COMPLETELY AND TOTALLY DISTRIBUTIVE CATEGORIES I

DEDICATED TO OUR MENTOR AND FRIEND, BOB PARÉ

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ABSTRACT. In 1978, Street and Walters defined a locally small category $\mathscr{K}$ to be *totally cocomplete* if its Yoneda functor Y has a left adjoint X. Such a $\mathscr{K}$ is *totally distributive* if X has a left adjoint W. Small powers of the category of small sets are totally distributive, as are certain sheaf categories. A locally small category $\mathscr{K}$ is small cocomplete if it is a $\mathscr{P}$-algebra, where $\mathscr{P}$ is the small-colimit completion monad on **Cat**. In 2007, Day and Lack showed that $\mathscr{P}$ lifts to $\mathscr{R}$-algebras, where $\mathscr{R}$ is the small-limit completion monad on **Cat**. It follows that there is a distributive law $\mathscr{R}\mathscr{P} \longrightarrow \mathscr{P}\mathscr{R}$ and we say that $\mathscr{K}$ is *completely distributive* if $\mathscr{K}$ is a $\mathscr{P}\mathscr{R}$-algebra, meaning that $\mathscr{K}$ is small cocomplete, small complete, and $\mathscr{P}\mathscr{K} \longrightarrow \mathscr{K}$ preserves small limits. Totally distributive implies completely distributive. We show that there is a further supply of totally distributive categories provided by categories of interpolative bimodules between small *taxons* as introduced by Koslowski in 1997.

1. Introduction

Many notions of classes of limits distributing over classes of colimits have been studied. To give just two examples, consider the case of finite products distributing over finite coproducts and that of finite limits distributing over all small colimits. A category that is distributive in the sense we discuss has: colimits of a given class, limits of a given class, and the functor which "assigns" the colimits in question preserves the limits in question. This statement warrants clarification.

First, the class of colimits needs, for our purposes, to be given by a *KZ-doctrine*. (We remark that K stands for Kock and Z for Zöberlein, who discovered independently the important condition on 2-dimensional monads that we recall below. The term "KZ-doctrine" seems to have been coined by Street in [ST1].) This means that there is a pseudomonad, say $\mathscr{C} = (\mathscr{C}, \mu, \eta)$, on a suitable 2-category of categories, say **Cat**, with the property that $\mathscr{C}\eta \dashv \mu \dashv \eta\mathscr{C}$. (Actually, the pseudomonad structure is secondary to the adjoint string but the details would take us too far afield and we refer the reader to

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[MAR1].) For a $\mathscr{K}$ in **Cat**, we require that $\mathscr{C}\mathscr{K}$ be the free colimit completion of $\mathscr{K}$ with respect to the colimits in question and then $\mathscr{K}$ has such colimits precisely if the unit component $\eta\mathscr{K}:\mathscr{K}\to\mathscr{C}\mathscr{K}$ has a left adjoint, $\mathscr{C}\mathscr{K}\to\mathscr{K}$, which is seen as *assignment of $\mathscr{C}$-colimits for $\mathscr{K}$*.

Next, the class of limits should also be given by a pseudomonad $\mathscr{L}=(\mathscr{L},\mu,\eta)$ on **Cat**, this with what is known as the *coKZ* property, meaning that here $\eta\mathscr{L}\dashv\mu\dashv\mathscr{L}\eta$. For such monads it follows that $\mathscr{K}$ has $\mathscr{L}$-structure, meaning $\mathscr{L}$-limits, if and only if the unit component $\eta\mathscr{K}:\mathscr{K}\to\mathscr{L}\mathscr{K}$ has a right adjoint $\mathscr{L}\mathscr{K}\to\mathscr{K}$ which is then seen as assignment of $\mathscr{L}$-limits for $\mathscr{K}$.

So, if $\mathscr{K}$ has $\mathscr{C}$-colimits and $\mathscr{L}$-limits and we further demand that assignment of $\mathscr{C}$-colimits for $\mathscr{K}$ preserve $\mathscr{L}$-structure, then we need to know that $\mathscr{K}$ having $\mathscr{L}$-limits implies that $\mathscr{C}\mathscr{K}$ has $\mathscr{L}$-limits too. Stated properly this means that $\mathscr{C}$ lifts to the category of $\mathscr{L}$-algebras and it is standard that liftings of a monad $\mathscr{C}$ to the category of $\mathscr{L}$-algebras for a monad $\mathscr{L}$ are in bijective correspondence with distributive laws $\mathscr{L}\mathscr{C}\to\mathscr{C}\mathscr{L}$, in the sense of [BEK]. However, in the case of pseudomonads this well-known basic theory has to be carefully adjusted, for the equational aspects are then at the level of coherence of isomorphisms which replace Beck's equations. Moreover, the heightened complexity of the general pseudo case is then reduced, in another direction, when the pseudomonads are KZ or coKZ. Fortunately, the required theory for such distributive laws has been worked out in [MAR2] and its sequel [M&W].

In the cases at hand, because the structures in question are unique to within isomorphism, distributive laws are also essentially unique. One speaks of a category $\mathscr{K}$ with $\mathscr{C}$-colimits and $\mathscr{L}$-limits and $\mathscr{C}\mathscr{K}\to\mathscr{K}$ preserving $\mathscr{L}$-limits as being an algebra for the distributive law $\mathscr{L}\mathscr{C}\to\mathscr{C}\mathscr{L}$.

Examples of the monads $\mathscr{C}$ mentioned above are provided by the *finite families* construction, whose algebras are categories with finite coproducts, and by the small-colimit completion monad $\mathscr{P}$ that sends a locally small category $\mathscr{K}$ to the full subcategory of $\mathbf{set}^{\mathscr{K}^{\mathrm{op}}}$ determined by all small colimits of representables. Examples of the monads $\mathscr{L}$ are finite-product completion and finite-limit completion. It is fairly easy to see that if a category $\mathscr{K}$ has finite products then its finite colimit completion $\mathrm{fam}(\mathscr{K})$ does too and that $\sum:\mathrm{fam}(\mathscr{K})\to\mathscr{K}$ preserves finite products if and only if the canonical arrows $(A\times B)+(A\times C)\to A\times(B+C)$ are invertible.

It is somewhat harder to see that if a category $\mathscr{K}$ has finite limits then $\mathscr{P}\mathscr{K}$ does too. Fortunately, this and more is made explicit in [D&L] and we will make further use of their results. In the other example we mentioned at the beginning of this Introduction, a category $\mathscr{K}$ with finite limits and small-colimits has finite limits distributing over small colimits if and only if assignment of small-colimits $\mathscr{P}\mathscr{K}\to\mathscr{K}$ preserves finite limits. It is known, perhaps not widely, that a category $\mathscr{K}$ is a Grothendieck topos if and only if it is an algebra for this distributive law and has a small set of generators.

Other distributive laws of the kind we speak been given serious attention by other authors, notably [ALR1] and [ALR2]. The distributive law on which we focus:

$$\lim\cdot\mathrm{colim}\to\mathrm{colim}\cdot\lim$$

appears fleetingly at the end of [ABLR] but we prefer to establish it here by building on the work of [D&L]. This is distributivity of all small limits over all small colimits and we propose that an algebra for it should be called a *completely distributive category*.

From the two paragraphs above it follows that, modulo generators, a completely distributive category is a very special Grothendieck topos. It is well known that a Grothendieck topos $\mathscr{K}$ is also *lex total*, which is to say that the Yoneda functor $Y_{\mathscr{K}}: \mathscr{K} \to \mathbf{set}^{\mathscr{K}^{\mathrm{op}}}$ has a left exact left adjoint. A lex total category with a small set of generators is certainly a Grothendieck topos but Peter Freyd has shown, see [ST2], that a lex total category, with a rather subtle condition on the size of its set of objects, necessarily has a small set of generators. There is no similar result for categories that are merely total but it is natural to ask if a total category for which the left adjoint to the Yoneda functor has itself a left adjoint, subject to no further conditions, has a small set of generators. Such categories were called *totally distributive* in [R&W].

