Quantum Channels for Mix Unitary Categories

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Dagger compact closed categories

Dagger compact closed categories (†-KCC) provide a categorical framework for finite-dimensional quantum mechanics.

Dagger (†) is a contravariant functor which is stationary on objects ($A = A^{\dagger}$) and is an involution ($f^{\dagger\dagger} = f$).

In a †-KCC, quantum processes are represented by completely positive maps.

The CPM construction on a †-KCC chooses exactly the completely positive maps from the category.

FHilb, the category of finite-dimensional Hilbert Spaces and linear maps is an example of †-KCC.

CPM[FHilb] is precisely the category of quantum processes.

Finite versus infinite dimensions

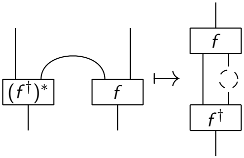
Dagger compact closed categories $\Rightarrow$ Finite-dimensionality on Hilbert Spaces. Because infinite-dimensional Hilbert spaces are not compact closed.

However, infinite-dimensional systems occur in many quantum settings including quantum computation and quantum communication.

There have been attempts to generalize the existing structures and constructions to infinite-dimensions.

The CP^∞ construction

CP^∞ construction generalized the CPM construction to †-symmetric monoidal categories (†-SMC) by rewriting the completely positive maps as follows:



Is there a way to generalize the CPM construction to arbitrary dimensions and still retain the goodness of the compact closed structure?

Linearly distributive categories

$*$-autonomous categories or more generally, linearly distributive categories generalize compact closed categories and allow for infinite dimensions.

A **linearly distributive category (LDC)** has two monoidal structures $(\otimes, \top, a_\otimes, u^L_\otimes, u^R_\otimes)$ and $(\oplus, \bot, a_\oplus, u^L_\oplus, u^R_\oplus)$ linked by natural transformations called the linear distributors:

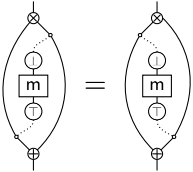
$$\partial_L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$
$$\partial_R : (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

LDCs are equipped with a graphical calculus.

Mix categories

A **mix category** is a LDC with a map $m : \bot \to \top$ in $\mathbb{X}$ such that

$$mx_{A,B} : A \otimes B \to A \oplus B :=$$



mx is called a **mix map**. The mix map is a natural transformation.

It is an **isomix** category if m is an isomorphism.

m being an isomorphism does not make the mx map an isomorphism.

The Core of mix category

The **core of a mix category**, $\mathrm{Core}(\mathbb{X}) \subseteq \mathbb{X}$, is the full subcategory determined by objects $U \in \mathbb{X}$ for which the natural transformation is also an isomorphism:

$$U \otimes (\_) \xrightarrow{\mathsf{mx}_{U,(\_)}} U \oplus (\_)$$

The core of a mix category is closed to $\otimes$ and $\oplus$.

The core of an isomix category contains the monoidal units $\top$ and $\bot$.

Roadmap

$$\begin{array}{c}
\text{LDC} \\
\Downarrow \\
\text{Define } \dagger\text{-LDC} \\
\Downarrow \\
\text{Define unitary isomorphisms in } \dagger\text{-LDCs} \\
\Downarrow \\
\text{Generalize } CP^\infty \text{ construction for } \dagger\text{-LDCs}
\end{array}$$

Forging the †

The definition of $\dagger : \mathbb{X}^{op} \longrightarrow \mathbb{X}$ cannot be directly imported to LDCs because the dagger minimally has to flip the tensor products: $(A \otimes B)^\dagger = A^\dagger \oplus B^\dagger$.

Why? If the dagger is identity-on-objects, then the linear distributor denegenerates to an associator:

$$\frac{(\delta_R)^\dagger : (A \oplus (B \otimes C))^\dagger \longrightarrow ((A \oplus B) \otimes C)^\dagger}{(\delta^R)^\dagger : A^\dagger \oplus (B^\dagger \otimes C^\dagger) \longrightarrow (A^\dagger \oplus B^\dagger) \otimes C^\dagger}$$

†-LDCs

A †-**LDC** is a LDC $\mathbb{X}$ with a dagger functor $\dagger : \mathbb{X}^{op} \to \mathbb{X}$ and the natural isomorphisms:

$$
\begin{aligned}
\text{tensor laxtors: } & \lambda_\oplus : A^\dagger \oplus B^\dagger \to (A \otimes B)^\dagger \\
& \lambda_\otimes : A^\dagger \otimes B^\dagger \to (A \oplus B)^\dagger \\
\text{unit laxtors: } & \lambda_\top : \top \to \bot^\dagger \\
& \lambda_\bot : \bot \to \top^\dagger \\
\text{involutor: } & \iota : A \to A^{\dagger\dagger}
\end{aligned}
$$

such that certain coherence conditions hold.

