A Tangent Category Alternative to the Faà di Bruno Construction
(Or How I Avoided Learning the Faà di Bruno Construction)

JS Lemay

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Brief Intro on the Faà di Bruno Construction

- **Cartesian differential categories**, introduced by Blute, Cockett, and Seely, come equipped with a **differential combinator** whose axioms are based on the basic properties of the directional derivative from multivariable calculus. (Full definition soon)
- Many interesting examples of Cartesian differential categories originating from a wide variety of different areas such as: classical differential calculus, functor calculus, and linear logic.

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- Many interesting examples of Cartesian differential categories originating from a wide variety of different areas such as: classical differential calculus, functor calculus, and linear logic.
- Cockett and Seely also introduced a construction of cofree Cartesian differential categories, which they called the **Faà di Bruno construction**.
- The Faà di Bruno construction provides a comonad **Faà** on the category of Cartesian left additive categories such that the **Faà**-coalgebras are precisely Cartesian differential categories.
- Composition in these cofree Cartesian differential categories **Faà**$(\mathbb{X})$ are based on the Faà di Bruno formula for higher-order chain rule.

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- Cockett and Seely also introduced a construction of cofree Cartesian differential categories, which they called the **Faà di Bruno construction**.
- The Faà di Bruno construction provides a comonad **Faà** on the category of Cartesian left additive categories such that the **Faà**-coalgebras are precisely Cartesian differential categories.
- Composition in these cofree Cartesian differential categories $\mathbf{Faà}(\mathbb{X})$ are based on the Faà di Bruno formula for higher-order chain rule.
- However, because the Faà di Bruno formula itself is very combinatorial in nature, this composition is also very combinatorial (making use of symmetric trees) and making it somewhat complex and very notation-heavy. So its not easy to work with $\mathbf{Faà}(\mathbb{X})$.
- **TODAY:** Provide an alternative construction of cofree Cartesian differential categories inspired by tangent categories (and hopefully easier to work with!).

A **Cartesian differential category** is a

Cartesian Differential Categories

A **Cartesian differential category** is a **Cartesian left additive category**

Cartesian Left Additive Categories

A **left additive category** is a category such that each hom-set is a commutative monoid, with $+$ and 0, such that composition[1] on the *LEFT* preserves the additive structure:

$$f(g+h) = fg + fh \qquad f0 = 0$$

A map h is **additive** if composition on the right by h preserves the additive structure:

$$(f+g)h = fh + gh \qquad 0h = 0$$

A **Cartesian left additive category** is a left additive category with finite products such that all projection maps π_i are additive.

[1] Composition is written diagramaticaly throughout this presentation: so fg is f then g.

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Cartesian Differential Categories

A **Cartesian differential category** is a Cartesian left additive category with a combinator D on maps – called the **differential combinator** – which written as an inference rule:

$$\frac{f : A \to B}{\mathsf{D}[f] : A \times A \to B}$$

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such that D satisfies the following:

[CD.1] $\mathsf{D}[f+g] = \mathsf{D}[f] + \mathsf{D}[g]$ and $\mathsf{D}[0] = 0$

[CD.2] $(1 \times (\pi_0 + \pi_1))\,\mathsf{D}[f] = (1 \times \pi_0)\mathsf{D}[f] + (1 \times \pi_1)\mathsf{D}[f]$ and $\langle 1, 0\rangle \mathsf{D}[f] = 0$

[CD.3] $\mathsf{D}[1] = \pi_1$ and $\mathsf{D}[\pi_j] = \pi_1 \pi_j$ (where $j \in \{0, 1\}$)

[CD.4] $\mathsf{D}[\langle f, g\rangle] = \langle \mathsf{D}[f], \mathsf{D}[g]\rangle$

[CD.5] $\mathsf{D}[fg] = \langle \pi_0 f, \mathsf{D}[f]\rangle \mathsf{D}[g]$ (**CHAIN RULE**)

[CD.6] $\ell \mathsf{D}^2[f] = \mathsf{D}[f]$ where $\ell := \langle 1, 0\rangle \times \langle 0, 1\rangle$

[CD.7] $c\mathsf{D}^2[f] = \mathsf{D}^2[f]$ where $c := 1 \times \langle \pi_1, \pi_0\rangle \times 1$

In a Cartesian differential category, a map f is said to be **linear** if $\mathsf{D}[f] = \pi_1 f$.

