Differential forms in non-linear Cartesian differential categories

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FMCS 2018
Sackville, Canada, June 1, 2018

Overview

- Cartesian differential categories (CDCs)
- Non-linear CDCs
  - Motivation for non-linear CDCs
  - Examples
  - Constructions on categories
- Differential Forms
  - New definition for non-linear CDCs
- Cohomology
  - Examples
- Non-linear tangent categories

Cartesian differential category definition

Definition (Blute, Cockett, Seely, 2009)

A **Cartesian differential category** is a left additive category with chosen products which has, for each map $f : X \to Y$, a map

$$D(f) : X \times X \longrightarrow Y$$

such that:

[CD.1] $D(f + g) = D(f) + D(g)$ and $D(0) = 0$;

[CD.2] $\langle a + b, c\rangle D(f) = \langle a, c\rangle D(f) + \langle b, c\rangle D(f)$ and $\langle 0, a\rangle D(f) = 0$;

[CD.3] $D(\pi_0) = \pi_0\pi_0$, $D(\pi_1) = \pi_0\pi_1$, and $D(1) = \pi_0$;

[CD.4] $D(\langle f, g\rangle) = \langle D(f), D(g)\rangle$;

[CD.5] $D(fg) = \langle D(f), \pi_1 f\rangle D(g)$ ("Chain rule");

[CD.6] $\langle\langle a, 0\rangle, \langle 0, d\rangle\rangle D(D(f)) = \langle a, d\rangle D(f)$;

[CD.7] $\langle\langle a, b\rangle, \langle c, d\rangle\rangle D(D(f)) = \langle\langle a, c\rangle, \langle b, d\rangle\rangle D(D(f))$.

Examples of CDCs

Example

- **Smooth** is a CDC, with $\langle v, x\rangle D(f) = [J(f(x))] \cdot v$, where $[J(f)]$ is the Jacobian of f.
- **Poly** is a CDC, with the same derivative
  - **Poly** has the $\mathbb{R}^n$s as objects and polynomial functions as arrows
- The category of abelian groups with group homomorphisms as arrows, with $\langle v, x\rangle D(f) = f(v)$.

The forward difference operator

- The category $\mathbf{ab}_{fun}$ (objects: abelian groups, arrows: functions) with $\langle v, x\rangle D(f) = f(x+v) - f(x)$ is not an example of a CDC.
- It satisfies every axiom except for the first part of **[CD.2]**.
- What kind of structure do we get if we simply remove the first part of **[CD.2]**?

Non-linear Cartesian differential category definition

Definition (Bradet-Legris, Cruttwell, Reid)

A **non-linear Cartesian differential category** is a left additive category with chosen products which has, for each map $f : X \to Y$, a map

$$D(f) : X \times X \longrightarrow Y$$

such that:

[NLCD.1] $D(f+g) = D(f) + D(g)$ and $D(0) = 0$;

[NLCD.2] $\langle 0, a \rangle D(f) = 0$;

[NLCD.3] $D(\pi_0) = \pi_0\pi_0$, $D(\pi_1) = \pi_0\pi_1$, and $D(1) = \pi_0$;

[NLCD.4] $D(\langle f, g \rangle) = \langle D(f), D(g) \rangle$;

[NLCD.5] $D(fg) = \langle D(f), \pi_1 f \rangle D(g)$;

[NLCD.6] $\langle\langle a, 0\rangle, \langle 0, d\rangle\rangle D(D(f)) = \langle a, d\rangle D(f)$;

[NLCD.7] $\langle\langle a, b\rangle, \langle c, d\rangle\rangle D(D(f)) = \langle\langle a, c\rangle, \langle b, d\rangle\rangle D(D(f))$.

A few examples of Non-linear CDCs

Example

- All CDCs are non-linear CDCs.
- The category $\mathbf{ab}_{fun}$, which has abelian groups as objects and functions as arrows, and the D arrow is $\langle v, x\rangle D(f) = f(x+v) - f(x)$.
- **Smooth**, but changing D to be $\langle v, x\rangle D(f) = f(x+v) - f(x)$.

Simple slice categories

Definition (Blute, Cockett, Seely, 2009)

For a category with products $\mathbb{X}$ and a fixed $A \in \mathbb{X}$, the following structure is called a **simple slice category**, and is denoted $\mathbb{X}[\mathbb{A}]$.