We show that totally distributive categories are completely distributive and provide examples of the former. We do not know of an example of a completely distributive category that is not totally distributive. We needed to widen the scope of total categories somewhat to include categories that are not necessarily locally small. We call these *prototal categories* and after introducing them in Section 2 find some results that appear to be of independent interest. In Sections 3 and 4 we study totally distributive categories and completely distributive categories, respectively. In Section 5 we recall Koslowski's notions of *taxon* and of *i-modules* between taxons. We show, for small taxons $\mathbf{T}$ and $\mathbf{S}$, that the category $\textbf{i-mod}(\mathbf{T},\mathbf{S})$ of i-modules between them is a totally distributive category. Along the way to this major result of the paper we find it necessary to relate taxon functors $\mathbf{T}^{\mathrm{op}} \to I\mathbf{set}$ and i-modules $I\mathbf{1} \to\!\!\!\!\!+ \mathbf{T}$, where I is the interpretation of a category as a taxon. We establish an equivalence between these which, unlike the situation for functors $\mathscr{C}^{\mathrm{op}} \to \mathbf{set}$ and profunctors $\mathbf{1} \to\!\!\!\!\!+ \mathscr{C}$, is not a mere isomorphism. We also in Lemma 5.12 prove a "Yoneda Lemma" for taxons and we think that these results about taxons are also of independent interest.

2. Total and Prototal Categories

2.1 For a category [bicategory] $\mathscr{K}$, we write $|\mathscr{K}|$ for the set of objects of $\mathscr{K}$. If X and A are objects of $\mathscr{K}$, $\mathscr{K}(X,A)$ denotes the set [category] of arrows from X to A. If $\mathscr{K}$ is a category of sets we say that $\mathscr{K}$ is a full category of sets if, for all sets X and A in $|\mathscr{K}|$, $\mathscr{K}(X,A)$ is the set of all functions from X to A. We assume the existence of full categories of sets called **set** and **SET**, both toposes, with **set** contained in **SET**, and $|\mathbf{set}|$ an object of **SET**. The sets in **set** are called *small* sets. We assume that **set** has all sums indexed by objects of **set** and **SET** has all sums indexed by objects of **SET**. We will write $i: \mathbf{set} \to \mathbf{SET}$ for the inclusion. We write **CAT** for the 2-category of category objects in **SET** and **cat** for the 2-category of category objects in **set**. Note that **set** is an object of **CAT**. The objects of **cat** are called *small* categories. If a category $\mathscr{K}$ in **CAT** has all its hom-sets $\mathscr{K}(A,B)$ in **set**, we say that $\mathscr{K}$ is *locally small* and **Cat** denotes the full sub-2-category of **CAT** determined by the locally small categories. For $\mathscr{X}$ and $\mathscr{A}$ in **CAT**, we write $\mathbf{Prof}(\mathscr{X},\mathscr{A})$ for $\mathbf{CAT}(\mathscr{A}^{\mathrm{op}} \times \mathscr{X}, \mathbf{set})$ and refer to its objects as small profunctors from $\mathscr{X}$ to $\mathscr{A}$. We write $P : \mathscr{X} \nrightarrow \mathscr{A}$ for an object of $\mathbf{Prof}(\mathscr{X},\mathscr{A})$. Note that **Prof** is not a bicategory but we also write $\mathbf{PROF}(\mathscr{X},\mathscr{A})$ for the category of all functors $\mathscr{A}^{\mathrm{op}} \times \mathscr{X} \to \mathbf{SET}$. With the usual compositions, **PROF** is a bicategory and we have the usual proarrow equipment $(-)_* : \mathbf{CAT} \to \mathbf{PROF}$. (Note that **SET** is not an object of any of the bicategories under consideration.)

2.2 A locally small category $\mathscr{K}$ has a Yoneda functor $Y_{\mathscr{K}} = Y : \mathscr{K} \to \mathbf{Cat}(\mathscr{K}^{\mathrm{op}}, \mathbf{set})$. Henceforth, we will often write $\widehat{\mathscr{K}}$ for $\mathbf{CAT}(\mathscr{K}^{\mathrm{op}}, \mathbf{set})$, for *any* category $\mathscr{K}$. (It follows that $\mathbf{Prof}(\mathscr{X},\mathscr{A}) = \widehat{\mathscr{A} \times \mathscr{X}^{\mathrm{op}}}$.) Note that if $\mathscr{K}$ is locally small then $\widehat{\mathscr{K}}$ is locally small if and only if $\mathscr{K}$ is small — the "only if" clause being the celebrated result of [F&S] — but, in any event, $\widehat{\mathscr{K}}$ is in **CAT**. Following [S&W] we say that the locally small $\mathscr{K}$ is *totally cocomplete* (usually abbreviated to *total*) if $Y_{\mathscr{K}}$ has a left adjoint, which will then be called X. For a small profunctor P in $\mathbf{Prof}(\mathscr{X},\mathscr{A})$ and a functor F in $\mathbf{CAT}(\mathscr{A},\mathscr{K})$, a *$P$-weighted colimit for F* is a functor $F \bullet P : \mathscr{X} \to \mathscr{K}$ and a natural transformation $\iota : P(F \bullet P)^* \to F^*$ which exhibits $(F \bullet P)^*$ as a right lifting of F^* through P in **PROF** as in

$$\begin{tikzcd}
& \mathscr{K} \arrow[dl, "(F\bullet P)^*"'] \arrow[dr, "F^*"] \arrow[d, phantom, "\xrightarrow{\iota}"] & \\
\mathscr{X} \arrow[rr, "P"', "\shortmid" marking] & {} & \mathscr{A}
\end{tikzcd}$$

It follows that, if $F \bullet P$ exists then it is given uniquely, to within isomorphism, by the requirement that, for all X in $\mathscr{X}$ and K in $\mathscr{K}$, we have

$$\mathscr{K}((F \bullet P)(X), K) \cong \int_A \mathscr{K}(FA, K)^{P(A,X)}$$

where the right hand side is the (possibly large) set of natural transformations

$$\text{from} \qquad \mathscr{A}^{\mathrm{op}} \xrightarrow{P(-,X)} \mathbf{set} \xrightarrow{i} \mathbf{SET} \qquad \text{to} \qquad \mathscr{A}^{\mathrm{op}} \xrightarrow{\mathscr{K}(F-,K)} \mathbf{SET}$$

which is $\mathbf{SET}^{\mathscr{A}^{\mathrm{op}}}(iP(-,X), \mathscr{K}(F-,K))$. For any $\mathscr{K}$ in **CAT** (not necessarily locally small), we say that $\mathscr{K}$ is *prototal* if, for every small $P : \mathscr{X} \nrightarrow \mathscr{K}$, $1_{\mathscr{K}} \bullet P$ exists. From the description above, it is clear that we can take $\mathscr{X} = \mathbf{1}$ so that $\mathscr{K}$ is prototal if and only if, for every small $P : \mathbf{1} \nrightarrow \mathscr{K}$, $1_{\mathscr{K}} \bullet P$ exists. In more primitive terms, this last means that, *for every P in $\widehat{\mathscr{K}}$, there is an object $1_{\mathscr{K}} \bullet P$ in $\mathscr{K}$ and, for every K in $\mathscr{K}$, for every $p \in PK$, arrows $\iota_p : K \to 1_{\mathscr{K}} \bullet P$, natural in K, such that given any family $\langle \phi_p : K \to L \rangle_{p \in PK, K \in \mathscr{K}}$, natural in K, there is a unique arrow $f : 1_{\mathscr{K}} \bullet P \to L$ such that, for all K, for all $p \in PK$, $f\iota_p = \phi_p$. In other words, the ι_p mediate bijections, for every L in $\mathscr{K}$, between $\mathscr{K}(1_{\mathscr{K}} \bullet P, L)$ and the the set of natural transformations*

$$\textit{from} \qquad \mathscr{K}^{\mathrm{op}} \xrightarrow{P} \mathbf{set} \xrightarrow{i} \mathbf{SET} \qquad \textit{to} \qquad \mathscr{K}^{\mathrm{op}} \xrightarrow{\mathscr{K}(-,L)} \mathbf{SET}$$

2.3. Lemma. *For a locally small category $\mathscr{K}$, $\mathscr{K}$ is total if and only if $\mathscr{K}$ is prototal.*

Proof. Trivial, because if $\mathscr{K}$ is locally small then each $\mathscr{K}(-,L)$ factors through the full and faithful $i:\mathbf{set}\to\mathbf{SET}$. For total $\mathscr{K}$, we have $X(P)=1_{\mathscr{K}}\bullet P$. ∎

For any $\mathscr{K}$, we will write $X(P)$ for $1_{\mathscr{K}}\bullet P$ when this particular weighted colimit exists. In fact it is clear that, for a prototal category $\mathscr{K}$, we have $X:\widehat{\mathscr{K}}\to\mathscr{K}$ left adjoint to $Y:\mathscr{K}\to\mathbf{SET}^{\mathscr{K}^{\mathrm{op}}}$, relative to $i^{\mathscr{K}^{\mathrm{op}}}:\mathbf{set}^{\mathscr{K}^{\mathrm{op}}}\to\mathbf{SET}^{\mathscr{K}^{\mathrm{op}}}$, in the sense of [S&W].