Coherences for †-LDCs

Coherences for the interaction between the tensor laxtors and the basic natural isomorphisms (6 coherences):

$$\begin{tikzcd}
A^\dagger \otimes (B^\dagger \otimes C^\dagger) \arrow[r, "a_\otimes"] \arrow[d, "1\otimes\lambda_\otimes"'] & (A^\dagger \otimes B^\dagger) \otimes C^\dagger \arrow[d, "\lambda_\otimes\otimes 1"] \\
(A^\dagger \otimes (B \oplus C)^\dagger) \arrow[d, "\lambda_\otimes"'] & (A \oplus B)^\dagger \otimes C^\dagger \arrow[d, "\lambda_\otimes"] \\
(A \oplus (B \oplus C))^\dagger \arrow[r, "(a_\oplus^{-1})^\dagger"'] & ((A \oplus B) \oplus C)^\dagger
\end{tikzcd}$$

Coherences for †-LDCs (cont.)

Interaction between the unit laxtors and the unitors (2 coherences):

$$\begin{tikzcd}
\top \otimes A^\dagger \arrow[r, "\lambda_\top \otimes 1"] \arrow[d, "u^l_\otimes"'] & \bot^\dagger \otimes A^\dagger \arrow[d, "\lambda_\otimes"] \\
A^\dagger \arrow[r, "(u^l_\oplus)^\dagger"'] & (\bot \oplus A)^\dagger
\end{tikzcd}
\qquad
\begin{tikzcd}
\bot \oplus A^\dagger \arrow[r, "\lambda_\bot \oplus 1"] \arrow[d, "u^l_\oplus"'] & \top^\dagger \oplus A^\dagger \arrow[d, "\lambda_\oplus"] \\
A^\dagger \arrow[r, "(u^l_\otimes)^\dagger"'] & (\top \otimes A)^\dagger
\end{tikzcd}$$

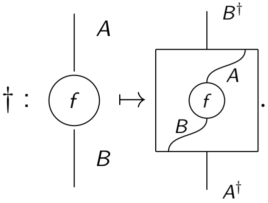
Interaction between the involutor and the laxtors (4 coherences):

$$\begin{tikzcd}
A \oplus B \arrow[r, "\iota"] \arrow[d, "i \oplus i"'] & ((A \oplus B)^\dagger)^\dagger \arrow[d, "\lambda_\otimes^\dagger"] \\
(A^\dagger)^\dagger \oplus (B^\dagger)^\dagger \arrow[r, "\lambda_\oplus"'] & (A^\dagger \otimes B^\dagger)^\dagger
\end{tikzcd}
\qquad
\begin{tikzcd}
\bot \arrow[r, "\iota"] \arrow[dr, "\lambda_\bot"'] & (\bot^\dagger)^\dagger \arrow[d, "\lambda_\top^\dagger"] \\
 & \top^\dagger
\end{tikzcd}$$

Diagrammatic calculus for †-LDC

Extends the diagrammatic calculus of LDCs

The action of dagger is represented diagrammatically using dagger boxes:



Isomix †-LDCs

A **mix †-LDC** is a †-LDC with $m : \perp \to \top$ such that:

$$\begin{tikzcd} \perp \arrow[r, "m"] \arrow[d, "\lambda_\perp"'] & \top \arrow[d, "\lambda_\top"] \\ \top^\dagger \arrow[r, "m^\dagger"'] & \perp^\dagger \end{tikzcd}$$

If m is an isomorphism, then $\mathbb{X}$ is an **iso-mix †-LDC**.