- objects: those of $\mathbb{X}$
- arrows: an arrow f from X to Y is an arrow $f : X \times A \to Y$
- composites: the composite $X \xrightarrow{f} Y \xrightarrow{g} Z$ is

  $$X \times A \xrightarrow{\langle \pi_0 f, \pi_1 \rangle} Y \times A \xrightarrow{g} Z$$
- identity: $1_X : X \times A \xrightarrow{\pi_0} X$

Simple slice categories results

Theorem (From Blute, Cockett, Seely, 2009)

Let $\mathbb{C}$ be a Cartesian differential category. Then $\mathbb{C}[A]$ is a Cartesian differential category, with D arrow $D_A(f) = \langle \pi_0, 0, \pi_1, \pi_2 \rangle D(f)$, where $D(f)$ is the D arrow for $\mathbb{C}$.

Theorem

Let $\mathbb{C}$ be a non-linear Cartesian differential category. Then $\mathbb{C}[A]$ is a non-linear Cartesian differential category, with D arrow $D_A(f) = \langle \pi_0, 0, \pi_1, \pi_2 \rangle D(f)$, where $D(f)$ is the D arrow for $\mathbb{C}$.

Idempotent splitting categories

Definition

An **idempotent** in a category is an arrow $e : X \to X$ such that $ee = e$.

Definition

The **idempotent splitting category** of a category $\mathbb{C}$, denoted $\text{Idem}(\mathbb{C})$ has

- objects: (X, e_X) where e_X is an idempotent on X.
- arrows: an arrow $f : (X, e_X) \to (Y, e_Y)$ is an arrow $f : X \to Y$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[d, "e_X"'] \arrow[drr, "f"] & & \\ X \arrow[r, "f"'] & Y \arrow[r, "e_Y"'] & Y \end{tikzcd}$$

- identity: $e_X : (X, e_X) \to (X, e_X)$ is the identity on (X, e_x).
- composites: defined as in $\mathbb{C}$.

Idempotent splitting categories results

Definition

In a Cartesian differential category, a map f is linear if $D(f) = \pi_0 f$.

Definition

The **linear idempotent splitting category** of a Cartesian differential category $\mathbb{C}$, denoted $idemLin(\mathbb{C})$, is the full subcategory of $idem(\mathbb{C})$ consisting of objects (X, e) such that e linear.

Theorem

Let $\mathbb{C}$ be a Cartesian differential category. Then $idemLin(\mathbb{C})$ is a Cartesian differential category, with the same D arrow as $\mathbb{C}$.

Idempotent splitting category results

Definition

The **non-linear idempotent splitting category** of a category $\mathbb{C}$, denoted $\mathrm{idemNLin}(\mathbb{C})$, is the full subcategory of $\mathrm{idem}(\mathbb{C})$ consisting of objects (X,e) such that e is linear and additive.

Theorem

Let $\mathbb{C}$ be a non-linear Cartesian differential category. The idemNLin($\mathbb{C}$) is a non-linear Cartesian differential category, with the same D arrow as $\mathbb{C}$.

Differential forms

- Differential forms and exterior differentiation for CDCs were defined by Cruttwell in 2013.
- This definition required the use of the linearity condition ([CD.2]) to prove the naturality of the exterior derivative.
- We needed a new definition for differential forms and exterior differentiation for the non-linear CDCs.

Important Definitions

Based on the definitions from [5]

(i) Functor $Q_n : \mathbb{X} \to \mathbb{X}$

(ii) Linear Objects

(iii) Non-linear differential forms

- quasi-multilinear (preserves the 0 map)
- skew-symmetric

(iv) Quasi exterior Derivative

Q Functor and Linear Objects

Definition. Given a non-linear Cartesian differential category $\mathbb{X}$, for any $n \geq 1$, there is an endofunctor $Q_n : \mathbb{X} \to \mathbb{X}$.

- given an object M in $\mathbb{X}$: $Q_n(M) = Q(M)^n \times M$ where $Q(M)^n = \underbrace{M \times M \times \ldots \times M}_{\text{n times}}$
- given a map $f : M \to M'$: $Q_n(f) = \langle \langle \pi_0, 0 \rangle D(f), \langle \pi_1, 0 \rangle D(f), \ldots, \langle \pi_{n-1}, 0 \rangle D(f), \pi_n f \rangle$

Definition. In a **Non-Linear Cartesian Differential Category**, say that an object A is *linear* if $Q(A) = A \times A$.