It was observed as far back as [S&W] that a locally small category $\mathscr{K}$ is total if and only if every discrete fibration, with small fibres, into $\mathscr{K}$ has a colimit. Since this characterization does not mention local smallness and extends to prototal categories, it provides another proof of Lemma 2.3.

2.4. Lemma. *Full reflective subcategories of [pro]total categories are [pro]total.*

Proof. By Lemma 2.3 it suffices to prove the "pro" version. So assume $a\dashv j:\mathscr{L}\to\mathscr{K}$ with j fully faithful and $\mathscr{K}$ prototal. Take P in $\widehat{\mathscr{L}}$. We claim $a(X(Pa^{\mathrm{op}}))$ provides the weighted colimit $1_{\mathscr{L}}\bullet P$. (We remark that Pa^{op} is indeed a small profunctor from $\mathbf{1}$ to $\mathscr{K}$.) For L in $\mathscr{L}$ we have

$$\mathscr{L}(a(X(Pa^{\mathrm{op}})),L)\cong\mathscr{K}(X(Pa^{\mathrm{op}}),jL)\cong\mathbf{SET}^{\mathscr{K}^{\mathrm{op}}}(iPa^{\mathrm{op}},\mathscr{K}(-,jL))$$

$$\cong\mathbf{SET}^{\mathscr{L}^{\mathrm{op}}}(iP,\mathscr{K}(-,jL)j^{\mathrm{op}})\cong\mathbf{SET}^{\mathscr{L}^{\mathrm{op}}}(iP,\mathscr{K}(j-,jL))\cong\mathbf{SET}^{\mathscr{L}^{\mathrm{op}}}(iP,\mathscr{L}(-,L))$$

∎

2.5. Theorem. *If $\mathscr{K}$ is locally small then $\widehat{\mathscr{K}}$ is prototal. In fact $X_{\widehat{\mathscr{K}}}=\widehat{Y}_{\mathscr{K}}$.*

Proof. Let $P:(\widehat{\mathscr{K}})^{\mathrm{op}}\to\mathbf{set}$ be given. We claim that $X(P)$ is given by the composite

$$\mathscr{K}^{\mathrm{op}}\xrightarrow{Y^{\mathrm{op}}}(\widehat{\mathscr{K}})^{\mathrm{op}}\xrightarrow{P}\mathbf{set}$$

(which is $\widehat{Y}_{\mathscr{K}}(P)$). Take F in $\widehat{\mathscr{K}}$. To give a natural transformation α from $(\widehat{\mathscr{K}})^{\mathrm{op}}\xrightarrow{iP}\mathbf{SET}$ to $\widehat{\mathscr{K}}(-,F):(\widehat{\mathscr{K}})^{\mathrm{op}}\to\mathbf{SET}$ is to give, for each G, components $\alpha_G:P(G)\to\widehat{\mathscr{K}}(G,F)$. In $\mathbf{SET}^{\mathscr{K}^{\mathrm{op}}}$, G is a colimit with injections $\iota_x:\mathscr{K}(-,K)\to G$ for each element $x\in G(K)$. It follows that $\widehat{\mathscr{K}}(G,F)$ is a limit in $\mathbf{SET}$ with projections $\widehat{\mathscr{K}}(G,F)\to\widehat{\mathscr{K}}(\mathscr{K}(-,K),F)$ given by the functions $\widehat{\mathscr{K}}(\iota_x,F)$. Since α is natural we have, for each $x\in G(K)$,

$$\begin{tikzcd} P(G) \arrow[r, "\alpha_G"] \arrow[d, "P(\iota_x)"'] & \widehat{\mathscr{K}}(G,F) \arrow[d, "\widehat{\mathscr{K}}(\iota_x,F)"] \\ P(\mathscr{K}(-,K)) \arrow[r, "\alpha_{\mathscr{K}(-,K)}"'] & \widehat{\mathscr{K}}(\mathscr{K}(-,K),F) \end{tikzcd}$$

It follows that α is uniquely determined by the components

$$\alpha_{\mathscr{K}(-,K)} : P(\mathscr{K}(-,K)) \to \widehat{\mathscr{K}}(\mathscr{K}(-,K), F)$$

which can equally be seen as the components of a natural tranformation

$$PY^{\mathrm{op}} \to F : \mathscr{K}^{\mathrm{op}} \to \mathbf{set}$$

The determination provides a bijection witnessing PY^{op} as the colimit $X(P)$. ∎

2.6 A category $\mathscr{K}$ is *cototal* if $\mathscr{K}^{\mathrm{op}}$ is total. In terms of the locally small $\mathscr{K}$ this means that the Yoneda functor $Z : \mathscr{K} \to \widehat{\mathscr{K}^{\mathrm{op}}}^{\mathrm{op}} = \left(\mathbf{set}^{\mathscr{K}}\right)^{\mathrm{op}}$ has a right adjoint, which will then be called A. For a small profunctor Q in $\mathbf{Prof}(\mathscr{A}, \mathscr{X})$ and a functor F in $\mathbf{CAT}(\mathscr{A}, \mathscr{K})$, a Q-weighted limit for F is a functor $\{Q, F\} : \mathscr{X} \to \mathscr{K}$ and a natural transformation $\pi : \{Q, F\}_* Q \to F_*$ which exhibits $\{Q, F\}_*$ as a right extension of F_* along Q in $\mathbf{PROF}$ as in

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "Q", "\shortmid"{marking}] \arrow[dr, "F_*"'] & \arrow[d, phantom, "\xleftarrow{\pi}"] & \mathscr{X} \arrow[dl, "\{Q,F\}_*"] \\
& \mathscr{K} &
\end{tikzcd}$$

For any $\mathscr{K}$ in $\mathbf{CAT}$, we say that $\mathscr{K}$ is *procototal* if $\mathscr{K}^{\mathrm{op}}$ is prototal. This is the case if and only if for every small $Q : \mathscr{K} \to\!\!\!\!\!\!\shortmid\,\, \mathscr{X}$, $\{Q, 1_{\mathscr{K}}\}$ exists if and only if, for every small $Q : \mathscr{K} \to\!\!\!\!\!\!\shortmid\,\, \mathbf{1}$, $\{Q, 1_{\mathscr{K}}\}$ exists. This last means that for every K in $\mathscr{K}$,

$$\mathscr{K}(K, \{Q, 1_{\mathscr{K}}\}) \cong \int_L \mathscr{K}(K, L)^{Q(L)}$$

so that $\mathscr{K}(K, \{Q, 1_{\mathscr{K}}\})$ is isomorphic to the set of natural transformations

$$\text{from} \qquad \mathscr{K} \xrightarrow{Q} \mathbf{set} \xrightarrow{i} \mathbf{SET} \qquad \text{to} \qquad \mathscr{K} \xrightarrow{\mathscr{K}(K,-)} \mathbf{SET}$$

For any $\mathscr{K}$, we will write $A(Q)$ for $\{Q, 1_{\mathscr{K}}\}$ when this particular weighted limit exists. It is now clear that $\mathscr{K}$ is procototal if and only if there exists $A : (\mathbf{set}^{\mathscr{K}})^{\mathrm{op}} \to \mathscr{K}$ right adjoint to $Z : \mathscr{K} \to (\mathbf{SET}^{\mathscr{K}})^{\mathrm{op}}$, relative to $(i^{\mathscr{K}})^{\mathrm{op}} : (set^{\mathscr{K}})^{\mathrm{op}} \to (\mathbf{SET}^{\mathscr{K}})^{\mathrm{op}}$, and that