Tangent Functor and the Higher Order Chain Rule

Theorem (Cockett and Cruttwell)

Every Cartesian differential category $\mathbb{X}$ *is a tangent category where the* **tangent functor** $\mathsf{T} : \mathbb{X} \to \mathbb{X}$ *is defined on objects as* $\mathsf{T}(A) := A \times A$ *and on morphisms as:*

$$\mathsf{T}(f) := \; A \times A \xrightarrow{\langle \pi_0 f, \mathsf{D}[f] \rangle} B \times B$$

Furthermore, if f *is linear then* $\mathsf{T}(f) = f \times f$*.*

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Furthermore, if f *is linear then* $\mathsf{T}(f) = f \times f$.

Functoriality of T follows from the chain rule **[CD.5]**, which itself can be re-expressed as:

$$\mathsf{D}[fg] = \mathsf{T}(f)\mathsf{D}[g]$$

This then gives a very clean expression for the higher-order chain rule for all $n \in \mathbb{N}$:

$$\mathsf{D}^n[fg] = \mathsf{T}^n(f)\mathsf{D}^n[g]$$

This higher-order version of the chain rule will be our inspiration for our composition and will allow us to avoid all of the combinatorial complexities of the Faà di Bruno formula.

Pre-D-Sequences

For a category $\mathbb{X}$ with finite products, consider the endofunctor $\mathsf{P} : \mathbb{X} \to \mathbb{X}$ (where P is for product) which is defined on objects as $\mathsf{P}(A) := A \times A$ and on maps as $\mathsf{P}(f) := f \times f$.

$$\mathsf{P}^n(A) = \underbrace{A \times A \times \ldots \times A}_{2^n \text{ times}}$$

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Definition

In a category with finite products, a **pre-D-sequence** from A to B, which we denote as $f_\bullet : A \to B$, is a sequence of maps $f_\bullet = (f_0, f_1, f_2, \ldots)$ where $f_n : \mathsf{P}^n(A) \to B$.

Pre-D-Sequences:	Intuition:
$f_0 : A \to B$	$f : A \to B$
$f_1 : A \times A \to B$	$\mathsf{D}[f] : A \times A \to B$
$f_2 : A \times A \times A \times A \to B$	$\mathsf{D}^2[f] : A \times A \times A \times A \to B$
$\vdots$	$\vdots$
$f_n : \mathsf{P}^n(A) \to B$	$\mathsf{D}^n[f] : \mathsf{P}^n(A) \to B$
$\vdots$	$\vdots$

In general for arbitrary pre-D-sequences there is no relation between the f_n.

A Differential Combinator Already?!

For a pre-D-sequence $f_\bullet : A \to B$ we define the following two pre-D-sequences:

(i) Its **differential** pre-D-sequence $\mathsf{D}[f_\bullet] : \mathsf{P}(A) \to B$ where:

$$\mathsf{D}[f_\bullet]_n := f_{n+1} \qquad \mathsf{D}[(f_0, f_1, \ldots)] = (f_1, f_2, \ldots)$$

Applying this to $(f, \mathsf{D}[f], \ldots)$ we get $(\mathsf{D}[f], \mathsf{D}^2[f], \mathsf{D}^3[f], \ldots)$ which of the right form!

(ii) Its **tangent** pre-D-sequence $\mathsf{T}(f_\bullet) : \mathsf{P}(A) \to \mathsf{P}(B)$ where:

$$\mathsf{T}(f_\bullet)_n := \ \mathsf{P}^{n+1}(A) \xrightarrow{\langle \mathsf{P}^n(\pi_0) f_n, f_{n+1} \rangle} \mathsf{P}(B)$$

This is the analogue of $(\mathsf{T}(f), \mathsf{D}[\mathsf{T}(f)], \mathsf{D}^2[\mathsf{T}(f)], \ldots)$ which would be:

$$(\langle \pi_0 f, \mathsf{D}[f] \rangle, \langle (\pi_0 \times \pi_0)\mathsf{D}[f], \mathsf{D}^2[f] \rangle, \big\langle (\pi_0 \times \pi_0 \times \pi_0 \times \pi_0)\mathsf{D}^2[f], \mathsf{D}^3[f] \big\rangle, \ldots)$$

Looking forward, these will be our differential combinator and its induced tangent functor.

