Seeing double
(https://www.mscs.dal.ca/~pare/FMCS2.pdf)

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Before we start

Double functors

\[ \text{Slice}(A) \longrightarrow \text{Slice}(B) \]

are in bijection with natural transformations

\[ \begin{array}{ccc}
A & \xrightarrow{\mathbf{t}} & B \\
\downarrow^f & & \downarrow^G \\
A' & \xleftarrow{\mathbf{t}'} & B'
\end{array} \]

The associated double functor is given (on the objects) by

\[ \begin{array}{ccc}
A & \xrightarrow{\mathbf{Ff}} & FA' \\
\downarrow^f & & \downarrow^{tA'} \\
A' & \xrightarrow{\mathbf{G}} & GA'
\end{array} \]
Words of wisdom

If you want something done right you have to do it yourself.
And, you have to do it right.

Micah McCurdy
The plan

- The theory of restriction categories is a nice, simply axiomatized theory of partial morphisms
- It is well motivated with many examples and has lots of nice results
- But it is somewhat tangential to mainstream category theory
- The plan is to bring it back into the fold by taking a double category perspective
- Every restriction category has a canonically associated double category
- What can double categories tell us about restriction categories?
- What can restriction categories tell us about double categories?
- References
  - D. DeWolf, Restriction Category Perspectives of Partial Computation and Geometry, Thesis, Dalhousie University, 2017
Definition

A restriction category is a category equipped with a restriction operator

\[\begin{align*}
A \xrightarrow{f} B & \rightsquigarrow A \xrightarrow{\bar{f}} A
\end{align*}\]

satisfying

R1. \(f \bar{f} = f\)
R2. \(\bar{f} \bar{g} = \bar{g} \bar{f}\)
R3. \(g \bar{f} = \bar{g} \bar{f}\)
R4. \(\bar{g} f = f g f\)
Example

Let \( A \) be a category and \( M \) a subcategory such that

1. \( m \in M \Rightarrow m \) monic
2. \( M \) contains all isomorphisms
3. \( M \) stable under pullback: for every \( m \in M \) and \( f \in A \) as below, the pullback of \( m \) along \( f \) exists and is in \( M \)

\[
\begin{array}{ccc}
P & \xrightarrow{\tilde{f}} & B \\
\downarrow m' & & \downarrow m \\
C & \xrightarrow{f} & A \\
\end{array}
\]

\( m \in M \Rightarrow m' \in M \)

\( \text{Par}_M A \) has the same objects as \( A \) but the morphisms are isomorphism classes of spans

\[
\begin{array}{ccc}
A_0 & \xrightarrow{m} & A \\
& \searrow f & \searrow \\
& A & B \\
\end{array}
\]

with \( m \in M \)
Composition is by pullback

The restriction operator is \((m, f) = (m, m)\)
The double category

Let $A$ be a restriction category

**Definition**

$f : A \to B$ is *total* if $\bar{f} = 1_A$

**Proposition**

The total morphisms form a subcategory of $A$

The double category $\mathbb{D}c(A)$ associated to a restriction category $A$ has

- The same objects as $A$
- Total maps as horizontal morphisms
- All maps as vertical morphisms
- There is a unique cell

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{v} & & \downarrow{m} \\
C & \xrightarrow{g} & D
\end{array}
$$

if and only if $gv = wf\bar{v}$
**Theorem**

\(Dc(A)\) is a double category

**Remark**

C & L define an order relation between \(f, g : A \to B\), \(f \leq g \iff f = g \bar{f}\)

Makes \(A\) into a 2-category. They say “seems to be less useful than one might expect”

There is a cell

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{v} & \Rightarrow & \downarrow{w} \\
C & \xrightarrow{g} & D
\end{array}
\]

if and only if \(gv \leq wf\). So our \(Dc(A)\) is not far from that 2-category. Perhaps it will turn out to be more useful than they might expect!
In $\mathcal{D}c\text{Par}_M(A)$ there is a cell if and only if there exists a (necessarily unique) morphism $h$
Companions

Proposition

In $\mathcal{D}c(A)$ every horizontal arrow has a companion, $f_* = f$

Proof.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & \Rightarrow & \downarrow 1 \\
B & \xrightarrow{1} & B
\end{array}
\quad 1 \cdot f = 1 \cdot f \cdot \bar{f}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{1} & A \\
\downarrow & \Rightarrow & \downarrow f \\
A & \xrightarrow{f} & B
\end{array}
\quad f \cdot 1 = f \cdot 1 \cdot \bar{1}
\]
Conjoints

Proposition

In \( \mathbb{DcPar}_M(A) \), \( f \) has a conjoint if and only if \( f \in M \)

Proof.