2.7. Corollary. *For a locally small category $\mathscr{K}$, $\mathscr{K}$ is cototal if and only if $\mathscr{K}$ is procototal.* ∎

Dualizing Lemma 2.4 we have

2.8. Lemma. *Full coreflective subcategories of [pro]cototal categories are [pro]cototal.* ∎

2.9. Lemma. *If $\mathscr{K}$ is a locally small category then, for all $Q : \widehat{\mathscr{K}} \to \mathbf{set}$ and all K in $\mathscr{K}$, the set of natural transformations from Q to $\widehat{\mathscr{K}}(\mathscr{K}(-,K),-)$ is small. In fact, it is bounded by $2^{Q(2^{\mathscr{K}(K,-)})}$.*

Proof. The functor $\widehat{\mathscr{K}}(\mathscr{K}(-,K),-)$ is evaluation at K in $\mathscr{K}$ so that we have $\widehat{\mathscr{K}}(K(-,K),-) = \widehat{K}$, where we see the object K as the functor $K : \mathbf{1} \to \mathscr{K}$. Since $\mathbf{1}$ is small and $\mathscr{K}$ is locally small, $\widehat{K}$ has both left and right adjoints. Writing K_* for the right adjoint, the usual argument shows that, for S in $\mathbf{set}$, $K_*(S)$ is given by $K_*(S)(L) = S^{\mathscr{K}(K,L)}$.

Next, recall that the canonical $\iota : 1_{\mathbf{set}} \rightarrowtail 2^{\mathbf{set}(-,2)} : \mathbf{set} \to \mathbf{set}$ is monic. For any $F, G : \mathscr{C} \to \mathbf{set}$, horizontal composition with ι provides a function

$$\iota \circ - : \mathbf{set}^{\mathscr{C}}(F,G) \to \mathbf{set}^{\mathscr{C}}(F, 2^{\mathbf{set}(G-,2)})$$

We can write $\iota \circ -$ as the vertical composite $\iota G.-$, which is one to one because ιG is monic. It follows that if $\mathbf{set}^{\mathscr{C}}(F, 2^{\mathbf{set}(G-,2)})$ is small then $\mathbf{set}^{\mathscr{C}}(F,G)$ is small.

Thus to show the statement of the Lemma, that $\mathbf{set}^{\widehat{\mathscr{K}}}(Q, \widehat{K})$ is small, it suffices to show that $\mathbf{set}^{\widehat{\mathscr{K}}}(Q, 2^{\mathbf{set}(\widehat{K}-,2)})$ is small. But

$$\begin{array}{c}
Q \to 2^{\mathbf{set}(\widehat{K}-,2)} \\
\hline
Q \to 2^{\widehat{\mathscr{K}}(-,K_*(2))} \\
\hline
\widehat{\mathscr{K}}(-,K_*(2)) \to 2^{Q-} \\
\hline
\widehat{\mathscr{K}}(-,2^{\mathscr{K}(K,-)}) \to 2^{Q-} \\
\hline
\bullet \in 2^{Q(2^{\mathscr{K}(K,-)})}
\end{array}$$

∎

2.10. Theorem. *If $\mathscr{K}$ is locally small then $\widehat{\mathscr{K}}$ is procototal.*

Proof. Let $Q : \widehat{\mathscr{K}} \to \mathbf{set}$ be given. We claim that $A(Q) : \mathscr{K}^{\mathrm{op}} \to \mathbf{set}$ is given by $A(Q)(K) = \mathbf{set}^{\widehat{\mathscr{K}}}(Q, \widehat{\mathscr{K}}(\mathscr{K}(-,K),-))$ which by Lemma 2.9 is small, an object of $\mathbf{set}$. Take F in $\widehat{\mathscr{K}}$ and consider

$$\widehat{\mathscr{K}}(F, A(Q)) \cong \int_K \mathbf{set}(F(K), A(Q)(K)) \cong \int_K \mathbf{set}(F(K), \mathbf{set}^{\widehat{\mathscr{K}}}(Q, \widehat{\mathscr{K}}(\mathscr{K}(-,K),-)))$$

$$\cong \int_K \mathbf{set}(F(K), \int_G \mathbf{set}(Q(G), G(K))) \cong \int_K \int_G \mathbf{set}(F(K), \mathbf{set}(Q(G), G(K)))$$

$$\cong \int_G \int_K \mathbf{set}(Q(G), \mathbf{set}(F(K), G(K))) \cong \int_G \mathbf{SET}(iQ(G), \int_K \mathbf{set}(F(K), G(K)))$$

$$\cong \int_G \mathbf{SET}(iQ(G), \widehat{\mathscr{K}}(F,G)) \cong \mathbf{SET}^{\widehat{\mathscr{K}}}(iQ, \widehat{\mathscr{K}}(F,-))$$

∎

Theorems 2.5 and 2.10 say, respectively, that locally small powers of **set** are prototal and procototal.

3. Totally Distributive Categories

The following definition first appeared in [R&W].

3.1. Definition. *A total category is* totally distributive *if* $\exists W \dashv X : \widehat{\mathscr{K}} \to \mathscr{K}$.

3.2. Proposition. *A totally distributive category is total and cototal and* $X : \widehat{\mathscr{K}} \to \mathscr{K}$ *preserves all limits.*

Proof. By definition $\mathscr{K}$ is total and X being a right adjoint preserves all limits. From the fully faithful adjoint string $W \dashv X \dashv Y : \mathscr{K} \rightarrowtail \widehat{\mathscr{K}}$ we have W fully faithful so that the adjunction $W \dashv X$ exhibits $\mathscr{K}$ as a full coreflective subcategory of $\widehat{\mathscr{K}}$, which by Theorem 2.10 is procototal. By Lemma 2.8, $\mathscr{K}$ is procototal and, since $\mathscr{K}$ is locally small, it is cototal by Corollary 2.7. ∎

3.3. Remark. By [WD2] Theorem 5, for a totally distributive category $\mathscr{K}$, we have

$$\begin{tikzcd}
\widehat{\mathscr{K}} \arrow[rr, shift right, "(-)^+"'] \arrow[dr, "X"'] & & (\widehat{\mathscr{K}^{\mathrm{op}}})^{\mathrm{op}} \arrow[ll, shift right, "(-)^-"'] \arrow[ll, phantom, "\top"] \arrow[dl, "A"] \\
& \mathscr{K} &
\end{tikzcd}$$

where $(-)^+ \dashv (-)^-$ is the Isbell conjugation adjunction.

A reasonable supply of totally distributive categories is provided by

3.4. Theorem. *For small* $\mathscr{C}$, $\widehat{\mathscr{C}}$ *is totally distributive.*

Proof. For small $\mathscr{C}$, $\widehat{\mathscr{C}}$ is locally small so that $Y_{\mathscr{C}} : \mathscr{C} \to \widehat{\mathscr{C}}$ has both left and right Kan extensions, giving the adjoint string $(Y_{\mathscr{C}})_! \dashv \widehat{Y_{\mathscr{C}}} \dashv (Y_{\mathscr{C}})_* : \widehat{\mathscr{C}} \to \widehat{\widehat{\mathscr{C}}}$. By [S&W], $(Y_{\mathscr{C}})_* = Y_{\widehat{\mathscr{C}}}$. ∎

Totally distributive categories are closed under "quotients" and "subobjects" in the following sense:

3.5. Lemma. *If* $\mathscr{K}$ *is totally distributive and either*

$$\begin{tikzcd}
\mathscr{L} \arrow[r, two heads, shift left=3, "k"] \arrow[r, two heads, shift right=3, "j"'] & \mathscr{K} \arrow[l, "a" description] \arrow[l, phantom, shift left=1.5, "\perp"] \arrow[l, phantom, shift right=1.5, "\perp"]
\end{tikzcd}
\quad or \quad
\begin{tikzcd}
\mathscr{L} \arrow[r, tail, "j" description] & \mathscr{K} \arrow[l, shift right=3, "a"'] \arrow[l, shift left=3, "b"] \arrow[l, phantom, shift left=1.5, "\perp"] \arrow[l, phantom, shift right=1.5, "\perp"]
\end{tikzcd}$$

(assuming in the second case the existence of $a_! \dashv \widehat{a}$*) then* $\mathscr{L}$ *is totally distributive.*

Proof. In either case, from $a \dashv j$ and using $j_! = \widehat{a}$ it is standard that $X_{\mathscr{L}} \cong aX_{\mathscr{K}}\widehat{a}$.