Category of Pre-D-Sequences

We want to build a category of pre-D-sequences.

- What is the composition of pre-D-sequences?
- What is the identity?

Recall that pre-D-sequences should be thought of as $(f, \mathsf{D}[f], \dots)$. Then:

- Composition of $(f, \mathsf{D}[f], \dots)$ and $(g, \mathsf{D}[g], \dots)$ should be $(fg, \mathsf{D}[fg], \dots)$, which is:

$$(fg, \mathsf{T}(f)\mathsf{D}[g], \dots, \mathsf{T}^n[f]\mathsf{D}^n[g], \dots)$$

- The identity should be $(1, \mathsf{D}[1], \dots)$, which would be:

$$(1, \pi_1, \dots, \underbrace{\pi_1\pi_1 \dots \pi_1}_{n \text{ times}}, \dots)$$

Category of Pre-D-Sequences

For a category $\mathbb{X}$ with finite products, define its category of pre-D-sequences $\overline{\mathcal{D}}[\mathbb{X}]$ where:

- Objects of $\overline{\mathcal{D}}[\mathbb{X}]$ are objects of $\mathbb{X}$;
- Maps of $\overline{\mathcal{D}}[\mathbb{X}]$ are pre-D-sequences $f_\bullet : A \to B$;
- The identity is the pre-D-sequence $i_\bullet : A \to A$ where $i_0 := 1_A$ and for $n \geq 1$:

$$i_n := \mathsf{P}^n(A) \xrightarrow{\pi_1} \mathsf{P}^{n-1}(A) \xrightarrow{\pi_1} \ldots \xrightarrow{\pi_1} \mathsf{P}(A) \xrightarrow{\pi_1} A$$

- Composition of $f_\bullet : A \to B$ and $g_\bullet : B \to C$ is the pre-D-sequence $f_\bullet * g_\bullet : A \to C$ where:

$$(f_\bullet * g_\bullet)_n := \mathsf{P}^n(A) \xrightarrow{\mathsf{T}^n(f_\bullet)_0} \mathsf{P}^n(B) \xrightarrow{g_n} C$$

  Notice that $\mathsf{D}^n[g_\bullet]_0 = g_n$ and so we have that $(f_\bullet * g_\bullet)_n = \mathsf{T}^n(f_\bullet)_0 \mathsf{D}^n[g_\bullet]_0$.

At first glance, $\mathsf{T}^n(f_\bullet)_0$ in the composition may seem intimidating...

However by the functorial properties of T, $*$ satisfies many nice properties which makes the composition of pre-D-sequences is easy to work with!

Product Structure of Pre-D-Sequences

We can also "scalar multiply" pre-D-sequences by maps of the base category.

(i) **Left**: Given a map $h : A' \to A$, define the pre-D-sequence $h \cdot f_\bullet : A' \to B$ where:

$$(h \cdot f_\bullet)_n := \ \mathrm{P}^n(A') \xrightarrow{\mathrm{P}^n(h)} \mathrm{P}^n(A) \xrightarrow{f_n} B$$

(ii) **Right**: Given a map $k : B \to C$, define the pre-D-sequence $f_\bullet \cdot k : A \to C$ where:

$$(f_\bullet \cdot k)_n := \ \mathrm{P}^n(A) \xrightarrow{f_n} B \xrightarrow{k} C$$

$\overline{\mathcal{D}}[\mathbb{X}]$ is also a category with finite product where:

- The product of objects is the product of objects in $\mathbb{X}$;
- The projections are the pre-D-sequences $i_\bullet \cdot \pi_0 : A \times B \to A$ and $i_\bullet \cdot \pi_1 : A \times B \to B$;
- The pairing of pre-D-sequences $f_\bullet : C \to A$ and $g_\bullet : C \to B$ is $\langle f_\bullet, g_\bullet \rangle : C \to A \times B$ where $\langle f_\bullet, g_\bullet \rangle_n := \langle f_n, g_n \rangle$