Lemma 1: The following diagram commutes in a mix †-LDC:

$$\begin{tikzcd} A^\dagger \otimes B^\dagger \arrow[r, "mx"] \arrow[d, "\lambda_\otimes"'] & A^\dagger \oplus B^\dagger \arrow[d, "\lambda_\oplus"] \\ (A \oplus B)^\dagger \arrow[r, "mx^\dagger"'] & (A \otimes B)^\dagger \end{tikzcd}$$

Isomix †-LDCs

Lemma 2: Suppose $\mathbb{X}$ is a mix †-LDC and $A \in \mathsf{Core}(\mathbb{X})$ then $A^\dagger \in \mathsf{Core}(\mathbb{X})$.

Proof: The natural transformation $A^\dagger \otimes X \xrightarrow{\mathsf{mx}} A^\dagger \oplus X$ is an isomorphism:

$$\begin{tikzcd}
A^\dagger \otimes X \arrow[r, "1 \otimes \iota"] \arrow[d, "\mathsf{mx}"'] \arrow[dr, phantom, "\text{nat. mx}"] & A^\dagger \otimes X^{\dagger\dagger} \arrow[r, "\lambda_\otimes"] \arrow[d, "\mathsf{mx}"'] \arrow[dr, phantom, "\text{Lemma 1}"] & (A \oplus X^\dagger)^\dagger \arrow[d, "\mathsf{mx}^\dagger"] \\
A^\dagger \oplus X \arrow[r, "1 \oplus \iota"'] & A^\dagger \oplus A^{\dagger\dagger} \arrow[r, "\lambda_\oplus"'] & (A \otimes X^\dagger)^\dagger
\end{tikzcd}$$

commutes.

Next step: Unitary structure

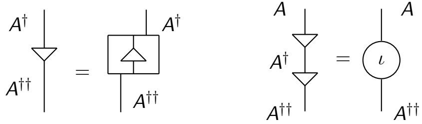
- ✓ Define †-LDC
- ? Define unitary isomorphisms

The usual definition of unitary maps ($f^\dagger : B^\dagger \to A^\dagger = f^{-1} : B \to A$) is applicable only when the † functor is stationary on objects.

Unitary structure

An isomix †-LDC has **unitary structure** in case there is an essentially small class of objects called **unitary objects** such that:

- Every unitary object, $A \in \mathcal{U}$, is in the core;
- Each unitary object $A \in \mathcal{U}$ comes equipped with an isomorphism, called the **unitary strucure** of A, : $A \xrightarrow{\varphi_A} A^\dagger$ such that



$$\varphi_{A^\dagger} = ((\varphi_A)^{-1})^\dagger \qquad (\varphi_A \varphi_{A^\dagger}) = \iota$$

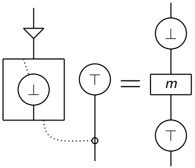
Unitary structure (cont.)

- $\top, \bot$ are unitary objects with:

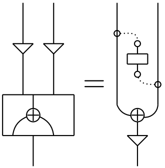
$$\varphi_\bot = \mathsf{m}\lambda_\top \qquad \varphi_\top = \mathsf{m}^{-1}\lambda_\bot$$

- If A and B are unitary objects then $A \otimes B$ and $A \oplus B$ are unitary objects such that:

$$(\varphi_A \otimes \varphi_B)\lambda_\otimes = \mathsf{mx}\varphi_{A\oplus B} : A \otimes B \to (A \otimes B)^\dagger$$
$$\varphi_{A\otimes B}\lambda_\oplus^{-1} = \mathsf{mx}(\varphi_A \oplus \varphi_B) : A \otimes B \to A^\dagger \oplus B^\dagger$$



$$\varphi_\bot\lambda_\top^{-1} = \mathsf{m}$$



$$(\varphi_A \otimes \varphi_B)\lambda_\otimes = \mathsf{mx}\varphi_{A\oplus B}$$

Mix Unitary Category (MUC)

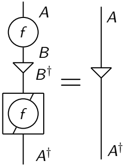
An iso-mix $\dagger$-LDC with unitary structure is called a **mixed unitary category**, **MUC**.

The unitary objects of a MUC, $\mathbb{X}$, determine a full subcategory, UCore($\mathbb{X}$), called the **unitary core**.

