Seeing double

(https://www.mscs.dal.ca/∼pare/FMCS2.pdf)

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Before we start

Double functors

$$\mathbb{S}\mathrm{lice}(\mathbf{A}) \longrightarrow \mathbb{S}\mathrm{lice}(\mathbf{B})$$

are in bijection with natural transformations

$$\begin{tikzcd} \mathbf{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow t"] & \mathbf{B} \end{tikzcd}$$

The associated double functor is given (on the objects) by

$$\begin{array}{ccc} A & & FA \\ f\downarrow & \longmapsto & \downarrow Ff \\ A' & & FA' \\ & & \downarrow tA' \\ & & GA' \end{array}$$

Words of wisdom

If you want something done right
you have to do it yourself.
And, you have to do it right.

Micah McCurdy

The plan

- The theory of restriction categories is a nice, simply axiomatized theory of partial morphisms
- It is well motivated with many examples and has lots of nice results
- But it is somewhat tangential to mainstream category theory
- The plan is to bring it back into the fold by taking a double category perspective
- Every restriction category has a canonically associated double category
- What can double categories tell us about restriction categories?
- What can restriction categories tell us about double categories?
- References

  - R. Cockett, S. Lack, Restriction Categories I: Categories of Partial Maps, Theoretical Computer Science 270 (2002) 223-259
  - R. Cockett, Introduction to Restriction Categories, Estonia Slides (2010)
  - D. DeWolf, Restriction Category Perspectives of Partial Computation and Geometry, Thesis, Dalhousie University, 2017

Restriction categories

Definition

A *restriction category* is a category equipped with a *restriction operator*

$$A \xrightarrow{f} B \rightsquigarrow A \xrightarrow{\bar{f}} A$$

satisfying

R1. $f\bar{f} = f$

R2. $\bar{f}\bar{g} = \bar{g}\bar{f}$

R3. $\overline{g\bar{f}} = \bar{g}\bar{f}$

R4. $\bar{g}f = f\,\overline{gf}$

Example

Let **A** be a category and **M** a subcategory such that

(1) $m \in \mathbf{M} \Rightarrow m$ monic

(2) **M** contains all isomorphisms

(3) **M** stable under pullback: for every $m \in \mathbf{M}$ and $f \in \mathbf{A}$ as below, the pullback of m along f exists and is in **M**

$$\begin{tikzcd} P \arrow[r, "\bar{f}"] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "m"] \\ C \arrow[r, "f"'] & A \end{tikzcd}$$

$$m \in M \Rightarrow m' \in M$$

$\mathrm{Par}_{\mathbf{M}}\mathbf{A}$ has the same objects as **A** but the morphisms are isomorphism classes of spans

$$\begin{tikzcd} & A_0 \arrow[dl, tail, "m"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

with $m \in M$

Composition is by pullback

The restriction operator is $\overline{(m, f)} = (m, m)$

$$
\begin{tikzcd}
& A_0 \arrow[dl, tail, "m"'] \arrow[dr, "f"] & & & & A_0 \arrow[dl, tail, "m"'] \arrow[dr, tail, "m"] & \\
A & & B & \arrow[r, mapsto, "\bar{(\ )}"] & A & & A
\end{tikzcd}
$$

Let **A** be a restriction category

Definition

$f : A \longrightarrow B$ is *total* if $\bar{f} = 1_A$

Proposition

The total morphisms form a subcategory of **A**

The double category $\mathbb{D}c(\mathbf{A})$ *associated to a restriction category* **A** *has*

- *The same objects as* **A**
- *Total maps as horizontal morphisms*
- *All maps as vertical morphisms*
- *There is a unique cell*
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "m"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$
if and only if $gv = wf\bar{v}$

Theorem

$\mathbb{D}c(\mathbf{A})$ *is a double category*

Remark

C & L define an order relation between $f, g : A \longrightarrow B$, $f \leq g \Leftrightarrow f = g\bar{f}$
Makes **A** *into a 2-category. They say "seems to be less useful than one might expect"*

There is a cell

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "w"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

if and only if $gv \leq wf$. *So our* $\mathbb{D}c(\mathbf{A})$ *is not far from that 2-category. Perhaps it will turn out to be more useful than they might expect!*