Cartesian Left Additive Structure of Pre-D-Sequences

If $\mathbb{X}$ is a Cartesian left additive category, then so is $\overline{\mathcal{D}}[\mathbb{X}]$ where:

- The zero map is the pre-D-sequence $0_\bullet : A \to B$ where $0_n = 0$

$$0_\bullet = (0, 0, \ldots)$$

- The sum of pre-D-sequences $f_\bullet : A \to B$ and $g_\bullet : A \to B$ is $f_\bullet + g_\bullet : A \to B$ where $(f_\bullet + g_\bullet)_n := f_n + g_n$.

$$(f_0, f_1, \ldots) + (g_0, g_1, \ldots) = (f_0 + g_0, f_1 + g_1, \ldots)$$

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For a Cartesian left additive category $\mathbb{X}$, is $\overline{\mathcal{D}}[\mathbb{X}]$ its cofree Cartesian differential category?

Answer: **NO!** ... but most of the work has been done!

How close are we to a Cartesian Differential category?

Remember that there were seven differential combinator axioms.

Here are some properties that D and T already satisfy (because of the simplicity of the differential of pre-D-sequences):

(i) $\mathrm{T}(f_\bullet) = \langle \pi_0 \cdot f_\bullet, \mathrm{D}[f_\bullet] \rangle$

(ii) $\mathrm{D}[0_\bullet] = 0_\bullet$ and $\mathrm{D}[f_\bullet + g_\bullet] = \mathrm{D}[f_\bullet] + \mathrm{D}[g_\bullet]$ (**[CD.1]**);

(iii) $\mathrm{D}[i_\bullet] = i_\bullet \cdot \pi_1$ and $\mathrm{D}[i_\bullet \cdot \pi_j] = i_\bullet \cdot (\pi_1 \pi_j)$ (**[CD.3]**);

(iv) $\mathrm{D}[\langle f_\bullet, g_\bullet \rangle] = \langle \mathrm{D}[f_\bullet], \mathrm{D}[g_\bullet] \rangle$ (**[CD.4]**);

(v) $\mathrm{D}[f_\bullet * g_\bullet] = \mathrm{T}(f_\bullet) * \mathrm{D}[g_\bullet]$ (Chain rule - **[CD.5]**).

Therefore the only axioms remaining are **[CD.2]**, **[CD.6]**, and **[CD.7]**. To obtaining these last three axioms, we will have to consider special kinds of pre-D-sequences...

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[CD.2] $(1 \times (\pi_0 + \pi_1))\,\mathsf{D}[f] = (1 \times \pi_0)\mathsf{D}[f] + (1 \times \pi_1)\mathsf{D}[f]$ and $\langle 1, 0\rangle \mathsf{D}[f] = 0$

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Note that $\langle 1, 0\rangle$, $(1 \times (\pi_0 + \pi_1))$, ℓ, and c can all be defined without the differential combinator.

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Definition

For a Cartesian left additive category, a D-**sequence** is a pre-D-sequence $f_\bullet$ such that for each $n \in \mathbb{N}$ and each $k \leq n$, $f_\bullet$ satisfies the following equalities:

[DS.1] $\mathsf{P}^k(\langle 1, 0\rangle)f_{n+1} = 0$;

[DS.2] $\mathsf{P}^k\,(1 \times (\pi_0 + \pi_1))\, f_{n+1} = \mathsf{P}^k(1 \times \pi_0)f_{n+1} + \mathsf{P}^k(1 \times \pi_1)f_{n+1}$ (multi-additivity in certain arguments);

[DS.3] $\mathsf{P}^k(\ell)f_{n+2} = f_{n+1}$ (relation between the f_n!);

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Where do these axioms come from?

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Where do these axioms come from?

