The Representability of Partial Recursive Functions in Arithmetical Theories and Categories

Yan Steimle

Department of Mathematics and Statistics
University of Ottawa

Foundational Methods in Computer Science
May 30 – June 2, 2018
First-order theories

- With equality
- $\Gamma \vdash \varphi$ satisfying the rules for intuitionistic sequent calculus
- Logical axioms:
  - For all theories, decidability of equality (DE):
    $$ x \neq y \lor x = y $$
  - To obtain classical theories, the excluded middle (EM):
    $$ \neg \varphi \lor \varphi, \text{ for all formulas } \varphi $$

First-order theories

With equality

$\Gamma \vdash \varphi$ satisfying the rules for intuitionistic sequent calculus

Logical axioms:

- For all theories, decidability of equality (DE):
  $$ x \neq y \lor x = y $$
- To obtain classical theories, the excluded middle (EM):
  $$ \neg \varphi \lor \varphi, \text{ for all formulas } \varphi $$
Let $\mathcal{L}_M$ be the first-order language with 0, $S$, $\cdot$, $\,+$.

Let $S^m(0)$ be the $n^{th}$ numeral, denoted $\overline{n}$.

Let $x < y$ abbreviate $(\exists w)(x + S(w) = y)$.

Let $(\exists! y)\varphi(x, y)$ abbreviate

$$(\exists y)\varphi(x, y) \land (\forall y)(\forall z)(\varphi(x, y) \land \varphi(x, z) \Rightarrow y = z).$$
The arithmetical theory $M$

**Definition**

$M$ is a theory over $\mathcal{L}_M$ with the nonlogical axioms

1. $(M1)$ $S(x) \neq 0$
2. $(M2)$ $S(x) = S(y) \Rightarrow x = y$
3. $(M3)$ $x + 0 = x$
4. $(M4)$ $x + S(y) = S(x + y)$
5. $(M5)$ $x \cdot 0 = 0$
6. $(M6)$ $x \cdot S(y) = (x \cdot y) + x$
7. $(M7)$ $x \neq 0 \Rightarrow (\exists y)(x = S(y))$
8. $(M8)$ $x < y \lor x = y \lor y < x$

We consider an arbitrary *arithmetical theory* $T$, i.e. a consistent r.e. extension of $M$. 
Recursive functions, brief overview

- Primitive recursive: basic functions; closed under substitution (S) and primitive recursion (PR)
- Total recursive: basic functions; closed under (S), (PR), and total $\mu$
- Partial recursive: basic functions; closed under (S), (PR), and partial $\mu$
**Definition**

A function \( f : \mathbb{N}^k \to \mathbb{N} \) is *numeralwise representable* in \( T \) as a total function if there exists a formula \( \varphi(x, y) \) satisfying

(a) for all \( m, n \in \mathbb{N}^{k+1} \), if \( f(m) = n \), then \( \vdash \varphi(\overline{m}, \overline{n}) \)

(b) for all \( m \in \mathbb{N}^k \), \( \vdash (\exists! y) \varphi(\overline{m}, y) \)

\( f \) is *strongly representable* in \( T \) as a total function if there exists a formula \( \varphi(x, y) \) satisfying (a) and

(b)' \( \vdash (\exists! y) \varphi(x, y) \)
Representability of partial functions

**Definition**

For $f : \mathbb{N}^k \rightarrow \mathbb{N}$ and $\varphi(x, y)$ consider the conditions

(P1) for all $m, n \in \mathbb{N}^{k+1}$, $f(m) \equiv n$ iff $\vdash \varphi(\overline{m}, \overline{n})$

(P2) for all $m \in \mathbb{N}^k$, $\vdash \varphi(\overline{m}, y) \land \varphi(\overline{m}, z) \Rightarrow y = z$

(P3) $\vdash \varphi(x, y) \land \varphi(x, y) \Rightarrow y = z$

(P4) $\vdash (\exists ! y) \varphi(x, y)$

For $f : \mathbb{N}^k \rightarrow \mathbb{N}$, if there exists $\varphi(x, y)$ in $T$ such that

- (P1) and (P2) hold, $f$ is *numeralwise representable* in $T$ as a partial function
- (P1) and (P3) hold, $f$ is *type-one representable* in $T$
- (P1) and (P4) hold, $f$ is *strongly representable* in $T$ as a partial function
Theorem (I)

*Let* $T$ *be any arithmetical theory. All partial recursive functions are type-one representable in* $T$.