If, further, we have $k \dashv a$ then we also have $\widehat{k} \dashv \widehat{a}$ so that the left adjoint of $X_{\mathscr{L}} \cong aX_{\mathscr{K}}\widehat{a}$ is $W_{\mathscr{L}} \cong \widehat{k}W_{\mathscr{K}}k$.

On the other hand, if $j \dashv b$ then j preserves colimits so that $X_{\mathscr{K}}j_! \cong jX_{\mathscr{L}}$. In this case we also have $X_{\mathscr{L}} \cong bjX_{\mathscr{L}} \cong bX_{\mathscr{K}}j_!$ and now, taking left adjoints, we get $W_{\mathscr{L}} \cong a_!W_{\mathscr{K}}j$ (assuming $a_! \dashv \widehat{a} = j_!$ exists). ∎

This provides a further class of examples of totally distributive categories.

3.6. Corollary. *If $\mathscr{L}$ is a lex CCD lattice, meaning that $\Downarrow$, the defining left adjoint to the supremum operation, is left exact; then $\mathbf{shv}(\mathscr{L})$ is totally distributive.*

Proof. The adjunction $\Downarrow \dashv \bigvee : D\mathscr{L} \to \mathscr{L}$, where $D\mathscr{L}$ is the lattice of down-closed subsets of $\mathscr{L}$, in the 2-category of frames corresponds to the adjunction $\bigvee \dashv \downarrow$ in the 2-category of locales so that we have

$$\begin{tikzcd} \mathbf{shv}(\mathscr{L}) \arrow[r, shift left=6] \arrow[r, phantom, shift left=3, "\perp"] & \mathbf{shv}(D\mathscr{L}) \arrow[l, "\mathbf{shv}(\bigvee)"'] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd} \simeq \widehat{\mathscr{L}}$$
$$\mathbf{shv}(\mathscr{L}) \xrightarrow{\mathbf{shv}(\downarrow)} \mathbf{shv}(D\mathscr{L})$$

The displayed equivalence results from the fact that any sheaf on $D\mathscr{L}$ seen as a functor $(D\mathscr{L})^{\mathrm{op}} \to \mathbf{set}$ is uniquely determined, within isomorphism, by its restriction along $\downarrow^{\mathrm{op}} : \mathscr{L}^{\mathrm{op}} \to (D\mathscr{L})^{\mathrm{op}}$. By Theorem 3.4, $\widehat{\mathscr{L}}$ is totally distributive so, by Lemma 3.5, $\mathbf{shv}(\mathscr{L})$ is totally distributive. ∎

3.7. Remark. The term *lex CCD lattice* was introduced in [R&W] but the concept had appeared earlier in [B&N] under the name *stably supercontinuous frame* where it was shown that such lattices are precisely the regular projectives in the category of frames. This condition on $\mathscr{L}$ is stronger than we need since the left adjoint to $\mathbf{shv}(\bigvee)$ is left exact, which is not required for application of Lemma 3.5.

4. Completely Distributive Categories

4.1 Following [D&L] and many others before, we write $\mathscr{P}\mathscr{K}$ for the full subcategory of $\widehat{\mathscr{K}}$ determined by all small colimits of representables. We recall that if $\mathscr{K}$ is locally small then $\mathscr{P}\mathscr{K}$ is locally small and that $\mathscr{P}\mathscr{K} = \widehat{\mathscr{K}}$ if and only if $\mathscr{K}$ is small. For $\mathscr{K}$ locally small, $\mathscr{P}\mathscr{K}$ is the free small-colimit completion of $\mathscr{K}$. By left Kan extension, $\mathscr{P}$ becomes a 2-functor $\mathscr{P} : \mathbf{Cat} \to \mathbf{Cat}$ which underlies a KZ-(pseudo)monad whose unit factors the Yoneda functor $Y : \mathscr{K} \to \widehat{\mathscr{K}}$. We will also write $Y : \mathscr{K} \to \mathscr{P}\mathscr{K}$ for the unit of the monad $\mathscr{P}$. We will write M for the multiplication on $\mathscr{P}$ and, by [MAR1], we have $\mathscr{P}Y_{\mathscr{K}} \dashv M_{\mathscr{K}} \dashv Y_{\mathscr{P}\mathscr{K}}$, for any locally small $\mathscr{K}$. The $\mathscr{P}$-algebras are the small-cocomplete categories and $\mathscr{K}$ is small-cocomplete if and only if $Y : \mathscr{K} \to \mathscr{P}\mathscr{K}$ has a left adjoint. If a small-complete $\mathscr{K}$ is total then its $\mathscr{P}$-structure is the restriction of $X : \widehat{\mathscr{K}} \to \mathscr{K}$ so we will generically use $X : \mathscr{P}\mathscr{K} \to \mathscr{K}$ for $\mathscr{P}$-structure functors. It is a formality that $\mathscr{R} = (\mathscr{P}((-)^{\mathrm{op}}))^{\mathrm{op}} : \mathbf{Cat} \to \mathbf{Cat}$ underlies the free small-limit completion (coKZ)-monad.

4.2 It is well known that $\widehat{\mathscr{K}}$ is small-complete and we have seen in Theorem 2.10 that, for $\mathscr{K}$ locally small, $\widehat{\mathscr{K}}$ is procototal. By contrast, $\mathscr{P}\mathscr{K}$ is not necessarily small-complete. A counterexample is given in [D&L]. We remark that when $\mathscr{P}\mathscr{K}$ is small-complete and $\mathscr{K}$ is small-cocomplete, $\mathscr{K}$ is a full reflective subcategory of a small-complete category and hence also small-complete. Since small-cocomplete is known to not imply small-complete, we have another demonstration that $\mathscr{P}\mathscr{K}$ is not small-complete, in general. The authors of [D&L] remark that because $\mathscr{P}\mathscr{K}$ contains all the representables, any limit in $\mathscr{P}\mathscr{K}$ is pointwise. Indeed if $P:\mathscr{I}\to\mathscr{P}\mathscr{K}$ and $\varprojlim P$ exists then, for any K in $\mathscr{K}$, we have:

$$(\varprojlim_{I\in\mathscr{I}} PI)(K)\cong\mathscr{P}\mathscr{K}(\mathscr{K}(-,K),\varprojlim_{I\in\mathscr{I}} PI)\cong\varprojlim_{I\in\mathscr{I}}\mathscr{P}\mathscr{K}(\mathscr{K}(-,K),PI)\cong\varprojlim_{I\in\mathscr{I}}(PI(K))$$

It follows that $\mathscr{P}\mathscr{K}\rightarrowtail\widehat{\mathscr{K}}$ preserves any limits that exist in $\mathscr{P}\mathscr{K}$. The following important theorem of [D&L] was foreshadowed by [PJF]:

4.3. THEOREM. *If $\mathscr{K}$ is small-complete then $\mathscr{P}\mathscr{K}$ is complete.* ∎

Building on this, [D&L] also proved:

4.4. THEOREM. *For $\mathscr{K}$ and $\mathscr{L}$ small-complete, $\mathscr{P}F:\mathscr{P}\mathscr{K}\to\mathscr{P}\mathscr{L}$ preserves small limits if and only if $F:\mathscr{K}\to\mathscr{L}$ preserves small limits.* ∎

Of course Yoneda functors preserve any limits that exist. So if $\mathscr{K}$ is small complete then $Y_{\mathscr{K}}:\mathscr{K}\to\mathscr{P}\mathscr{K}$ preserves small limits and the adjunction $\mathscr{P}Y_{\mathscr{K}}\dashv M_{\mathscr{K}}$ ensures that, for small-complete $\mathscr{K}$, $M_{\mathscr{K}}$ preserves small limits. These observations of [D&L] together with their Theorems 4.3 and 4.4 ensure that the monad $(\mathscr{P},Y,M)$ lifts from **Cat** to $\mathbf{Cat}^{\mathscr{R}}$, the 2-category of small-complete categories, small-limit preserving functors, and natural transformations.