Assume \( f \) has conjoint \((m, g)\), then there are \( \alpha, \beta \)

\[
\begin{align*}
A & \xrightarrow{f} B \\
\downarrow & \downarrow m \\
A & \xrightarrow{\alpha} B_0
\end{align*} \quad \text{and} \quad
\begin{align*}
B & \xleftarrow{m} B \\
\downarrow & \downarrow \beta \\
B_0 & \xrightarrow{g} B
\end{align*}
\]

So \( m\alpha g = fg = \beta = m \) which implies \( \alpha g = 1 \)

Thus \( \alpha \) is an isomorphism and \( f = m\alpha \in M \)
If we suspect that \( A \) is of the form \( \mathbb{D}c\text{Par}_M(A) \) we can recover \( M \) as those horizontal arrows having a conjoint.

Is the requirement of stability under pullback of conjoints a good double category notion?

In \( \mathbb{D}c(A) \), a horizontal arrow \( f : A \rightarrow B \) always has a companion \( f_* \), and if it also has a conjoint \( f^* \) then \( f_* \dashv f^* \) so

\[
\begin{array}{ccc}
A & \xrightarrow{f_*} & B \\
\downarrow & & \downarrow \\
A & & A
\end{array}
\]

\( f_* \bullet f^* \)

is a comonad, i.e. an idempotent \( \leq \text{id}_A \)

**Proposition**

In \( \mathbb{D}c(A) \), \( f_* \bullet f^* = \bar{f}^* \)
Proposition

$\mathbb{D}cPar_M(A)$ has tabulators and they are effective.

Proof.

Given $(m, v) : A \rightarrow B$, the tabulator is

\[
\begin{array}{c}
A_0 \xrightarrow{m} A \\
\downarrow \\
A_0 \xrightarrow{v} B \\
\end{array}
\]

\[
\begin{array}{c}
A_0 \xleftarrow{m} A \\
\uparrow \\
A_0 \xleftarrow{v} B \\
\end{array}
\]

Conjecture: In a general $\mathbb{D}c(A)$, $v : A \rightarrow B$ has a tabulator if and only if $\bar{v}$ splits.
Classification of vertical arrows

- The original definition of elementary topos was in terms of a partial map classifier
  \[
  \begin{array}{c}
  B \\ B \rightarrow \tilde{A}
  \end{array}
  \]

- In a topos, relations are classifiable
  \[
  \begin{array}{c}
  B \\ B \rightarrow \Omega^A
  \end{array}
  \]

- For profunctors
  \[
  \begin{array}{c}
  B \\ B \rightarrow (\text{Set}^A)^{op}
  \end{array}
  \]
  provided \( A \) is small

- How do we formalize this in a general double category?
• The desired bijection

\[
\begin{array}{c}
B \xrightarrow{\cdot} A \\
B \xrightarrow{\tilde{v}} \tilde{A}
\end{array}
\]

gives \(eA : \tilde{A} \rightarrow A\) and \(hA : A \rightarrow \tilde{A}\)

• We express our definition in terms of \(eA\)

**Definition**

Let \(A\) be a double category and \(A\) an object of \(A\). We say that \(A\) is *classifying* if we are given an object \(\tilde{A}\) and a vertical morphism \(eA : \tilde{A} \rightarrow A\) with the following universal properties:
(1) For every vertical arrow $\nu : B \rightarrow A$ there exist a horizontal arrow $\hat{\nu} : B \rightarrow \tilde{A}$ and a cell

\[ B \xrightarrow{\hat{\nu}} \tilde{A} \]

\[ \begin{array}{c}
\nu \\
\downarrow \\
A
\end{array} \]

\[ \begin{array}{c}
\epsilon \nu \\
\downarrow \\
eA
\end{array} \]

such that for every cell $\alpha$

\[ D \xrightarrow{g} B \xrightarrow{\hat{\nu}} \tilde{A} \]

