\mathbb{X})(G)(M) \\ \downarrow & & \downarrow \\ \mathbf{Mf}(\mathbb{X})(F)(M) & \longrightarrow & \mathbf{Mf}(\mathbb{X})(H)(M) \end{array}$$

is also a restriction pullback.

PROOF: The diagram commutes since \mathbf{Mf} is a functor. Thus, it suffices to show the universal property. Suppose we have

$$\begin{array}{ccccc} (V_m, \psi_{mn}) & & & & \\ & \searrow^B & & & \\ & & (PU_i, P\phi_{ij}) & \xrightarrow{\mathbf{Mf}(r)} & (GU_i, G\phi_{ij}) \\ & & \downarrow \mathbf{Mf}(l) & & \downarrow \mathbf{Mf}(\alpha) \\ & \searrow^A & (FU_i, F\phi_{ij}) & \xrightarrow{\mathbf{Mf}(\alpha)} & HU_i, H\phi_{ij} \end{array}$$

so that $A\mathbf{Mf}(\alpha) \smile B\mathbf{Mf}(\beta)$. Now, compatibility implies pointwise compatibility, so we have

$$A\mathbf{Mf}(\alpha)_{mk} \smile B\mathbf{Mf}(\beta)_{mk}$$

for each m and k . By the lemma about $\mathbf{Mf}(\alpha)$, this gives

$$A_{mk}\alpha_k \smile B_{mk}\alpha_k.$$

Then by the universal property of the pullback in \mathbb{Y} , we know there exists a map $V_m \xrightarrow{\langle A_{mk}, B_{mk} \rangle} PU_k$. We claim these maps together form a manifold map. For ATM2,

$$\psi_{mn}\langle A_{nk}, B_{nk} \rangle = \langle \psi_{mn}A_{nk}, \psi_{mn}B_{nk} \rangle \leq \langle A_{mk}, B_{mk} \rangle,$$

and ATM1 is similar. For ATM3,

$$\begin{aligned} \langle A_{mk}, B_{mk} \rangle P\phi_{kj} &= \langle A_{mk}, B_{mk} \rangle \langle l_k F(\phi_{kj}), r_k G(\phi_{kh}) \rangle \text{ by definition of } P, \\ &= \langle A_{mk} F(\phi_{kj}), B_{mk} G(\phi_{kj}) \rangle \\ &= \langle \overline{A_{mk}} A_{mj}, \overline{B_{mk}} B_{mj} \rangle \text{ by ATM3 for A and B,} \\ &= \overline{A_{mk}} \overline{B_{mk}} \langle A_{mj}, B_{mj} \rangle \\ &= \langle \overline{A_{mk}}, \overline{B_{mk}} \rangle \langle A_{mj}, B_{mj} \rangle \end{aligned}$$

so it is a manifold map.

For its restriction, recall that compatibility of $\mathbf{AMf}(\alpha)$ and $\mathbf{BMf}(\beta)$ also implies that we have

$$\bigvee_i \overline{(\mathbf{AMf}(\alpha))_{mi}} \overline{(\mathbf{BMf}(\beta))_{mj}} = \bigvee_i \overline{(\mathbf{BMf}(\beta))_{mi}} \overline{(\mathbf{AMf}(\alpha))_{mj}}$$

that is,

$$\bigvee_i \overline{A_{mi}\alpha_i} \overline{B_{mj}\alpha_j} = \bigvee_i \overline{B_{mi}\beta_i} \overline{A_{mj}\beta_j},$$

but since α and β are total, this reduces to

$$\bigvee_i A_{mi} B_{mj} = \bigvee_i B_{mi} A_{mj}.$$

Now, we want to show that $\overline{\langle A, B \rangle}_{mn} = \overline{(\overline{A} \overline{B})}_{mn}$. Indeed, consider

$$\begin{aligned} \overline{(\overline{A} \overline{B})}_{mn} &= \bigvee_i \overline{A_{mi}} \bigvee_j \overline{B_{mj}} \psi_{mn} \text{ by definition of manifold map restriction,} \\ &= \bigvee_i \overline{A_{mi}} \bigvee_j \overline{B_{mj}} \overline{A_{mi}} \psi_{mn} \\ &= \bigvee_i \overline{A_{mi}} \bigvee_j \overline{A_{mj}} \overline{B_{mi}} \psi_{mn} \text{ by the above calculation,} \\ &= \bigvee_i \overline{A_{mi}} \overline{B_{mi}} \psi_{mn} \\ &= \bigvee_i \overline{\langle A_{mi}, B_{mi} \rangle} \psi_{mn} \\ &= \overline{\langle A, B \rangle}_{mn} \end{aligned}$$

as required.