4.5. COROLLARY. *There is a distributive law $\rho:\mathscr{R}\mathscr{P}\to\mathscr{P}\mathscr{R}$.* ∎

4.6. DEFINITION. *A locally small category $\mathscr{K}$ is* completely distributive *if $\mathscr{K}$ is a ρ-algebra, meaning that $\mathscr{K}$ is a $\mathscr{P}$-algebra and $\mathscr{K}$ is an $\mathscr{R}$-algebra and $\mathscr{P}\mathscr{K}\to\mathscr{K}$ is an $\mathscr{R}$-homomorphism. In other words, $\mathscr{K}$ is completely distributive if $\mathscr{K}$ is small-cocomplete and small-complete and assignment of colimits, $X:\mathscr{P}\mathscr{K}\to\mathscr{K}$ preserves all small limits.*

4.7. REMARK. A stronger definition might require that $X:\mathscr{P}\mathscr{K}\to\mathscr{K}$ have a left adjoint, equivalently that $(\mathscr{K},X:\mathscr{P}\mathscr{K}\to\mathscr{K})$ have a $\mathscr{P}$-coalgebra structure for $\mathscr{P}$ seen as a KZ comonad on $\mathbf{Cat}^{\mathscr{P}}$.

4.8. REMARK. A ρ-algebra $\mathscr{K}$ has *limits distributing* over *colimits* in the terminology of Beck. There is also a distributive law $\lambda:\mathscr{P}\mathscr{R}\to\mathscr{R}\mathscr{P}$ and a λ-algebra has *colimits distributing over limits.* $\mathscr{K}$ is a λ-algebra if and only if $\mathscr{K}^{\mathrm{op}}$ is a ρ-algebra. However, if both $\mathscr{K}$ and $\mathscr{K}^{\mathrm{op}}$ are ρ-algebras then both $\mathscr{K}$ and $\mathscr{K}^{\mathrm{op}}$ are cartesian closed so that $\mathscr{K}$ is an ordered set.

4.9. THEOREM. *A totally distributive category is completely distributive.*

Proof. Let $\mathscr{K}$ be totally distributive then, as we already remarked, it has $\mathscr{P}$-structure given by restricting X (and $\mathscr{R}$ structure given by restricting A). Since $\mathscr{P}\mathscr{K} \twoheadrightarrow \widehat{\mathscr{K}}$ preserves all limits that exist and so does the right adjoint functor $X : \widehat{\mathscr{K}} \to \mathscr{K}$, $\mathscr{K}$ is completely distributive. ■

5. Taxons

5.1 Recall that a category $\mathscr{K}$ is a (possibly large) set $|\mathscr{K}|$ together with a monad $\mathscr{K}$ on $|\mathscr{K}|$ in **MAT**, the bicategory with objects those of **SET** and arrows given by **SET**-valued matrices. We will make several calculations in **MAT** so we recall that **MAT** can be seen as the full subbicategory of **PROF** determined by the discrete objects, which are the objects of **SET**. We note that, like **PROF**, **MAT** *has local coequalizers which are preserved by composition with 1-cells from either side.*

A *taxon* $\mathbf{T}$, as in [KOS] is a (possibly large) set $|\mathbf{T}|$, whose elements are called *objects*, together with an *interpolad* $\mathbf{T}$ on $|\mathbf{T}|$ in **MAT**. This means that $\mathbf{T}$ is a pair $\mathbf{T} = (\mathbf{T} : |\mathbf{T}| \nrightarrow |\mathbf{T}|, \mu : \mathbf{TT} \to \mathbf{T})$ in **MAT**, where

$$\mathbf{TTT} \underset{\mu\mathbf{T}}{\overset{\mathbf{T}\mu}{\rightrightarrows}} \mathbf{TT} \xrightarrow{\mu} \mathbf{T}$$

is a coequalizer in $\mathbf{MAT}(|\mathbf{T}|, |\mathbf{T}|)$. To understand the definition, let T and U be objects of $\mathbf{T}$ and, as for categories, write the elements of $\mathbf{T}(T, U)$ as arrows $f : T \to U$. From the definition of matrix multiplication, it follows that $\mathbf{TT}(T, U)$ is the set of *composable pairs*

$$T \xrightarrow{h} M \xrightarrow{g} U$$

from T to U. The value of the (T, U)-component of the 2-cell μ at the pair (h, g) can, and will, be denoted $gh : T \to U$, and called the *composite* of (h, g). We will freely use other categorical vocabulary when it makes sense to do so.

Provisionally, write $\mathbf{T}_{\mathbf{T}}\mathbf{T}$ for the coequalizer of $\mathbf{T}\mu$ and $\mu\mathbf{T}$. It now follows, merely from our notational conventions, that $\mathbf{T}_{\mathbf{T}}\mathbf{T}(T, U)$ is the set of equivalence classes of composable pairs of arrows from T to U, for the equivalence relation that identifies

$$T \xrightarrow{h} M \xrightarrow{g} U \qquad \text{and} \qquad T \xrightarrow{v} N \xrightarrow{u} U$$

if there exists a (finite) path from (h, g) to (v, u) in the sense suggested by:

$$\begin{tikzcd}
& M \arrow[d] \arrow[ddr, "g"] & \\
& \bullet \arrow[dr] & \\
T \arrow[uur, "h"] \arrow[ur] \arrow[r] \arrow[ddr, "v"'] & \bullet \arrow[u] \arrow[r] & U \\
& \vdots & \\
& N \arrow[uur, "u"'] &
\end{tikzcd}$$

It is helpful, and natural, to write $g\otimes h$ for the equivalence class of the pair (h,g). Because μ *coequalizes* $\mathbf{T}\mu$ and $\mu\mathbf{T}$, it follows that there is a unique 2-cell $\bar{\mu}:\mathbf{T}_{\mathbf{T}}\mathbf{T}\to\mathbf{T}$ such that the following equation holds:

$$\begin{tikzcd} \mathbf{TT} \arrow[r] \arrow[dr, "\mu"'] & \mathbf{T}_{\mathbf{T}}\mathbf{T} \arrow[d, "\bar{\mu}"] \\ & \mathbf{T} \end{tikzcd}$$

Clearly, we have $\bar{\mu}(g\otimes h)=gh$. The requirement that μ be a coequalizer is the requirement that $\bar{\mu}$ be an isomorphism. In other words, for all $f:T\to U$ in $\mathbf{T}$, there is a unique $g\otimes h$ such that $f=gh$. Thus every arrow in a taxon can be factored and any two factorizations of an arrow are connected.

A category can be interpreted as a taxon. Indeed, the composition data for a category gives rise to a local coequalizer in **MAT** which is both reflexive and contractible via the identity data.

5.2 For $\mathbf{T}$ and $\mathbf{S}$ taxons, a *taxon functor* $F:\mathbf{T}\to\mathbf{S}$ consists of an object function $F:|\mathbf{T}|\to|\mathbf{S}|$ and functions $F_{T,U}:\mathbf{T}(T,U)\to\mathbf{S}(FT,FU)$ which preserve compostion. A *natural transformation* $\tau:F\to G:\mathbf{T}\to\mathbf{S}$ consists of functions $\tau_{T,U}:\mathbf{T}(T,U)\to\mathbf{S}(FT,GU)$, with $f:T\to U\mapsto\tau_f:FT\to GU$ such that, for all $T\xrightarrow{h}M\xrightarrow{g}U$,

$$\begin{tikzcd} FT \arrow[r, "Fh"] \arrow[d, "\tau_h"'] \arrow[dr, "\tau_{gh}"] & FM \arrow[d, "\tau_g"] \\ GM \arrow[r, "Gg"'] & GU \end{tikzcd}$$

In Exercise 5 of Section 4 of Chapter 1 of [MAC], the reader is asked to show that the definition of natural transformation given above agrees with the usual one if F and G are functors between categories. With most composites being the evident ones, taxons, taxon functors, and natural transformations form a 2-category that we call **TAX**. Note that if we have functors $F,G,H:\mathbf{T}\to\mathbf{S}$ between taxons and natural transformations $\tau:F\to G$ and $\sigma:G\to H$, then the *vertical composite* $\sigma.\tau$ has $(\sigma.\tau)_f=\sigma_g\tau_h$ for any composable pair (h,g) with $f=gh$. That this is well defined follows from the defining coequalizer condition for taxons.