UCore($\mathbb{X}$) is always a compact linearly distributive subcategory of $\mathbb{X}$.

Unitary isomorphisms

Suppose A and B are unitary objects. An isomorphism $A \xrightarrow{f} B$ is said to be a **unitary isomorphism** if the following diagram commutes:



$$f \varphi_B f^\dagger = \varphi_A$$

Lemma: In a MUC

- $f^\dagger$ is a unitary map iff f is;
- $f \otimes g$ and $f \oplus g$ are unitary maps whenever f and g are.
- $a_\otimes, a_\oplus, c_\otimes, c_\oplus, \delta^L$, m, and mx are unitary isomorphisms.
- $\lambda_\otimes, \lambda_\oplus, \lambda_\top, \lambda_\bot$, and ι are unitary isomorphisms.
- φ_A is a unitary isomorphisms for for all unitary objects A.

Example of a MUC

Category of finite-dimensional framed vector spaces, FFVec_K

Objects: The objects are pairs $(V, \mathcal{V})$ where V is a finite dimensional K-vector space and $\mathcal{V} = \{v_1, ..., v_n\}$ is a basis;

Maps: These are vectors space homomorphisms which ignore the basis information;

Tensor product:

$$(V, \mathcal{V}) \otimes (W, \mathcal{W}) = (V \otimes W, \{v \otimes w | v \in \mathcal{V}, w \in \mathcal{W}\})$$

Tensor unit: $(K, \{e\})$ where e is the unit of the field K.

Example (cont.)

To define the "dagger" we assume that the field has an involution $\overline{(\_)} : K \to K$, that is a field homomorphism with $k = \overline{(\overline{k})}$.

This involution then can be extended to a (covariant) functor:

$$\overline{(\_)} : \mathsf{FFVec}_K \to \mathsf{FFVec}_K; \quad \begin{array}{c}(V, \mathcal{V}) \\ \downarrow f \\ (W, \mathcal{W})\end{array} \mapsto \begin{array}{c}\overline{(V, \mathcal{V})} \\ \downarrow \overline{f} \\ \overline{(W, \mathcal{W})}\end{array}$$

where $\overline{(V, \mathcal{V})}$ is the vector space with the same basis but the conjugate action $c \mathbin{\bar{\cdot}} v = \overline{c} \cdot v$. $\overline{f}$ is the same underlying map.

Example (cont.)

FFVec_K is also a compact closed category with $(V, \mathcal{B})^* = (V^*, \{\widetilde{b_i} | b_i \in \mathcal{B}\})$ where

$$V^* = V \multimap K \quad \text{and} \quad \widetilde{b_i} : V \to K; \left(\sum_j \beta_j \cdot b_j \right) \mapsto \beta_i$$

Hence, we have a contravariant functor $(\_)^* : \mathsf{FFVec}_K^{\mathrm{op}} \to \mathsf{FFVec}_K$.

$$(V, \mathcal{B})^\dagger = \overline{(V, \mathcal{B})^*}$$

$$\iota : (V, \mathcal{V}) \to ((V, \mathcal{V})^\dagger)^\dagger ; v \mapsto \lambda f . f(v)$$

FFVec_K is a compact LDC: $\otimes$ and $\oplus$ coincides.

Unitary structure of FFVec_K

Unitary structure for FFVec_K is $\varphi_{(V,\mathcal{V})} : (V,\mathcal{V}) \longrightarrow (V,\mathcal{V})^{\dagger}; v_i \mapsto \widetilde{v_i}$

Define a functor $U : \mathrm{FFVec}_K \longrightarrow \mathrm{Mat}(K)$

- for each object in FFVec_K we choose a total order on the elements of the basis
- any map is given by a matrix acting on the bases

Lemma: An isomorphism $u : (A,\mathcal{A}) \longrightarrow (B,\mathcal{B})$ in FFVec_K is unitary if and only if $U(f)$ is unitary in $\mathrm{Mat}(K)$.