\[ \begin{array}{c}
w \\
\downarrow \\
C
\end{array} \]

\[ \begin{array}{c}
\alpha \\
\downarrow \\
eA
\end{array} \]

\[ C \xrightarrow{f} A \]

there exists a unique cell $\tilde{\alpha}$ such that

\[ D \xrightarrow{g} B \xrightarrow{\hat{\nu}} \tilde{A} \]

\[ \begin{array}{c}
w \\
\downarrow \\
C
\end{array} \]

\[ \begin{array}{c}
\tilde{\alpha} \\
\downarrow \\
eA
\end{array} \]

\[ \begin{array}{c}
\nu \\
\downarrow \\
A
\end{array} \]

\[ \begin{array}{c}
\epsilon \nu \\
\downarrow \\
eA
\end{array} \]

\[ = \]

\[ D \xrightarrow{g} B \xrightarrow{\hat{\nu}} \tilde{A} \]

\[ \begin{array}{c}
w \\
\downarrow \\
C
\end{array} \]

\[ \begin{array}{c}
\alpha \\
\downarrow \\
eA
\end{array} \]

\[ C \xrightarrow{f} A \]
(2) For every cell

\[ D \xrightarrow{\beta} A \]

there exists a unique cell \( \bar{\beta} \) such that

\[
\begin{align*}
D & \xrightarrow{\hat{w}} \tilde{A} \\
A & \xrightarrow{w} B \\
B & \xrightarrow{g} D
\end{align*}
\]

\[
\begin{align*}
D & \xrightarrow{\hat{v}} \tilde{A} \\
\tilde{A} & \xrightarrow{id} \tilde{A} \\
\tilde{A} & \xrightarrow{id} \tilde{A}
\end{align*}
\]

\[
\begin{align*}
D & \xrightarrow{\epsilon w} \tilde{A} \\
\tilde{A} & \xrightarrow{\epsilon A} \tilde{A} \\
\tilde{A} & \xrightarrow{\epsilon v} \tilde{A}
\end{align*}
\]

\[
\begin{align*}
D & \xrightarrow{\beta} A \\
A & \xrightarrow{\epsilon w} \tilde{A} \\
\tilde{A} & \xrightarrow{\epsilon A} \tilde{A}
\end{align*}
\]

\[
\begin{align*}
D & \xrightarrow{\epsilon v} \tilde{A} \\
\tilde{A} & \xrightarrow{\epsilon A} \tilde{A} \\
\tilde{A} & \xrightarrow{\epsilon v} \tilde{A}
\end{align*}
\]
Complete classification

- How do we understand this?
- Take a more global approach
  Assume $\mathbb{A}$ is companionable, i.e. every horizontal arrow $f$ has a companion $f_*$
  Then we get a (pseudo) double functor

$$\left( \_ \right)_* : \mathcal{Q} \mathcal{H}or\mathbb{A} \to \mathbb{A}$$

```
\begin{tikzcd}
A \arrow{r}{f} \arrow{d}{h} & B \arrow{d}{k} \\
C \arrow{r}{g} & D
\end{tikzcd} \quad \quad \quad \\
\begin{tikzcd}
A \arrow{r}{f} \arrow{d}{h_*} & B \arrow{d}{k} \\
C \arrow{r}{g} \arrow{r}[swap]{\alpha} & D
\end{tikzcd}
```

Exercise!

Definition

Say that $\mathbb{A}$ is **classifying** if $(\_)_*$ has a **down adjoint** $(\sim)$, i.e. a right adjoint in the vertical direction
Bijections

The adjunction can be formalized in terms of bijections

\[
\begin{array}{c}
B \\
\downarrow v \\
A
\end{array}
\rightarrow
\begin{array}{c}
\overset{\hat{v}}{\longrightarrow} \\
\end{array}
\begin{array}{c}
\tilde{A}
\end{array}
\]

More precisely, for \( v : B \rightarrow A \) there exists a \( \hat{v} : B \rightarrow \tilde{A} \) and an isomorphism

\[
\begin{array}{c}
B \\
\overset{(\hat{v})_{*}}{\longrightarrow} \\
\tilde{A}
\end{array}
\sim
\begin{array}{c}
\overset{v}{\longrightarrow} \\
\end{array}
\begin{array}{c}
A
\end{array}
\]