Non-Linear Differential Forms

Definition. For any $n \leq 1$ and $0 \leq i \leq n-1$, define the map $q_i : L(M) \times Q_n(M) \to Q(Q_n(M))$ by

$$q_i = \langle 0, 0, \ldots, 0, \pi_0, 0, \ldots, 0 | \pi_1, \ldots, \pi_i, 0, \pi_{i+2}, \ldots, \pi_{n+1} \rangle$$

For a map $f : T_n M \to A$, say f is **quasi-multilinear** if for all $0 \leq i \leq n-1$:

$$\begin{tikzcd}
M \times Q_n(M) \arrow[r, "q_i"] \arrow[d, "\langle \pi_1, \pi_2, \ldots, \pi_i, \pi_0, \pi_{i+2}, \ldots, \pi_{n+1} \rangle"'] & Q(Q_n(M)) \arrow[d, "D(f)"] \\
Q_n(M) \arrow[r, "f"'] & A
\end{tikzcd}$$

.

Definition Say a map f is **skew-symmetric** if for any $0 \leq i, j \leq n-1$, the following is true :

$$\langle \pi_0, \ldots, \pi_i, \ldots, \pi_j, \ldots, \pi_n \rangle f + \langle \pi_0, \ldots, \pi_j, \ldots, \pi_i, \ldots, \pi_n \rangle f = 0$$

Non-Linear Differential Forms

Let $\mathbb{X}$ be a Non-Linear CDC. For an object $M \in \mathbb{X}$, a linear object $A \in \mathbb{X}$ and $n \geq 1$, define a **non-linear differential n-form on M with values in A** to be a map

$$\omega : Q_n(M) \to A$$

which is quasi-multilinear and skew-symmetric. Denote the set of n-forms on M with values in A by $\Psi_n^A(M)$. Define $\Psi_0^A(M)$ to be the hom-set $\mathbb{X}(M, A)$.

Quasi Exterior Derivative

Definition. For $n \geq 1$ and $0 \leq i \leq n-1$ and M an object, define the the map r_i to be

$$M \times Q_n(M) \xrightarrow{r_i = \langle 0, \ldots, 0, \pi_i | \pi_0, \ldots, \hat{\pi}_i, \ldots, \pi_n, 0 \rangle} Q(Q_n(M))$$

where $\hat{\pi}_i$ indicates the exclusion of that term.

Suppose A is a linear group, and $\omega \in \Psi_n^A(M)$. For $n \geq 1$, define the **quasi exterior derivative of** ω, denoted $\partial_n(\omega)$, to be the map given by

$$\partial_n(\omega) := \sum_{i=0}^{n} (-1)^i r_i D(\omega)$$

Key Results

- Let $f : M' \to M$, and $\omega : M \to A$ be a non-linear differential n-form, then $Q(f)\omega : M' \to A$ is also a non-linear differential n-form.
- The quasi exterior derivative applied to a non-linear differential n-form gives a non-linear differential $(n+1)$-form.
- The quasi exterior derivative is a natural transformation.
- Applying the quasi exterior derivative twice to a non-linear differential n-form gives the 0 map : $\partial(\partial(\omega)) = 0$.

Key Results (1) & (2)

Lemma. Let $f : M' \to M$, and $\omega \in \Psi_n^A(M)$. Then the composite

$$Q_n(M') \xrightarrow{Q_n(f)} Q_n(M) \xrightarrow{\omega} A$$

is in $\Psi_n^A(M')$.
(This allows us to view $\Psi_n^A(-)$ as a functor from $\mathbb{X}^{op}$ to *set*.)

Lemma. For any $\omega \in \Psi_n^A(M)$, its exterior derivative $\partial_n(\omega)$ is in $\Psi_{n+1}^A(M)$.

Key Results (3) & (4)

Lemma. For any $n \geq 0$ and differential group A, exterior differentiation

$$\partial_n : \Psi_n^A \longrightarrow \Psi_{n+1}^A$$

is a natural transformation.

$$\begin{tikzcd}
\Psi_n^A(M) \arrow[r, "\Psi_n^A(f)"] \arrow[d, "\partial_n(\omega)"'] & \Psi_n^A(M') \arrow[d, "\partial_n(Q_n(f)\omega)"] \\
\Psi_{n+1}^A(M) \arrow[r, "\Psi_{n+1}^A(f)"'] & \Psi_{n+1}^A(M')
\end{tikzcd}$$

Lemma. For any $n \geq 0$ and linear group A, the following composition is the 0 map :

$$\Psi_n^A(-) \xrightarrow{\partial_n} \Psi_{n+1}^A(-) \xrightarrow{\partial_{n+1}} \Psi_{n+2}^A(-)$$

Cohomology

- The abelian groups $\Psi^A_n(M)$ and quasi exterior derivatives ∂_n for $n \leq 0$ form a cochain complex:

$$\{0\} \xrightarrow{0} \Psi^A_0(M) \xrightarrow{\partial_0} \Psi^A_1(M) \xrightarrow{\partial_1} \dots \xrightarrow{\partial_{n-1}} \Psi^A_n(M) \xrightarrow{\partial_n} \dots$$

- Call the cohomology groups of a cochain of this form the **quasi De Rahm cohomology of** M.
Let $H^i_{qdr}(M, A)$ denote the i$^{\text{th}}$ quasi De Rahm cohomology group and define $\partial_{-1} := 0$.