Finally, suppose that we have some manifold map $(V_m, \psi_{mn}) \xrightarrow{C} (PU_i, P\phi_{ij})$ such that

$$CMf(l) \leq A \text{ and } CMf(r) \leq B.$$

This gives, for each m and i ,

$$C_{mi}l_i \leq A_{mi} \text{ and } C_{mi}r_i \leq B_{mi},$$

so that by the universal property of the pullback in \mathbb{Y} , we have

$$C_{mi} \leq \langle A_{mi}, B_{mi} \rangle$$

so that $C \leq \langle A, B \rangle$. □

Thus, we have:

Corollary 6.25 *If (\mathbb{X}, T) is join restriction category with restriction tangent structure, then the pair $(Mf(\mathbb{X}), Mf(T))$ is also.*

In particular:

Corollary 6.26 *If \mathbb{X} is a join differential restriction category, then the total maps of $(Mf(\mathbb{X}), Mf(T))$ form a category with tangent structure.*

For example, applied to the join differential restriction category of smooth maps between Cartesian spaces, this gives the classical category of finite-dimensional smooth manifolds. This is discussed further in the next section. In particular we briefly show that the tangent functor arrived at through the above process coincides with the usual definition of the tangent bundle.

It is also useful to note that the result above actually applies to any differential restriction category: if \mathbb{X} does not have joins but has differential structure, then one can always join complete \mathbb{X} without destroying the differential structure (see [Cockett et al. 2011], section 5)

6.4 Comparison with other tangent bundle functor definitions

Finally, we give a brief comparison of the tangent bundle obtained, as above, through the manifold completion with the definitions of the tangent bundle of a smooth manifold often given in the literature.

Let M be a smooth manifold. The more geometric of the two standard definitions of the tangent bundle is the following:

Definition 6.27 (*Kinematic tangent bundle*) *If V is a vector space, a **kinematic tangent vector** is an equivalence class of smooth curves $f : \mathbb{R} \rightarrow V$ with $f \sim g$ if $f(0) = g(0)$ and $f'(0) = g'(0)$. The set of all kinematic tangent vectors forms the **kinematic tangent bundle** KM . Given a smooth map $f : X \rightarrow Y$, one defines $Kf : KX \rightarrow KY$ by $Kf(c) := cf$.*

The idea is that a tangent vector at a is an infinitesimally small curve through a ⁸.

However, this geometric definition is equivalent to the “local product” definition of the tangent structure of a differential restriction category that we gave above, which here we notate by T .

⁸Of course, synthetic differential geometry formalizes this by including infinitesimal spaces D , and defining a tangent vector to be a map $f : D \rightarrow X$.

Proposition 6.28 *In the category of smooth maps between Cartesian spaces, $K = T$.*

PROOF: Given a kinematic tangent vector $f : \mathbb{R} \rightarrow X$, we define a pair of elements of X by $(D(f)(1, 0), f(0))$. Given a pair of elements (x, a) of M , we define a kinematic tangent vector f by $f(r) := f(0) + r \cdot x$. It is clear that these two definitions are well-defined inverses of one another.

For the action on maps, the local product definition gives us that the result of applying $T(f)$ to c is

$$\langle c'(0), c(0) \rangle \langle Df, \pi_1 f \rangle$$

while the kinematic definition gives us

$$\langle (cf)'(0), f(c(0)) \rangle$$

(where $g'(x) = Df(1, x)$). These two definitions then agree by the chain rule. \square

Since these definitions agree on the base category, they also agree on the categories of manifolds, and hence we have:

Corollary 6.29 *In the category of smooth manifolds, $K = T$.*

Another standard bundle functor is the “operational” tangent bundle. For a smooth manifold M , let $C^\infty(M)$ denote the vector field of smooth maps from M to \mathbb{R} .

Definition 6.30 *Let x be a point in a smooth manifold M . An **operational tangent vector at x** is a linear map $\alpha : C^\infty(M) \rightarrow \mathbb{R}$ which satisfies*

$$\alpha(fg) = \alpha(f) \cdot g(x) + \alpha(g) \cdot f(x).$$

(These are known as linear derivations). The set of all operational tangent vectors over all points of M forms the **operational tangent bundle** DM . Given a smooth map $f : M \rightarrow N$, one defines $D(f)(\alpha)$ as $C^\infty(f)\alpha$.