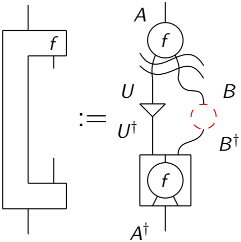
Next step: CP^{∞} construction on MUCs

- ✓ Define †-LDC
- ✓ Define unitary isomorphisms
- ✓ An example
- ? CP$^{\infty}$ construction on MUC

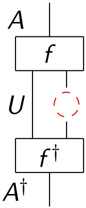
Krauss maps

In a MUC, a map $f : A \longrightarrow U \oplus B$ of $\mathbb{X}$ where U is a unitary object is called a **Krauss map** $f : A \to_U B$. U is called the **ancillary system** of f.

In a MUC, quantum processes are represented using Krauss maps as follows:



analogous to

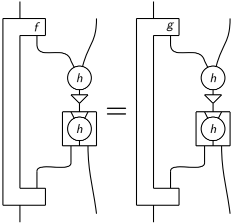


in †-SMCs.

$$A \xrightarrow{f} U \oplus B \xrightarrow{\mathrm{mx}^{-1}} U \otimes B \xrightarrow{\varphi \otimes 1} U^\dagger \otimes B$$
$$U^\dagger \otimes B^\dagger \xrightarrow{\lambda_\otimes} (U \oplus B)^\dagger \xrightarrow{f^\dagger} A^\dagger$$

Combinator and test maps

Two Krauss maps $f : A \to_{U_1} B$ and $g : A \to_{U_2} B$ are equivalent, $f \sim g$, if for all test maps $h : B \otimes X \to V$ where V is an unitary object, the following equation holds:



Lemma: Let $f : A \to_{U_1} B$ and $f' : A \to_{U_2} B$ be Krauss maps such that $U_1 \xrightarrow{\alpha} U_2$ is a unitary isomorphism with $f' = (\alpha \oplus 1)f$, then $f \sim f'$. In this case, f is said to be **unitarily isomorphic** to f'.

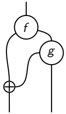
CP$^\infty$ construction

Given a MUC, $\mathbb{X}$, define $\mathrm{CP}^\infty(\mathbb{X})$ to have:

Objects: as of $\mathbb{X}$

Maps:

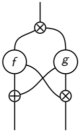
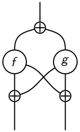
$$\mathrm{CP}^\infty(\mathbb{X})(A, B) := \{f \in \mathbb{X}(A, U \oplus B) | U \in \mathbb{X} \text{ and } U \text{ is unitary}\}/\sim$$

Composition: 

Identity: $A \xrightarrow{(u_\oplus^L)^{-1}} \bot \oplus A \in \mathbb{X}$

Tensor and Par

CP$^\infty(\mathbb{X})$ inherits tensor and par from $\mathbb{X}$:

$f \mathbin{\widehat{\otimes}} g :=$  $f \mathbin{\widehat{\oplus}} g :=$ 

$\widehat{\top} := \top$ $\widehat{\bot} := \bot$

CP^∞ construction on MUC (cont.)

Lemma: $\mathrm{CP}^\infty(\mathbb{X})$ is a well-defined category.

Proof sketch: Let $\mathbb{X}$ be a MUC, then there exists a functor $Q : \mathbb{X} \to \mathrm{CP}^\infty(\mathbb{X})$ as follows:

$$Q(A) := A$$
$$Q(f) := f(u_\oplus^L)^{-1}$$

Q is functorial since $f(u_\oplus^L)^{-1} \sim f \sim (u_\oplus^L)^{-1} f$.

$\mathrm{CP}^{\infty}(\mathbb{X})$ is an isomix category

Lemma: $\mathrm{CP}^{\infty}(\mathbb{X})$ is an isomix category.

Sketch of proof: The linear distribution maps, associators, unitors and symmetry isomorphisms - are inherited from $\mathbb{X}$ by composing each one of them with $(u_{\oplus}^{L})^{-1}$

$$\frac{A \otimes (B \otimes C) \xrightarrow{a_{\otimes}} (A \otimes B) \otimes C \xrightarrow{(u_{\oplus}^{L})^{-1}} \mathbb{X} \;/ \sim}{A \otimes (B \otimes C) \xrightarrow{a_{\widehat{\otimes}} := a_{\otimes}(u_{\oplus}^{L})^{-1}} (A \otimes B) \otimes C \in \mathrm{CP}^{\infty}(\mathbb{X})}$$

Naturality is proven by showing that the Krauss maps are unitarily isomorphic.