5.3. Remark. If $\tau:F\to G:\mathbf{T}\to\mathbf{S}$ and $\sigma:G\to F$, then τ and σ are inverse isomorphisms if and only if, for every composable pair (h,g) in $\mathbf{T}$, $\sigma_g.\tau_h=F(gh)$ and $\tau_g.\sigma_h=G(gh)$. Note that if $\tau:F\to G$ is an isomorphism and T is an object of $\mathbf{T}$ we cannot conclude that $F(T)$ and $G(T)$ are isomorphic, even if $\mathbf{S}$ is a category seen as a taxon. For that matter, evaluation at T does not necessarily provide a taxon functor.

5.4. Remark. To give an adjunction $\eta, \epsilon; F \dashv U : \mathbf{X} \to \mathbf{A}$ in $\mathbf{TAX}$ is to give, for all $f : X \to Y$ in $\mathbf{X}$, *natural* $\eta_f : X \to UFY$, and, for all $h : A \to B$ in $\mathbf{A}$, *natural* $\epsilon_h : FUA \to B$, such that for all composable (f, g) in $\mathbf{X}$ and all composable (h, k) in $\mathbf{A}$,

$$\begin{tikzcd} FX \arrow[r, "F(\eta_f)"] \arrow[dr, "F(gf)"'] & FUFY \arrow[d, "\epsilon_{F(g)}"] \\ & FZ \end{tikzcd} \qquad \begin{tikzcd} UFUB \arrow[d, "U\epsilon_k"'] & UA \arrow[l, "\eta_{Uh}"'] \arrow[dl, "U(kh)"] \\ UC & \end{tikzcd}$$

It is interesting to note that without the coequalizer condition it seems difficult to obtain a meaningful 2-categorical stucture on "categories without identities".

In addition to $\mathbf{TAX}$ there are 2-categories $\mathbf{Tax}$ and $\mathbf{tax}$, building on the size conventions of subsection 2.1. For the objects of $\mathbf{Tax}$ we require that the hom-sets $\mathbf{T}(X, A)$ be small and we speak of *locally small taxons.* For the objects of $\mathbf{tax}$ we require both $|\mathbf{T}|$ and the $\mathbf{T}(X, A)$ to be small and call such objects *small taxons.*

There is a 2-functor $I : \mathbf{CAT} \to \mathbf{TAX}$ which interprets categories as taxons. Notice that it is not an inclusion because it is not full — a taxon functor between categories does not necessarily preserve identities. For a "real" example let $\mathscr{C}$ be a category with non-identity idempotents and consider the underlying object taxon functor from the idempotent splitting completion $K\mathscr{C}$ to $\mathscr{C}$. It does not preserve identities. In fact, K extends to taxons and provides a right 2-adjoint, 2-functor to $I : \mathbf{CAT} \to \mathbf{TAX}$. By contrast, freely adjoining identities to a taxon provides a functor from the underlying category of $\mathbf{TAX}$ to the underlying category of $\mathbf{CAT}$ but it is *not* a 2-functor, although it is left adjoint to the underlying functor of I (from the underlying category of $\mathbf{CAT}$ to the underlying category of $\mathbf{TAX}$). The exercise in [MAC] mentioned above shows that I is locally fully faithful.

5.5 Many of the ideas of subsections 5.1, 5.2, and this one are taken from [KOS]. In particular, for taxons $\mathbf{T}$ and $\mathbf{S}$ we have from [KOS] the notions of an *i-module* from $\mathbf{T}$ to $\mathbf{S}$. These form an appropriate notion of proarrow, in the sense of [WD1], for taxons.

An *i-module* $\mathbf{M} : \mathbf{T} \twoheadrightarrow \mathbf{S}$ is a $\mathbf{SET}$-valued matrix $\mathbf{M} : |\mathbf{T}| \twoheadrightarrow |\mathbf{S}|$ together with mutually associative actions $\rho : \mathbf{MT} \to \mathbf{M}$ and $\lambda : \mathbf{SM} \to \mathbf{M}$ for which

$$\mathbf{MTT} \underset{\rho\mathbf{T}}{\overset{\mathbf{M}\mu}{\rightrightarrows}} \mathbf{MT} \xrightarrow{\rho} \mathbf{M} \qquad \text{and} \qquad \mathbf{SSM} \underset{\mathbf{S}\lambda}{\overset{\mu\mathbf{M}}{\rightrightarrows}} \mathbf{SM} \xrightarrow{\lambda} \mathbf{M}$$

are coequalizers. An i-module is small if its underlying matrix is $\mathbf{set}$-valued. It is sometimes convenient to speak of an associative action with associativity witnessed by a coequalizer as an *i-action.*

For i-modules $\mathbf{M}, \mathbf{N} : \mathbf{T} \twoheadrightarrow \mathbf{S}$, a 2-cell $\tau : \mathbf{M} \to \mathbf{N}$ is a 2-cell in $\mathbf{MAT}$ which is equivariant with respect to the actions. Taxons, i-modules, and 2-cells with the obvious compositions form a bicategory $\mathbf{i\text{-}MOD}$. In particular, note that if $\mathbf{M} : \mathbf{T} \twoheadrightarrow \mathbf{S}$ and $\mathbf{N} : \mathbf{S} \twoheadrightarrow \mathbf{R}$

then the composite module $\mathbf{N} \bullet \mathbf{M} : \mathbf{T} \twoheadrightarrow \mathbf{R}$ is given by the local coequalizer

$$\mathbf{NSM} \underset{\mathbf{N}\lambda}{\overset{\rho\mathbf{M}}{\rightrightarrows}} \mathbf{NM} \longrightarrow \mathbf{N} \bullet \mathbf{M}$$

in **MAT**. Note that the definition of $\mathbf{N} \bullet \mathbf{M}$ makes sense for general actions, not just i-actions. We employ such usage below.

Restricting to small taxons and small i-modules we obtain a bicategory **i-mod**. We write $\mathbf{i\text{-}Mod}(\mathbf{T}, \mathbf{S})$ for the category of small i-modules from $\mathbf{T}$ to $\mathbf{S}$ irrespective of the size of $\mathbf{T}$ and $\mathbf{S}$. **i-Mod** is not a bicategory for the same reason that **Prof** is not a bicategory — a composite of smalls over a large set of objects is not small in general.

If $F : \mathbf{T} \to \mathbf{S}$ is a taxon functor, we get matrices $F_+ : |\mathbf{T}| \twoheadrightarrow |\mathbf{S}|$ and $F^+ : |\mathbf{S}| \twoheadrightarrow |\mathbf{T}|$ defined by $F_+(S, T) = \mathbf{S}(S, FT)$ and $F^+(T, S) = \mathbf{S}(FT, S)$ respectively, each admitting actions on both sides. However, the actions of $\mathbf{T}$ can fail to be i-actions. We define F_* and F^* by the composites $F_* = F_+ \bullet \mathbf{T}$ and $F^* = \mathbf{T} \bullet F^+$. For future reference, note that $F^*(T, S)$ has elements of the form $s \otimes t$, for $t : T \to T'$ in $\mathbf{T}$ and $s : FT' \to S$ in $\mathbf{S}$, where $s \otimes t$ again denotes the evident equivalence class. Now F_* and F^* are i-modules — small if $F : \mathbf{T} \to \mathbf{S}$ is in **tax** — with $F_* \dashv F^*$ in **i-MOD** (in **i-mod** if F is in **tax**). We get proarrow equipments $(-)_* : \mathbf{TAX} \to \mathbf{i\text{-}MOD}$ and $(-)_* : \mathbf{tax} \to \mathbf{i\text{-}mod}$. Like **PROF** and **prof**, **i-MOD** and **i-mod** have all right liftings and all right extensions. See [KOS] for details.

Just as a category can be interpreted as a taxon, so a profunctor between categories can be interpreted as an i-module. The coequalizer requirements are again met by using the identity data to exhibit the actions as reflexive contractible coequalizers. Thus we have $I : \mathbf{PROF} \to \mathbf{i\text{-}MOD}$ — which also restricts to the small case — but, unlike the situation for $I : \mathbf{CAT} \to \mathbf{TAX}$, $I : \mathbf{PROF} \to \mathbf{i\text{-}MOD}$ is full and faithful. This observation will be important for our use of taxons.