Theorem (II)

*Let* $T$ *be a* \textit{classical} *arithmetical theory. All partial recursive functions are strongly representable in* $T$ *as partial functions.*
Consequences of the Existence Property (EP)

The Existence Property (EP)

For every formula $\varphi$ in $T$ and any variable $x$ occurring free in $\varphi$,

$$\text{if } \vdash (\exists x)\varphi, \text{ then } \exists n \in \mathbb{N} \text{ such that } \vdash \varphi \left[ \frac{n}{x} \right].$$
The Kleene normal form theorem (alternate version)

**Theorem (Kleene normal form)**

*For each $k \in \mathbb{N}$, $k > 0$, there exist primitive recursive functions $U : \mathbb{N} \to \mathbb{N}$ and $T_k : \mathbb{N}^{k+2} \to \mathbb{N}$ such that, for any partial recursive function $f : \mathbb{N}^k \to \mathbb{N}$, there exists a number $e \in \mathbb{N}$ such that*

$$f(m) \simeq U(\mu_n(T_k(e, m, n) = 0))$$

*for all $m \in \mathbb{N}^k$.***
The strong representability of primitive recursive functions in arithmetical theories

**Theorem**

Let $T$ be any arithmetical theory. All primitive recursive functions are strongly representable in $T$ as total functions.

**Proof.**

It suffices to express the basic functions and the recursion schemes $(S)$ and $(PR)$ by formulas in $T$. For example:

- $y = S(x)$ strongly represents the successor function.
- If $g, h : \mathbb{N} \to \mathbb{N}$ are primitive recursive and strongly representable by $\psi(y, z)$ and $\varphi(x, y)$, respectively, then

$$ (\exists y)(\varphi(x, y) \land \psi(y, z)) $$

strongly represents $f = g(h) : \mathbb{N} \to \mathbb{N}$. 
Lemma (1)

Let $g : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ ($k \geq 0$) be a total function that is numeralwise representable in $T$ as a total function, and let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be obtained from $g$ by partial $\mu$. Then, $f$ is type-one representable in $T$.

Proof.

g is numeralwise representable in $T$ by $\sigma(x, y, z)$ and $f$ is defined by

$$f(m) \simeq \mu_n(g(m, n) = 0).$$

Thus, $f$ is type-one representable in $T$ by the formula

$$\sigma(x, y, 0) \land (\forall u)(u < y \Rightarrow \neg\sigma(x, u, 0)).$$
Weak representability of r.e. relations

**Definition (Weak representability)**

A relation $E \subseteq \mathbb{N}^k$ is *weakly representable* in $T$ if there exists a formula $\psi(x)$ with exactly $k$ free variables such that, for all $\mathbf{m} \in \mathbb{N}^k$,

$$E(\mathbf{m}) \text{ iff } \vdash \psi(\overline{\mathbf{m}}).$$

**Lemma (2)**

*All* $k$-ary r.e. relations on $\mathbb{N}$ ($k \geq 0$) *are weakly representable* in $T$.

(long technical proof)
Proof of Theorem (I)

Theorem (I)

Let $T$ be any arithmetical theory. All partial recursive functions are type-one representable in $T$.

Proof.

Let $f : \mathbb{N}^k \to \mathbb{N}$ be a partial recursive function.

$k = 0$: If $f$ is the constant $n$ in $\mathbb{N}$, take the formula $\overline{n} = y$. If $f$ is completely undefined, take the formula $y = y \land 0 \neq 0$. 
Proof of Theorem (I)

Proof (continued).

\( k \geq 1 \): By the Kleene normal form theorem, we obtain

\[ f(m) \simeq U(\mu_n(T_k(e, m, n) = 0)) \quad \forall m \in \mathbb{N}^k. \]

As \( T_k \) is primitive recursive, by Lemma 1 there exists a

formula \( \sigma(x, z) \) that type-one represents the partial function

given by

\[ \mu_n(T_k(e, m, n) = 0) \quad \forall m \in \mathbb{N}^k. \]

As \( U \) is primitive recursive, there exists a formula \( \varphi(z, y) \)

that strongly represents \( U \) as a total function. \( \varphi \) also
type-one represents \( U \).
Proof of Theorem (I)

Proof (continued).

By Lemma 2, there exists a formula $\eta(x)$ that weakly represents the r.e. domain $D_f$ of $f$. Then, $f$ is type-one representable in $T$ by the formula $\theta(x, y)$ defined by

$$
\eta(x) \land (\exists z)(\sigma(x, z) \land \varphi(z, y)).
$$

Indeed,

- (P3) for $\theta$ follows from (P3) for $\sigma$ and $\varphi$.
- For (P1), since $\eta$ weakly represents $D_f$, we only have to consider inputs on which $f$ is defined. Hence, we can show that $\vdash \theta(m, \bar{p})$ implies $f(m) \simeq p$ by (P3) for $\theta$ and the fact that $\vdash f(m) = \bar{p}$ iff $f(m) = p$. 