Finding Specific Examples of Cohomology Groups

Lemma. For any pair of groups G, H, the first quasi De Rahm cohomology group is $H^0_{qdr}(G, H) = H$.

Proposition. The quasi De Rahm cohomology groups of non-linear differential n-forms in $\mathbf{ab}_{\mathrm{fun}}$ from $\mathbb{Z}_2$ to $\mathbb{Z}_2$:

$$\{0\} \xrightarrow{0} \Psi_0^{\mathbb{Z}_2}(\mathbb{Z}_2) \xrightarrow{\partial_0} \Psi_1^{\mathbb{Z}_2}(\mathbb{Z}_2) \xrightarrow{\partial_1} \ldots \xrightarrow{\partial_{n-1}} \Psi_n^{\mathbb{Z}_2}(\mathbb{Z}_2) \xrightarrow{\partial_n} \ldots$$

are all $\mathbb{Z}_2$: $H_{qdr}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2, \mathbb{Z}_2 \ldots \mathbb{Z}_2 \ldots$

Integers and polynomial Functions

Consider the category of abelian groups with $\mathbb{Z}^n$'s as objects and polynomial functions as arrows

$$\{0\} \xrightarrow{0} \Psi_0^{\mathbb{Z}}(\mathbb{Z}) \xrightarrow{\partial_0} \Psi_1^{\mathbb{Z}}(\mathbb{Z}) \xrightarrow{\partial_1} \Psi_2^{\mathbb{Z}}(\mathbb{Z}) \xrightarrow{\partial_2} \dots$$

By the previous lemma : $H^0_{qdr}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$

Integers and polynomial Functions

Consider the category of abelian groups with $\mathbb{Z}^n$'s as objects and polynomial functions as arrows

$$\{0\} \xrightarrow{0} \Psi_0^{\mathbb{Z}}(\mathbb{Z}) \xrightarrow{\partial_0} \Psi_1^{\mathbb{Z}}(\mathbb{Z}) \xrightarrow{\partial_1} \Psi_2^{\mathbb{Z}}(\mathbb{Z}) \xrightarrow{\partial_2} \ldots$$

By the previous lemma : $H^0_{qdr}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}$

$H^1_{qdr}(\mathbb{Z}, \mathbb{Z})$

- Found a basis for the kernel of ∂_2 and image of ∂_1
- Kernel:

$$\{v, xv, v^2, x^2v + xv^2, v^3, x^3v + xv^3, x^2v^2, v^4, \ldots\}$$

- Image:

$$\{v, (2xv + 2v^2), (3x^2v + 3xv^2 + v^3), \ldots, v^n\}$$

Non-Linear Tangent Categories

Can define a "non-linear" version of tangent categories such that any non-linear CDCs can be an example, with the following changes:

- Remove the "$+$" natural transformation.
- Additive bundles are replaced with pointed bundles
- We require the following triple equalizer diagram to hold instead of an equalizer for ℓ involving $+$:

$$\begin{tikzcd} QM \arrow[r, "\ell"] & Q^2M \arrow[r, shift left=2, "Q(p)"] \arrow[r, "p" description] \arrow[r, shift right=2, "ppz"'] & QM \end{tikzcd}$$

Some structures in non-linear tangent categories

- Connections
  - removed conditions involving "$+$"
  - examples in $\mathbf{ab}_{fun}$
- Sector forms
  - same definition as regular tangent categories
  - examples in **smooth** and $\mathbf{ab}_{fun}$

Future Work

- More examples of non-linear CDCs
- More cohomology examples in $\mathbf{ab}_{\mathrm{fun}}$.
- Find out what Quasi De Rahm cohomology is in **smooth** and if it is the same as the De Rahm cohomology.
- H_{qdr} in idempotent and simple slice of non-linear CDCs.
- More sector form calculations
- More connections examples
- Applications for non-linear differential structure

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