Example

In $\mathbb{D}c\mathrm{Par}_{\mathbf{M}}(\mathbf{A})$ there is a cell if and only if there exists a (necessarily unique) morphism h

$$
\begin{tikzcd}
& A \arrow[rr, "f"] & & B \\
A_0 \arrow[ur, tail, "m"] \arrow[rr, dashed, "h"] \arrow[dr, "v"'] & & B_0 \arrow[ur, tail, "n"'] \arrow[dr, "w"] & \\
& C \arrow[rr, "g"'] & & D
\end{tikzcd}
$$

Companions

Proposition

In $\mathbb{D}c(\mathbf{A})$ every horizontal arrow has a companion, $f_* = f$

Proof.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "1"] \\
B \arrow[r, "1"'] & B
\end{tikzcd}
\qquad 1 \cdot f = 1 \cdot f \cdot \bar{f}
$$

$$
\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\Rightarrow"] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad f \cdot 1 = f \cdot 1 \cdot \bar{1}
$$

□

Conjoints

Proposition

In $\mathbb{D}cPar_{\mathbf{M}}(\mathbf{A})$, f has a conjoint if and only if $f \in M$

Proof.

Assume f has conjoint (m, g), then there are α, β

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \\
A \arrow[r, "\alpha"] \arrow[d, equal] & B_0 \arrow[u, "m"'] \arrow[d, "g"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
B \arrow[r, equal] & B \arrow[d, equal] \\
B_0 \arrow[u, "m"] \arrow[r, "\beta"] \arrow[d, "g"'] & B \arrow[d, equal] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

So $m\alpha g = fg = \beta = m$ which implies $\alpha g = 1$
Thus α is an isomorphism and $f = m\alpha \in \mathbf{M}$

□

- If we suspect that $\mathbb{A}$ is of the form $\mathbb{D}c\mathrm{Par}_{\mathbf{M}}(\mathbf{A})$ we can recover $\mathbf{M}$ as those horizontal arrows having a conjoint
- Is the requirement of stability under pullback of conjoints a good double category notion?
- In $\mathbb{D}c(\mathbf{A})$, a horizontal arrow $f : A \longrightarrow B$ always has a companion f_*, and if it also has a conjoint f^* then $f_* \dashv f^*$ so

$$\begin{tikzcd} A \arrow[d, "f_* \bullet f^*"', "\bullet" marking] \\ A \end{tikzcd}$$

is a comonad, i.e. an idempotent $\leq \mathrm{id}_A$

Proposition

In $\mathbb{D}c(\mathbf{A})$, $f_* \bullet f^* = \bar{f}^*$

Tabulators

Proposition

$\mathbb{D}cPar_{\mathbf{M}}(\mathbf{A})$ *has tabulators and they are effective*

Proof.

Given $(m, v) : A \longrightarrow\!\!\!\!\bullet\!\!\!\to B$, the tabulator is

$$\begin{tikzcd}
A_0 \arrow[r, "m"] \arrow[d, equal] & A \\
A_0 \arrow[r, equal] \arrow[d, equal] & A_0 \arrow[u, tail, "m"'] \arrow[d, "v"] \\
A_0 \arrow[r, "v"'] & B
\end{tikzcd}$$

□

Conjecture: In a general $\mathbb{D}c(\mathbf{A})$, $v : A \longrightarrow\!\!\!\!\bullet\!\!\!\to B$ has a tabulator if and only if $\bar{v}$ splits

Classification of vertical arrows

- The original definition of elementary topos was in terms of a partial map classifier

$$\frac{B \longrightarrow\!\!\!\!\!\!\bullet\;\; A}{B \longrightarrow \tilde{A}}$$

- In a topos, relations are classifiable

$$\frac{B \longrightarrow\!\!\!\!\!\!\bullet\;\; A}{B \longrightarrow \Omega^A}$$

- For profunctors

$$\frac{\mathbf{B} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{A}}{\mathbf{B} \longrightarrow (\mathbf{Set}^{\mathbf{A}})^{op}}$$

  provided **A** is small

- How do we formalize this in a general double category?