This can be expressed without mention of \( (\ )_{*} \) because we have a bijection
Yonedafication now yields the single-object definition
Given a monad \( (T, \eta, \mu) \) on \( A \) we get a double category \( \mathbb{Kl}(T) \)

- Objects are those of \( A \)
- Horizontal arrows are morphisms of \( A \)
- Vertical arrows are Kleisli morphisms i.e.

\[
\begin{array}{ccc}
A & \rightarrow & TB \\
\downarrow \quad \hat{v} & \quad & \downarrow \\
B & \rightarrow & A
\end{array}
\]

is a unique one if

\[
\begin{array}{ccc}
A & \rightarrow & TB \\
\downarrow \quad \hat{v} & \quad & \downarrow \\
B & \rightarrow & TB
\end{array}
\]

commutes
Kleisli (cont.)

- \( \mathcal{K}l(T) \) is companionable
  For \( f : B \rightarrow A \),

\[
\begin{array}{c}
B \\
\downarrow f_* \\
\downarrow \\
A
\end{array} \quad \text{is given by} \quad \begin{array}{c}
B \\
\downarrow f \\
\downarrow \\
A \\
\downarrow \eta_A \\
TA
\end{array} \quad \text{i.e.} \quad f_* = (\hat{\eta}_A \cdot f)
\]

- \( \mathcal{K}l(T) \) is classifiable

\[
\begin{array}{c}
B \\
\downarrow \hat{v} \\
\downarrow \\
A
\end{array} \quad \begin{array}{c}
B \\
\rightarrow TA
\end{array}
\]

- \( eA : TA \rightarrow A \) is \( \hat{id}_{TA} \)
- \( hA : A \rightarrow TA \) is \( \eta_A \)
• Double functors $\mathbb{K}l(T) \to \mathbb{K}l(S)$ correspond to monad morphisms $(F, \phi)$

$$A \xrightarrow{F} B$$

$$\phi : FT \to SF$$

such that ...

• Horizontal transformations correspond to the 2-cells in Street’s 1972 JPAA paper, *Formal theory of monads*

• Vertical transformations correspond to the 2-cells in Lack & Street’s 2002 paper, *Formal theory of monads II*
Restriction functors

- A restriction functor $F : A \to B$ is a functor that preserves the restriction operator, $F(\bar{f}) = \overline{F(f)}$

**Proposition**

A restriction functor $F$ gives a double functor $Dc(F) : Dc(A) \to Dc(B)$

**Question:** Is every double functor $F : Dc(A) \to Dc(B)$ of this form? $F$ is determined by a unique functor $A \to B$ which preserves the order and totality. Does this mean it preserves restriction? Probably not. Does $Dc$ at least reflect isos?

**Theorem**

A double functor $DcPar_M A \to DcPar_N B$ comes from a unique functor $F : A \to B$ which restricts to $M \to N$ and preserves pullbacks of $m \in M$ by arbitrary $f \in A$. Thus it does come from a restriction functor.
Transformations

Recall that a horizontal transformation $t : F \rightarrow G$ between double functors $\mathcal{A} \rightarrow \mathcal{B}$ consists of assignments:

1. For every $A$ in $\mathcal{A}$ a horizontal morphism $tA : FA \rightarrow GA$
2. For every vertical morphism $\nu : A \rightarrow \bar{A}$ a cell $G \bar{A} \rightarrow G\bar{A}$

satisfying

3. Horizontal naturality (for horizontal arrows and cells)
4. Vertical functoriality (for identities and composition)
Let $F, G : A \to B$ be restriction functors. Then a horizontal transformation

$$t : Dc(F) \to Dc(G)$$

(1) assigns to each $A$ in $A$ a total morphism

$$tA : FA \to GA$$

(2) such that for every $f : A \to \bar{A}$ in $A$ we have

$$\begin{array}{ccc}
FA & \to & GA \\
\downarrow Ff & \leq & \downarrow Gf \\
F\bar{A} & \to & G\bar{A}
\end{array}$$

(3) and $t$ is natural for horizontal arrows (i.e. for $f$ total, we have equality in (2))

This is what C & L call a lax restriction transformation
Proposition

Let \( M \subseteq A \) and \( N \subseteq B \) be stable systems of monics and \( F, G : A \rightarrow B \) functors that preserve the given monics and their pullbacks.