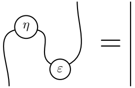
Linear adjoints

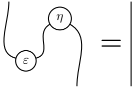
Suppose $\mathbb{X}$ is a LDC and $A, B \in \mathbb{X}$. Then, B is **left linear adjoint** to A $(\eta, \varepsilon) : B \dashv\!\vdash A$, if there exists

$$\eta : \top \to B \oplus A \qquad \varepsilon : A \otimes B \to \bot$$

such that the following triangle equalities hold:

$$\begin{tikzcd}
B \arrow[r, "(u_\otimes^L)^{-1}"] \arrow[d, equal] & \top \otimes B \arrow[r, "\eta \otimes 1"] & (B \oplus A) \otimes B \arrow[d, "\partial_R"] \\
B & B \oplus \bot \arrow[l, "u_\oplus^R"] & B \oplus (A \otimes B) \arrow[l, "1 \oplus \varepsilon"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "(u_\otimes^R)^{-1}"] \arrow[d, equal] & A \otimes \top \arrow[r, "1 \otimes \eta"] & A \otimes (B \oplus A) \arrow[d, "\partial_L"] \\
A & \bot \oplus A \arrow[l, "u_\oplus^L"] & (A \otimes B) \oplus A \arrow[l, "\varepsilon \oplus 1"]
\end{tikzcd}$$





When every object of a MUC has a linear adjoint, it is called a $*$- **MUC**.

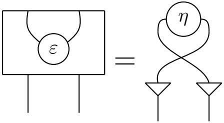
Unitary linear adjoints

Lemma: Let $\mathbb{X}$ be $\dagger$-LDC. If $A \dashv\!\!\vdash B$ then $B^\dagger \dashv\!\!\vdash A^\dagger$.

Proof: If $(\eta, \varepsilon) : A \dashv\!\!\vdash B$ then $(\lambda_\top \varepsilon^\dagger \lambda_\oplus^{-1}, \lambda_\otimes \eta^\dagger \lambda_\perp^{-1}) : B^\dagger \dashv\!\!\vdash A^\dagger$.

A **unitary linear adjoint** $(\eta, \varepsilon) : A \dashv\!\!\vdash_u B$ is a linear adjoint, $A \dashv\!\!\vdash B$ with A and B being unitary objects satisfying:

$$\eta_A(\varphi_A \oplus \varphi_B)c_\oplus = \lambda_\top \varepsilon^\dagger \lambda_\oplus^{-1} \qquad (\varphi_A \otimes \varphi_B)\lambda_\otimes \eta_A^\dagger = c_\otimes \varepsilon_A \lambda_\perp$$



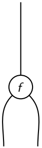
$$\lambda_\top \varepsilon^\dagger \lambda_\oplus^{-1} = \eta c_\oplus(\varphi_A \oplus \varphi_B)$$

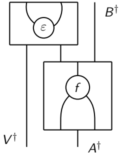
A MUC in which every unitary object has a unitary linear adjoint is called a **MUdC**.

Dagger functor for $\mathrm{CP}^{\infty}(\mathbb{X})$

Lemma: If $\mathbb{X}$ is a $*$-MUdC, then $\mathrm{CP}^{\infty}(\mathbb{X})$ is a $*$-MUdC.

Sketch of proof: Suppose $f : A \longrightarrow U \oplus B$ and $(\eta, \varepsilon) : V \dashv\!\!\dashv_u U$

$\dagger : \mathrm{CP}^{\infty}(\mathbb{X})^{\mathrm{op}} \longrightarrow \mathrm{CP}^{\infty}(\mathbb{X});$ 

$\mapsto$ 

Unitary structure and unitary linear adjoints are preserved due to the functoriality of Q.

Summary: Mix Unitary Categories

Mix Unitary Categories are †-LDCs with unitary structure.

There is a diagrammatic calculus for MUCs.

If every unitary object has a unitary linear adjoint, then the unitary core is analogous to a dagger compact closed category.

Summary: CP^∞ construction on MUCs

CP^∞ on MUCs strictly generalizes CP^∞ construction on †-SMCs.

The construction produces an isomix category.

The construction is functorial when every object has a linear adjoint.

The construction produces a *-MUdC when every unitary object has a unitary linear adjoint.

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