Classification (Beta version)

- The desired bijection

$$\frac{B \xrightarrow{\;v\;} A}{B \xrightarrow{\;\widehat{v}\;} \tilde{A}}$$

gives $eA : \tilde{A} \longrightarrow A$ and $hA : A \longrightarrow \tilde{A}$

- We express our definition in terms of eA

Definition

Let $\mathbb{A}$ be a double category and A an object of $\mathbb{A}$. We say that A is *classifying* if we are given an object $\tilde{A}$ and a vertical morphism $eA : \tilde{A} \longrightarrow A$ with the following universal properties:

(1) For every vertical arrow $v : B \xrightarrow{\ \bullet\ } A$ there exist a horizontal arrow $\widehat{v} : B \longrightarrow \tilde{A}$ and a cell

$$\begin{tikzcd}
B \arrow[rr, "\widehat{v}"] \arrow[dr, "v"', "\bullet" marking] \arrow[drr, phantom, "\epsilon v"] & & \tilde{A} \arrow[dl, "eA", "\bullet" marking] \\
& A &
\end{tikzcd}$$

such that for every cell α

$$\begin{tikzcd}
D \arrow[rr, "g"] \arrow[dr, "w"', "\bullet" marking] & & B \arrow[rr, "\widehat{v}"] \arrow[d, phantom, "\alpha"] & & \tilde{A} \arrow[dl, "eA", "\bullet" marking] \\
& C \arrow[rr, "f"'] & & A &
\end{tikzcd}$$

there exists a unique cell $\bar{\alpha}$ such that

$$\begin{tikzcd}
D \arrow[rr, "g"] \arrow[dr, "w"', "\bullet" marking] \arrow[drr, phantom, "\bar{\alpha}"] & & B \arrow[rr, "\widehat{v}"] \arrow[dr, "v"', "\bullet" marking] \arrow[drr, phantom, "\epsilon v"] & & \tilde{A} \arrow[dl, "eA", "\bullet" marking] \\
& C \arrow[rr, "f"'] & & A &
\end{tikzcd}
=
\begin{tikzcd}
D \arrow[rr, "g"] \arrow[dr, "w"', "\bullet" marking] & & B \arrow[rr, "\widehat{v}"] \arrow[d, phantom, "\alpha"] & & \tilde{A} \arrow[dl, "eA", "\bullet" marking] \\
& C \arrow[rr, "f"'] & & A &
\end{tikzcd}$$

(2) For every cell

$$\begin{tikzcd}
D \arrow[rr, "g"] \arrow[dr, "w"', "\bullet" marking] & {} \arrow[d, phantom, "\beta"] & B \arrow[dl, "v", "\bullet" marking] \\
& A &
\end{tikzcd}$$

there exists a unique cell $\bar{\bar{\beta}}$ such that

$$\begin{tikzcd}
D \arrow[rr, "g"] \arrow[dr, "\mathrm{id}"', "\bullet" marking] & & B \arrow[rr, "\widehat{v}"] \arrow[d, phantom, "\bar{\bar{\beta}}"] & & \tilde{A} \arrow[dl, "\mathrm{id}", "\bullet" marking] \\
& D \arrow[rr, "\widehat{w}"] \arrow[dr, "w"', "\bullet" marking] & {} \arrow[d, phantom, "\epsilon w"] & \tilde{A} \arrow[dl, "eA", "\bullet" marking] & \\
& & A & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
D \arrow[r, "g"] \arrow[dr, "w"', "\bullet" marking] \arrow[r, phantom, shift right=6, "\beta"] & B \arrow[r, "\widehat{v}"] \arrow[d, "\bullet" marking] \arrow[r, phantom, shift right=6, "\epsilon v"] & \tilde{A} \arrow[dl, "eA", "\bullet" marking] \\
& A &
\end{tikzcd}$$

Complete classification

- How do we understand this?
- Take a more global approach
  Assume $\mathbb{A}$ is companionable, i.e. every horizontal arrow f has a companion f_*
  Then we get a (pseudo) double functor

$$(\ )_* : \mathbb{Q}\,\mathcal{H}\mathrm{or}\mathbb{A} \longrightarrow \mathbb{A}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h_*"', "\bullet" marking] \arrow[dr, phantom, "\alpha_*"] & B \arrow[d, "k", "\bullet" marking] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Exercise!

Definition

Say that $\mathbb{A}$ is *classifying* if $(\ )_*$ has a *down adjoint* $\tilde{(\ )}$,
i.e. a right adjoint in the vertical direction

Bijections

The adjunction can be formalized in terms of bijections

$$
\begin{array}{c|c}
\begin{tikzcd} B \arrow[d, "v"', "\bullet" marking] \\ A \end{tikzcd} & B \xrightarrow{\widehat{v}} \tilde{A}
\end{array}
$$