Then horizontal transformations \( \mathcal{D}c(F) \rightarrow \mathcal{D}c(G) \) correspond to arbitrary natural transformations \( F \rightarrow G \).

Restriction transformations correspond to cartesian ones.

There is a notion of commuter cell in double categories, and requiring the cells in (2) to be commuter cells makes them equalities.
Vertical transformations

A vertical transformation $\phi : \mathbb{D}c(F) \to \mathbb{D}c(G)$

1. assigns to each object $A$ of $\mathbf{A}$ an arbitrary morphism of $\mathbf{B}$

$$tA : FA \to GA$$

2. will be automatic

3. is natural with respect to all morphisms

4. is vacuous

Question: Is this any good?

There are other notions of vertical transformation, e.g. the modules of

  which generalize to double categories the modules of

Project: Investigate the significance of lax (oplax) double functors and modules for restriction categories
A restriction category $\mathbf{A}$ is *cartesian* if for every pair of objects $A, B$ there is an object $A \times B$ and morphisms $p_1 : A \times B \to A$, $p_2 : A \times B \to B$ with the following universal property

For every $f, g$ there exists a unique $h$ such that

$$p_1 h = f \bar{g}$$

$$p_2 h = g \bar{f}$$

There is also a terminal object condition
Double products

Recall that $A$ has binary products if

1. for every $A, B$ there is an object $A \times B$ and horizontal arrows $p_1 : A \times B \to A$, $p_2 : A \times B \to B$ which have the usual universal property with respect to horizontal arrows

2. for every pair of vertical arrows $v : A \to C$ and $w : B \to D$ there is a vertical arrow $v \times w : A \times B \to C \times D$ and cells

```
\[
\begin{array}{c}
A \times B \xrightarrow{p_1} A \\
\downarrow v \times w \downarrow \pi_1 \downarrow v \\
C \times D \xrightarrow{q_2} C \\
\end{array}
\quad \quad \quad
\begin{array}{c}
A \times B \xrightarrow{p_2} B \\
\downarrow v \times w \downarrow \pi_2 \downarrow w \\
C \times D \xrightarrow{q_2} D \\
\end{array}
\]
```

with the usual universal property with respect to cells
Proposition

A is a cartesian restriction category if and only if $\mathbb{D}(A)$ has finite double products

Proof *

(1) Suppose $A$ is a cartesian restriction category. The universal property of product is the usual one when restricted to total maps.

Given vertical arrows $v : A \rightarrow C$, $w : B \rightarrow D$ we get a unique $v \times w$

\[
\begin{align*}
A & \xleftarrow{p_1} A \times B \xrightarrow{p_2} B \\
C & \xleftarrow{q_1} C \times D \xrightarrow{q_2} D
\end{align*}
\]

and

\[
\begin{align*}
X & \xrightarrow{f} A & X & \xrightarrow{h} B & X & \xrightarrow{(f,h)} A \times B \\
Y & \xrightarrow{g} C & Y & \xrightarrow{k} D & Y & \xrightarrow{(g,k)} C \times D
\end{align*}
\]

so $\mathbb{D}(A)$ has binary double products
Suppose $\mathbb{D}_c(A)$ has finite double products.

Given

$$
\begin{array}{c}
\xymatrix{ C \ar[r]^f & A \\
\ar@{.>}[u]^g & B }
\end{array}
$$

we have

$$
\begin{array}{c}
\xymatrix{ h = C \ar[r]^\Delta_* & C \times C \ar[r]^{f \times g} & A \times B }
\end{array}
$$

and cells

$$
\begin{array}{c}
\xymatrix{ C \ar[r]^{1_C} & C \\
\ar@{.>}[d]_{\Delta_*} & \ar@{.>}[d]^{\text{id}} \\
C \times C \ar[r]^q & C \\
\ar@{.>}[u]_{f \times g} & \ar@{.>}[u]_f \\
A \times B \ar[r]_{p_1} & A }
\end{array}
$$

so $p_1 h = f(f \times g \bullet \Delta_*) = f \bar{g}$

*Warning: Some details may not have been checked*