Consider again $(f, \mathsf{D}[f], \mathsf{D}^2[f], \ldots)$. By higher order chain rule, for each $k \leq n$ we have:

$$\mathsf{T}^k(\ell)\mathsf{D}^{n+2}[f] = \mathsf{T}^k(\ell)\mathsf{D}^k\left[\mathsf{D}^2\left[\mathsf{D}^{n-k}[f]\right]\right] = \mathsf{D}^k\left[\ell\mathsf{D}^2\left[\mathsf{D}^{n-k}[f]\right]\right] = \mathsf{D}^k\left[\mathsf{D}\left[\mathsf{D}^{n-k}[f]\right]\right] = \mathsf{D}^{n+1}[f]$$

and since ℓ is linear, $\mathsf{T}(\ell) = \ell \times \ell = \mathsf{P}(\ell)$, and we obtain **[DS.3′]** that $\mathsf{P}^k(\ell) f_{n+2} = f_{n+1}$.

Comonad of D-Sequences

- For a Cartesian left additive category $\mathbb{X}$, define $\mathcal{D}[\mathbb{X}]$ to be the sub-category of $\overline{\mathcal{D}}[\mathbb{X}]$ of D-sequences of $\mathbb{X}$. $\mathcal{D}[\mathbb{X}]$ is a Cartesian left additive category which is closed under D and T.
- Let CLAC be the category of Cartesian left additive category and functors them which preserve the product structure and additive structure strictly.
- There is a comonad $(\mathcal{D}, \delta, \varepsilon)$ on CLAC where $\mathcal{D}$: CLAC $\to$ CLAC is the functor which maps a Cartesian left additive category to its category of D-sequences.
- In fact, all the hard work of showing that $(\mathcal{D}, \delta, \varepsilon)$ is a comonad is actually done by showing that pre-D-sequences also give a comonad on CLAC.

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- In fact, all the hard work of showing that $(\mathcal{D}, \delta, \varepsilon)$ is a comonad is actually done by showing that pre-D-sequences also give a comonad on CLAC.

Theorem

$\mathcal{D}$-coalgebras of the comonad $(\mathcal{D}, \delta, \varepsilon)$ are precisely Cartesian differential categories.

All these proofs are actually quite simple and not combinatorial! (Robert Seely approved!)

In particular, for a Cartesian left additive category $\mathbb{X}$, its category of D-sequences $\mathcal{D}[\mathbb{X}]$ is the cofree Cartesian differential category over $\mathbb{X}$.

The differential combinator of $\mathcal{D}[\mathbb{X}]$ is given by the differential of pre-D-sequences:

$$\mathsf{D}[f_\bullet]_n := f_{n+1} \qquad \mathsf{D}[(f_0, f_1, \ldots)] = (f_1, f_2, \ldots)$$

And also, Faà$(\mathbb{X})$ and $\mathcal{D}[\mathbb{X}]$ are equivalent as Cartesian differential categories.

Concluding Remarks

- We've given an alternative construction of cofree Cartesian differential categories which avoids the combinatorics of the Faà di Bruno formula;
- As Robert Seely told me: "this construction clears away all the (symmetric) trees that hid the real structure".
- Hopefully this new construction will pave the way for future study on cofree Cartesian differential categories (if we have the time!)
- For Geoff: This approach also generalizes to constructing cofree generalized Cartesian differential categories (but notation isn't as nice...)

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END.

Thanks for listening!

(For more details, see 'A Tangent Category Alternative to the Faà di Bruno Construction' on arxiv)

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[DS.3] $\ell \cdot \mathrm{D}^{n+2}[f_\bullet] = \mathrm{D}^{n+1}[f_\bullet]$ where recall $\ell = \langle 1, 0 \rangle \times \langle 0, 1 \rangle$;

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[DS.1] to **[DS.4]** are analogues of higher-order versions of **[CD.2]**, **[CD.6]**, and **[CD.7]**

For a Cartesian left additive category $\mathbb{X}$, $\mathcal{D}[\mathbb{X}]$ is the sub-category of $\overline{\mathcal{D}}[\mathbb{X}]$ of D-sequences of $\mathbb{X}$.

$\mathcal{D}[\mathbb{X}]$ is a Cartesian left additive category which is closed under D and T.