More precisely, for $v : B \longrightarrow\!\!\!\!\!\!\bullet\;\;\; A$ there exists a $\widehat{v} : B \longrightarrow \tilde{A}$ and an isomorphism

$$
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "(\widehat{v})_*"', "\bullet" marking] & B \arrow[dd, "v", "\bullet" marking] \\
\tilde{A} \arrow[d, "eA"', "\bullet" marking] \arrow[r, phantom, "\cong"] & \\
A \arrow[r, equal] & A
\end{tikzcd}
$$

This can be expressed without mention of $(\ )_*$ because we have a bijection

Bijections (cont.)

$$
\begin{tikzcd}
D \arrow[r, "g"] \arrow[dd, "w"' {marking, pos=0.5}, "\bullet" marking] & B \arrow[d, "(\widehat{v})_*", "\bullet" marking] \\
\arrow[r, phantom, "\Rightarrow"] & \tilde{A} \arrow[d, "eA"] \\
C \arrow[r, "f"'] & A
\end{tikzcd}
\qquad \Bigg| \qquad
\begin{tikzcd}
D \arrow[rr, "g"] \arrow[dr, "w"', "\bullet" marking] & & B \arrow[rr, "\widehat{v}"] \arrow[d, phantom, "\Rightarrow"] & & \tilde{A} \arrow[dl, "eA", "\bullet" marking] \\
& C \arrow[rr, "f"'] & & A &
\end{tikzcd}
$$

Yonedafication now yields the single-object definition

Kleisli

- Given a monad (T, η, μ) on $\mathbf{A}$ we get a double category $\mathbb{K}\mathrm{l}(T)$
  - Objects are those of $\mathbf{A}$
  - Horizontal arrows are morphisms of $\mathbf{A}$
  - Vertical arrows are Kleisli morphisms i.e.

$$\begin{tikzcd} A \arrow[d, "v"'] \\ B \end{tikzcd} \quad \text{is} \quad A \xrightarrow{\widehat{v}} TB \quad \text{in} \quad \mathbf{A}$$

  - Cells

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\Rightarrow"] & A' \arrow[d, "v'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd} \quad \text{a unique one if} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\widehat{v}"'] & A' \arrow[d, "\widehat{v'}"] \\ TB \arrow[r, "Tg"'] & TB' \end{tikzcd} \quad \text{commutes}$$

Kleisli (cont.)

- $\mathbb{K}\mathrm{l}(T)$ is companionable
  For $f : B \longrightarrow A$,

$$\begin{tikzcd} B \arrow[d, "f_*"' , "\bullet" marking] \\ A \end{tikzcd} \quad \text{is given by} \quad \begin{tikzcd} B \arrow[d, "f"] \\ A \arrow[d, "\eta A"] \\ TA \end{tikzcd} \quad \text{i.e.} \quad f_* = (\widehat{\eta A \cdot f})$$

- $\mathbb{K}\mathrm{l}(T)$ is classifiable

$$\frac{\begin{tikzcd} B \arrow[d, "v"', "\bullet" marking] \\ A \end{tikzcd}}{B \xrightarrow{\widehat{v}} TA}$$

  - $eA : TA \longrightarrow\!\!\bullet\!\!\longrightarrow A$ is $\widehat{\mathrm{id}_{TA}}$
  - $hA : A \longrightarrow TA$ is ηA

- Double functors $\mathbb{K}\mathrm{l}(T) \longrightarrow \mathbb{K}\mathrm{l}(S)$ correspond to monad morphisms (F, ϕ)

$$\mathbf{A} \xrightarrow{F} \mathbf{B}$$

$$\phi : FT \longrightarrow SF$$

such that ...

- Horizontal transformations correspond to the 2-cells in Street's 1972 JPAA paper, *Formal theory of monads*
- Vertical transformations correspond to the 2-cells in Lack & Street's 2002 paper, *Formal theory of monads II*

Restriction functors

- A *restriction functor* $F : \mathbf{A} \longrightarrow \mathbf{B}$ is a functor that preserves the restriction operator, $F(\bar{f}) = \overline{F(f)}$

Proposition

A restriction functor F gives a double functor $\mathbb{D}c(F) : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$

Question: Is every double functor $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$ of this form? F is determined by a unique functor $\mathbf{A} \longrightarrow \mathbf{B}$ which preserves the order and totality. Does this mean it preserves restriction? Probably not. Does $\mathbb{D}c$ at least reflect isos?