This “functional analysis” version of the tangent bundle is popular because it is typically easier to manipulate. Unfortunately, for the level of abstraction at which we are working this is a problematic definition. First, it requires a base object \mathbb{R} , and so is impossible to define in a general differential restriction category. However, even when this object exists, as, for example, in the category of smooth maps between convenient vector spaces, this definition is not equivalent to the kinematic definition. There is an obvious map from KM to DM : given a curve c , one can define

$$\alpha_c(f) := cf'(0)$$

which is easily checked to be a linear derivation. However, in general this map is only invertible under special circumstances: for more information, see 28.7 of [Kriegl and Michor 1997]. Furthermore, D does not preserve all products (see the remarks after 28.16 in [Kriegl and Michor 1997]), so that it is not an example of Cartesian differential structure.

Thus, the kinematic tangent bundle definition seems to be the appropriate definition for generalizations of the category of smooth manifolds. However, it would be interesting to know more precisely what role the operational tangent bundle plays.

7 Concluding remarks

We initially developed our ideas on tangent structure in complete ignorance of Rosický’s work. Our motivation was to explain the residual differential structure which is inherited by the manifolds of a differential restriction category. We are in debt to Anders Kock for drawing our attention to Rosický’s paper: we had been struggling with the axiomatization of tangent structure. The definition of tangent structure we had arrived at was close to Rosický’s, but significantly we had not demanded all the coherence conditions. In particular, while we had become aware that the universality of the vertical lift implied that tangent spaces had differential structure, we did not realize the importance of this universality in other contexts, and had not included it in our axiomatization. Reading Rosický’s paper, however, immediately put beyond doubt the importance of this condition. His paper also clarified the relationship to synthetic differential geometry, which had been quite unclear to us.

With this changed perspective, we decided that it would be valuable to provide a paper which gave a broader viewpoint on Rosický’s axiomatization. While Rosický’s paper describes with masterful brevity the basic consequences of the theory, it left to the reader the details of most of the proofs. Here we have filled in many of these details.

In particular, this led to us developing further some of the aspects of the relationship between synthetic differential geometry and representable tangent structure. Rosický initiated these ideas, and we simply pursued them to give a tighter correspondence. The perspective of representable tangent structure strongly suggests that infinitesimal objects (rather than the rig of line type) be regarded as primary. This is hardly a new idea, as it can certainly be found in Lawvere’s writings (see, for example [Lawvere 2011]). Here we did manage to suggest a way of formalizing a “synthetic model” in a manner which relies only on the structure of infinitesimal objects. However, our formalism had a very limited aim: namely to support the basic requirements we had identified for having an infinitesimal object. It did not address the larger question of characterizing at this level of abstraction the more general use of Weil objects (or algebras) in these settings (see [Nishimura 2012]) and in this regard there is still work to be done.

For an infinitesimal object we identified three potentially different line objects. However, we did not exhibit any example in which these line objects are distinct. Our feeling is that there should be such models and that it would be nice to have separating examples. Furthermore in our synthetic models we did not suppose $\mathcal{D}(n) \subseteq \mathcal{D}(1)^n$, but again we did not exhibit any examples where these maps are not monic. Lastly, we supposed that the infinitesimal object need not be all the 2-nilpotent elements of the rig, but we did not provide an example of when it is not. Thus, there remain many open issues with these models.

A contribution of this work has been to connect tangent structure to differential structure in a manner which was simply not possible when Rosický wrote his paper. The theory of differential categories was partly motivated from computer science and combinatorics. It is to be hoped that by revealing structural relationships between these systems and those used in more traditional areas of mathematics – e.g. differential geometry – that this will stimulate further synergistic developments.

Our original motivation for this work was to explore the effect of applying the manifold construction to a differential restriction category. Indeed, that this construction does produce tangent structure is a contribution of this work. As this construction covers all the standard atlas-based manifold constructions, it brings into the reach of this theory the majority of “standard” examples from differential geometry. Significantly, it also covers a wide variety of examples which have

not been explored. In particular, additive differential manifolds are still a completely unexplored subject.

Using Rosický's axiomatization as a basis we have been able to collect under one axiomatization a wide variety of differential structures encompassing almost all the major approaches to differential geometry. There are many more examples which might yet be collected under this umbrella – and, undoubtedly, there are many which do not fit. Both will be useful to circumscribe the reach of these ideas.

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