Definition (Exact separability)

Two relations $E, F \subseteq \mathbb{N}^k$ are exactly separable in $T$ if there exists a formula $\psi(x)$ in $T$ with exactly $k$ free variables such that, for all $m \in \mathbb{N}^k$,

\[
E(m) \iff \vdash \psi(m) \\
F(m) \iff \vdash \neg \psi(m)
\]

Lemma (3)

Let $T$ be a classical arithmetical theory. Any two disjoint $k$-ary r.e. relations on $\mathbb{N}$ ($k \geq 0$) are exactly separable in $T$.

(long technical proof)
Proof of Theorem (II)

**Theorem (II)**

Let $T$ be a *classical* arithmetical theory. All partial recursive functions are strongly representable in $T$ as partial functions.

**Proof.**

Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a partial recursive function.

$k = 0$: If $f$ is completely undefined, let $G$ be a closed undecidable formula in $T$ and take

$$(y = 0 \Rightarrow \neg G) \land (y \neq 0 \Rightarrow G) \land y < 2$$
Proof of Theorem (II)

Proof (continued).

$k \geq 1$: Let $n_0, n_1 \in \mathbb{N}$ be distinct. By Lemma 3, we obtain a formula $\sigma(x)$ that exactly separates $f^{-1}(\{n_0\})$ and $f^{-1}(\{n_1\})$. By Theorem (I), we obtain a formula $\varphi(x, y)$ that type-one represents $f$. Consider

$$
\psi(x) \equiv (\exists z) \varphi(x, z) \land \neg \varphi(x, n_0) \land \neg \varphi(x, n_1)
$$

$$
\theta(x, y) \equiv (\psi(x) \land \varphi(x, y)) \lor (\neg \psi(x) \land \sigma(x) \land y = n_0)
$$

$$
\lor (\neg \psi(x) \land \neg \sigma(x) \land y = n_0).
$$

By (EM), $\vdash (\neg \psi(x) \land \neg \sigma(x)) \lor (\neg \psi(x) \land \sigma(x)) \lor \psi(x)$, from which (P4) follows. (P1) is obtained by cases.
Representability of total recursive functions

**Corollary (of Theorem (I))**

*Let \( T \) be an arithmetical theory. All total recursive functions are numeralwise representable in \( T \) as total functions.*

**Corollary (of Theorem (II))**

*Let \( T \) be a *classical* arithmetical theory. All total recursive functions are strongly representable in \( T \) as total functions.*
Given a theory $T$, we construct a classifying category $\mathcal{C}(T)$:
- objects: formulas of $T$
- morphisms: equivalence classes of provably functional relations between formulas

For a general theory $T$, $\mathcal{C}(T)$ is regular.

If $T$ is an intuitionistic arithmetical theory, we claim that in $\mathcal{C}(T)$:
- there is at least a weak NNO;
- numerals are standard;
- 1 is projective and indecomposable.
For an arithmetical theory $T$:

1. Consider the formulas representing recursive functions in $C(T)$ (for all possible variations). What sub-categories do we obtain?

2. Construct a partial map category associated with $C(T)$ and show it is a Turing category.

3. Ultimately, we want to consider partial recursive functionals of higher type using a notion of $\lambda$-calculus with equalisers and a construction of the free CCC with equalisers.
I would like to thank my supervisor, Professor Scott, the conference organisers, the University of Ottawa, and NSERC.
References


To deal with partialness, we use Kleene Equality. If $e_1$ and $e_2$ are two expressions on $\mathbb{N}$ that may or may not be defined, then

$$e_1 \simeq e_2 \text{ iff } (e_1, e_2 \text{ are defined and equal})$$

$$\text{OR } (e_1, e_2 \text{ are undefined}).$$

For example, if $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is a partial function and $m, n \in \mathbb{N}^{k+1}$,

$$f(m) \not\simeq n \text{ iff } (f(m) \text{ is defined but not equal to } n)$$

$$\text{OR } (f(m) \text{ is undefined}).$$
Consequences of the Existence Property (EP)

If $T$ is classical:

- $G$ is a closed undecidable formula in $T$, $f : \mathbb{N}^k \rightarrow \mathbb{N}$ the completely undefined function.

$$\varphi(x, y) \equiv x = x \land (y = 0 \Rightarrow \neg G) \land (y \neq 0 \Rightarrow G) \land y < 2$$

strongly represents $f$ in $T$ as a partial function.

- If $T$ were to satisfy EP, then $\vdash G$ or $\vdash \neg G$, a contradiction.
Consequences of the Existence Property (EP)

If $T$ is intuitionistic:

- Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a partial function undefined at $m \in \mathbb{N}$.
- Suppose that there exists $\varphi(x, y)$ satisfying (P1) and (P4).
- By (P4), $\vdash (\exists y)\varphi(m, y)$.
- By EP, there exists $n \in \mathbb{N}$ such that $\vdash \varphi(m, n)$.
- By (P1), $f(m) \simeq n$, and so $f(m)$ is defined.