Theorem

A double functor $\mathbb{D}c Par_{\mathbf{M}}\mathbf{A} \longrightarrow \mathbb{D}c Par_{\mathbf{N}}\mathbf{B}$ comes from a unique functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ which restricts to $\mathbf{M} \longrightarrow \mathbf{N}$ and preserves pullbacks of $m \in M$ by arbitrary $f \in \mathbf{A}$. Thus it does come from a restriction functor

Transformations

Recall that a horizontal transformation $t : F \longrightarrow G$ between double functors $\mathbb{A} \longrightarrow \mathbb{B}$ consists of assignments:

(1) For every A in $\mathbb{A}$ a horizontal morphism $tA : FA \longrightarrow GA$

(2) For every vertical morphism $v : A \longrightarrow\!\!\bullet\!\!\longrightarrow \bar{A}$ a cell

$$
\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "Fv"', "\bullet" marking] \arrow[dr, phantom, "tv"] & GA \arrow[d, "Gv", "\bullet" marking] \\
G\bar{A} \arrow[r, "t\bar{A}"'] & G\bar{A}
\end{tikzcd}
$$

satisfying

(3) Horizontal naturality (for horizontal arrows and cells)

(4) Vertical functoriality (for identities and composition)

Let $F, G : \mathbf{A} \longrightarrow \mathbf{B}$ be restriction functors. Then a horizontal transformation

$$t : \mathbb{D}c(F) \longrightarrow \mathbb{D}c(G)$$

(1) assigns to each A in $\mathbf{A}$ a total morphism

$$tA : FA \longrightarrow GA$$

(2) such that for every $f : A \longrightarrow\!\!\!\bullet\!\!\!\longrightarrow \bar{A}$ in $\mathbf{A}$ we have

$$\begin{tikzcd} FA \arrow[r, "tA"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\leq"] & GA \arrow[d, "Gf"] \\ F\bar{A} \arrow[r, "t\bar{A}"'] & G\bar{A} \end{tikzcd}$$

(3) and t is natural for horizontal arrows (i.e. for f total, we have equality in (2))

This is what C & L call a lax restriction transformation

Proposition

Let $\mathbf{M} \subseteq \mathbf{A}$ *and* $\mathbf{N} \subseteq \mathbf{B}$ *be stable systems of monics and* $F, G : \mathbf{A} \longrightarrow \mathbf{B}$ *functors that preserve the given monics and their pullbacks*

Then horizontal transformations $\mathbb{D}c(F) \longrightarrow \mathbb{D}c(G)$ *correspond to arbitrary natural transformations* $F \longrightarrow G$

Restriction transformations correspond to cartesian ones

There is a notion of commuter cell in double categories, and requiring the cells in (2) to be commuter cells makes them equalities

Vertical transformations

A vertical transformation $\phi : \mathbb{D}c(F) \longrightarrow \mathbb{D}c(G)$

(1) assigns to each object A of $\mathbf{A}$ an arbitrary morphism of B

$$tA : FA \longrightarrow\!\!\bullet\!\!\longrightarrow GA$$

(2) will be automatic

(3) is natural with respect to all morphisms

(4) is vacuous

Question: Is this any good?

There are other notions of vertical transformation, e.g. the *modules* of

- "Yoneda Theory for Double Categories", Theory and Applications of Categories, Vol. 25, No. 17, 2011, pp. 436-489
  which generalize to double categories the modules of
- Cockett, J.R.B., Koslowski, J., Seely, R.A.G., Wood, R.J., Modules, Theory Appl. Categ. 11 (2003), No. 17, pp. 375-396

Project: Investigate the significance of lax (oplax) double functors and modules for restriction categories

Cartesian restriction categories

A restriction category $\mathbf{A}$ is *cartesian* if for every pair of objects A, B there is an object $A \times B$ and morphisms $p_1 : A \times B \longrightarrow A, p_2 : A \times B \longrightarrow B$ with the following universal property

$$\begin{tikzcd}
 & & A \\
C \arrow[urr, "f"] \arrow[drr, "g"'] \arrow[r, dashed, "h"] & A \times B \arrow[ur, "p_1"] \arrow[dr, "p_2"'] \arrow[u, phantom, "\geq"] \arrow[d, phantom, "\geq"] & \\
 & & B
\end{tikzcd}$$

For every f, g there exists a unique h such that

$$p_1 h = f\bar{g}$$

$$p_2 h = g\bar{f}$$

There is also a terminal object condition

Double products

Recall that $\mathbb{A}$ has binary products if

(1) for every A, B there is an object $A \times B$ and horizontal arrows $p_1 : A \times B \longrightarrow A$, $p_2 : A \times B \longrightarrow B$ which have the usual universal property with respect to horizontal arrows

(2) for every pair of vertical arrows $v : A \longrightarrow\!\!\!\!\!\!\!\bullet\;\; C$ and $w : B \longrightarrow\!\!\!\!\!\!\!\bullet\;\; D$ there is a vertical arrow $v \times w : A \times B \longrightarrow\!\!\!\!\!\!\!\bullet\;\; C \times D$ and cells

$$\begin{tikzcd}
A \times B \arrow[r, "p_1"] \arrow[d, "v \times w"'] \arrow[dr, phantom, "\pi_1"] & A \arrow[d, "v"] \\
C \times D \arrow[r, "q_2"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[r, "p_2"] \arrow[d, "v \times w"'] \arrow[dr, phantom, "\pi_2"] & B \arrow[d, "w"] \\
C \times D \arrow[r, "q_2"'] & D
\end{tikzcd}$$

with the usual universal property with respect to cells

Proposition

A *is a cartesian restriction category if and only if* $\mathbb{D}c(\mathbf{A})$ *has finite double products*

Proof *

(1) Suppose **A** is a cartesian restriction category. The universal property of product is the usual one when restricted to total maps
Given vertical arrows $v : A \xrightarrow{\bullet} C$, $w : B \xrightarrow{\bullet} D$ we get a unique $v \times w$

$$\begin{tikzcd} A \arrow[d, "v"' , "\bullet" marking] & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "v \times w", "\bullet" marking] \arrow[dl, phantom, "\geq"] \arrow[dr, phantom, "\leq"] & B \arrow[d, "w", "\bullet" marking] \\ C & C \times D \arrow[l, "q_1"] \arrow[r, "q_2"'] & D \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "z"', "\bullet" marking] \arrow[dr, phantom, "\leq"] & A \arrow[d, "v", "\bullet" marking] \\ Y \arrow[r, "g"'] & C \end{tikzcd} \quad \& \quad \begin{tikzcd} X \arrow[r, "h"] \arrow[d, "z"', "\bullet" marking] \arrow[dr, phantom, "\leq"] & B \arrow[d, "w", "\bullet" marking] \\ Y \arrow[r, "k"'] & D \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} X \arrow[r, "{(f,h)}"] \arrow[d, "z"', "\bullet" marking] \arrow[dr, phantom, "\leq"] & A \times B \arrow[d, "v \times w", "\bullet" marking] \\ Y \arrow[r, "{(g,k)}"'] & C \times D \end{tikzcd}$$

so $\mathbb{D}c(\mathbf{A})$ has binary double products

(2) Suppose $\mathbb{D}c(\mathbf{A})$ has finite double products
Given

$$\begin{tikzcd} & A \\ C \arrow[ur, "f" bullet] \arrow[dr, "g"' bullet] & \\ & B \end{tikzcd} \qquad \text{we have} \qquad h = \; C \xrightarrow{\Delta_*} C \times C \xrightarrow{f \times g} A \times B$$

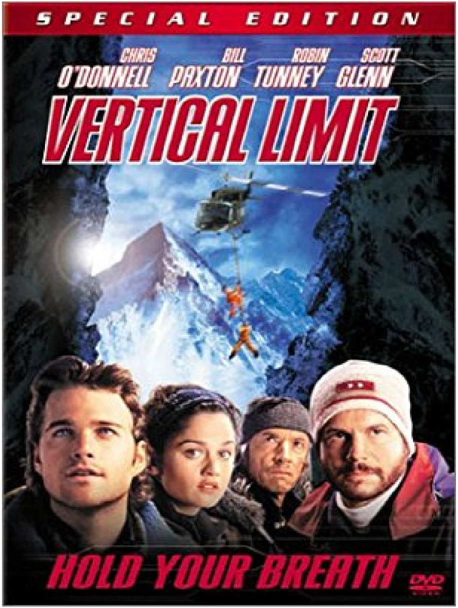
and cells

$$\begin{tikzcd} C \arrow[r, "1_C"] \arrow[d, "\Delta_*"'] \arrow[dr, phantom, "\leq"] & C \arrow[d, "\mathrm{id}"] \\ C \times C \arrow[r, "q"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\leq"] & C \arrow[d, "f"] \\ A \times B \arrow[r, "p_1"'] & A \end{tikzcd}$$

so $p_1 h = f\overline{(f \times g \bullet \Delta_*)} = f\bar{g}$

*Warning: Some details may not have been checked

Homework


SPECIAL EDITION
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