Contradiction.

So, strong representability of partial functions doesn’t make sense and Theorem (II) fails.
A technical lemma

**Lemma**

Let $E_1 \subseteq \mathbb{N}^k$ and $E_2 \subseteq \mathbb{N}^{k+j}$ ($k, j \geq 0$) be r.e. relations. There exists a formula $\varphi(x, u)$ in $T$ with $k+j$ free variables such that, for all $m \in \mathbb{N}^k$ and $p \in \mathbb{N}^j$,

- if $E_1(m)$ and $\neg E_2(m, p)$, then $\vdash \varphi(m, p)$
- if $\neg E_1(m)$ and $E_2(m, p)$, then $\not\vdash \varphi(m, p)$. 
Proof (idea).

(Adapted from the case for \( j = 0 \) in [S]) Let \( E_1 \subseteq \mathbb{N}^k \) and \( E_2 \subseteq \mathbb{N}^{k+j} \) \((k, j \geq 0)\) be r.e. relations. There exist primitive recursive relations \( F_1 \subseteq \mathbb{N}^{k+1} \), \( F_2 \subseteq \mathbb{N}^{k+j+1} \) such that, for all \( m \in \mathbb{N}^k \) and \( p \in \mathbb{N}^j \),

\[
E_1(m) \text{ iff } \exists n \in \mathbb{N} \text{ s.t. } F_1(m, n) \\
E_2(m, p) \text{ iff } \exists n \in \mathbb{N} \text{ s.t. } F_2(m, p, n).
\]

We obtain formulas \( \psi_1(x, y) \) and \( \psi_2(x, u, y) \) that numeralwise represent \( F_1 \) and \( F_2 \), respectively, in \( T \). Then, \( \varphi(x, u) \) given by

\[
(\exists y)(\psi_1(x, y) \land (\forall z)(z \leq y \Rightarrow \neg \psi_2(x, u, z))
\]

is the required formula.
Proof (Lemma 2).

\( k = 0: \mathbb{N}^0 = \{\ast\} \) is weakly representable by \( 0 = 0 \) and \( \emptyset \) is weakly representable by \( 0 \neq 0 \).

\( k \geq 1: \) Let \( E \subseteq \mathbb{N}^k \), let \( x, y \) be \( k + 1 \) distinct fixed variables. \( T \) has an associated Gödel numbering where \( \Gamma \psi \) denotes the Gödel number of \( \psi \) and \( \gamma_n \) is the formula with Gödel number \( n \). Then, we can construct a primitive recursive function \( g: \mathbb{N}^{k+1} \rightarrow \mathbb{N} \) such that

\[
g(m, n) = \begin{cases} 
\Gamma \gamma_n \left[ \overline{m} x, \overline{n} y \right] \Gamma & \text{if } \gamma_n \text{ exists} \\
n & \text{otherwise}
\end{cases}
\]
Proof (continued).

Since $T$ is an r.e. theory, $D \subseteq \mathbb{N}^{k+1}$ given by

$$D(m, n) \text{ iff } GTHM_T(g(m, n)) \text{ iff } \vdash \gamma_n \left[ \frac{m}{x}, \frac{n}{y} \right]$$

is an r.e. relation. By the technical lemma, we obtain $\varphi(x, y)$ in $T$ such that, for all $m, n \in \mathbb{N}^{k+1}$,

- if $E(m)$ and $\not\vdash \gamma_n \left[ \frac{m}{x}, \frac{n}{y} \right]$, then $\vdash \varphi(m, n)$
- if $\neg E(m)$ and $\vdash \gamma_n \left[ \frac{m}{x}, \frac{n}{y} \right]$, then $\not\vdash \varphi(m, n)$. 
Proof (continued).

Let \( p = \langle \varphi(x, y) \rangle \). Then, \( \gamma_p = \varphi \) and so, for all \( m \in \mathbb{N}^k \),

- if \( E(m) \) and \( \nvdash \varphi(\overline{m}, \overline{p}) \), then \( \vdash \varphi(\overline{m}, \overline{p}) \)
- if \( \neg E(m) \) and \( \vdash \varphi(\overline{m}, \overline{p}) \), then \( \nvdash \varphi(\overline{m}, \overline{p}) \).

It follows that, for all \( m \in \mathbb{N}^k \),

\[
E(m) \text{ iff } \vdash \varphi(\overline{m}, \overline{p}),
\]

and so \( \varphi(x, \overline{p}) \) weakly represents \( E \) in \( T \).