5.6. Proposition. *If $\mathbf{M} : \mathscr{C} \to \mathscr{D}$ is an i-module between categories then $\mathbf{M}$ is already a profunctor so that each $I_{\mathscr{C},\mathscr{D}} : \mathbf{PROF}(\mathscr{C}, \mathscr{D}) \to \mathbf{i\text{-}MOD}(I\mathscr{C}, I\mathscr{D})$ is an isomorphism of categories. In fact any i-action of a category is unitary.*

Proof. Let $(\mathscr{C}, \mu, \eta)$ be a category, seen as a monad, and let $\mathbf{M}$ be a right $\mathscr{C}$ i-module with i-action ρ so that

$$\mathbf{M}\mathscr{C}\mathscr{C} \underset{\rho\mathscr{C}}{\overset{\mathbf{M}\mu}{\rightrightarrows}} \mathbf{M}\mathscr{C} \xrightarrow{\rho} \mathbf{M}$$

is a coequalizer. Consider

$$
\begin{tikzcd}
\mathbf{M}\mathscr{C} \arrow[r, "\rho"] \arrow[d, "\mathbf{M}\mathscr{C}\eta"'] & \mathbf{M} \arrow[d, "\mathbf{M}\eta"] \\
\mathbf{M}\mathscr{C}\mathscr{C} \arrow[r, "\rho\mathscr{C}"] \arrow[d, "\mathbf{M}\mu"'] & \mathbf{M}\mathscr{C} \arrow[d, "\rho"] \\
\mathbf{M}\mathscr{C} \arrow[r, "\rho"] & \mathbf{M}
\end{tikzcd}
$$

The left hand vertical composite is $1_{\mathbf{M}\mathscr{C}}$. It follows by uniqueness that the right hand vertical composite is $1_{\mathbf{M}}$ so that ρ is unitary. ∎

5.7. Remark. This argument shows that, for any monad $(\mathscr{C}, \mu, \eta)$, on any object, in any bicategory, that if a putative algebra $\alpha : \mathscr{C}A \to A$ satisfies associativity by being a *coequalizer* of $\mathscr{C}\alpha$ and μA then α is an algebra (and the coequalizer is then contractible and reflexive).

We want to relate certain i-modules with certain taxon functors. The following proposition is helpful. Note that $\mathbf{T}^{\mathrm{op}}$, the *opposite* of $\mathbf{T}$, is the taxon with the same objects as $\mathbf{T}$, $\mathbf{T}^{\mathrm{op}}(T, U) = \mathbf{T}(U, T)$, and the obvious composition.

5.8. Proposition. *For a taxon* $\mathbf{T}$*, the category* $\mathbf{TAX}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$ *is isomorphic to the following category: the object are pairs* $(P : 1 \to\!\!\!\!\!\!+\, |\mathbf{T}|, \lambda : \mathbf{T}P \to P)$ *where* P *is a* small *matrix and* λ *is a matrix 2-cell that provides a merely associative action of* $\mathbf{T}$ *on* P*, meaning that we have the equation*

$$
\begin{tikzcd}
\mathbf{TT}P \arrow[r, "\mu P"] \arrow[d, "\mathbf{T}\lambda"'] & \mathbf{T}P \arrow[d, "\lambda"] \\
\mathbf{T}P \arrow[r, "\lambda"'] & P
\end{tikzcd}
$$

while the arrows $(P, \lambda) \to (Q, \lambda)$ *are matrix 2-cells* $\alpha : \mathbf{T}P \to Q$ *which satisfy the equations*

$$
\begin{tikzcd}
\mathbf{TT}P \arrow[r, "\mu P"] \arrow[d, "\mathbf{T}\alpha"'] & \mathbf{T}P \arrow[d, "\alpha"] \\
\mathbf{T}Q \arrow[r, "\lambda"'] & Q
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathbf{TT}P \arrow[r, "\mathbf{T}\lambda"] \arrow[d, "\mu P"'] & \mathbf{T}P \arrow[d, "\alpha"] \\
\mathbf{T}P \arrow[r, "\alpha"'] & Q
\end{tikzcd}
$$

Given $\alpha:\mathbf{T}P\to Q$ *and* $\beta:\mathbf{T}Q\to R$*, their composite* $\beta\star\alpha:\mathbf{T}P\to R$ *is the unique arrow satisfying the equation defined by the right hand square below.*

$$\begin{tikzcd}
\mathbf{TTT}P \arrow[r, shift left, "\mu\mathbf{T}P"] \arrow[r, shift right, "\mathbf{T}\mu P"'] \arrow[d, "\mathbf{TT}\alpha"'] & \mathbf{TT}P \arrow[r, "\mu P"] \arrow[d, "\mathbf{T}\alpha"] & \mathbf{T}P \arrow[d, "\beta\star\alpha"] \\
\mathbf{TT}Q \arrow[r, shift left, "\mu Q"] \arrow[r, shift right, "\mathbf{T}\lambda"'] & \mathbf{T}Q \arrow[r, "\beta"] & R
\end{tikzcd}$$

(The top row is of course a coequalizer; one of the serial equations of the left square is trivial, the other follows using both equations for α*; the second equation for* β *shows that the second row commutes.) The identity on* (P,λ) *is* λ*. A matrix 2-cell* $\tau:P\to Q$ *is* equivariant *with respect to left actions on* P *and* Q *if it satisfies the equation given by the square*

$$\begin{tikzcd}
\mathbf{T}P \arrow[r, "\mathbf{T}\tau"] \arrow[d, "\lambda"'] \arrow[dr, dashed, "\alpha"] & \mathbf{T}Q \arrow[d, "\lambda"] \\
P \arrow[r, "\tau"'] & Q
\end{tikzcd}$$

In this case the common value provided by the diagonal α *provides an arrow of the category at hand.*

Proof. Given a taxon functor $P:\mathbf{T}^{\mathrm{op}}\to I\mathbf{set}$, we regard the values $P(T)$ as "hom-sets" with elements $p:T\twoheadrightarrow P$, together with an associative composition assigning to $S\xrightarrow{g}T\xrightarrow{p}P$ a "composite" $pg:S\twoheadrightarrow P$. Of course, pg is just a convenient way of talking about $P(g)(p)\in P(S)$. Given $\alpha:P\to Q$ in $\mathbf{TAX}(\mathbf{T}^{\mathrm{op}},I\mathbf{set})$ we have functions $\alpha:\mathbf{T}(R,T)\times P(T)\to Q(R)$ sending the pair (f,p) to $\alpha_f(p)$ subject to, for $R\xrightarrow{h}S\xrightarrow{g}T$,

$$\alpha_g(p)h=\alpha_{gh}(p)=\alpha_h(pg)$$

The equations of the statement are just those above expressed in terms of matrices. We leave the other verifications as an exercise. ∎

5.9. Theorem. *For any small taxon* $\mathbf{T}$*, there is an equivalence of categories*

$$\mathbf{i}\text{-}\mathbf{Mod}(I\mathbf{1},\mathbf{T})\xrightarrow{\sim}\mathbf{Tax}(\mathbf{T}^{\mathrm{op}},I\mathbf{set})$$

and $\mathbf{i}\text{-}\mathbf{Mod}(I\mathbf{1},\mathbf{T})$ *is locally small.*

Proof. A *unitary* action by $\mathbf{1}$ is trivial, so by Proposition 5.6 the objects of $\textbf{i-Mod}(I\mathbf{1},\mathbf{T})$ are simply left $\mathbf{T}$ i-actions and the arrows are equivariant 2-cells. On the other hand, Proposition 5.8 allows us to regard the objects of $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$ as left $\mathbf{T}$ actions with arrows as described therein. Given $\tau : \mathbf{P} \to \mathbf{Q}$ in $\textbf{i-Mod}(I\mathbf{1},\mathbf{T})$, Proposition 5.8 shows that $\tau.\lambda = \lambda.\mathbf{T}\tau : \mathbf{TP} \to \mathbf{Q}$ provides an arrow from $\mathbf{P}$ to $\mathbf{Q}$ in $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$. Thus we have a functor $\Phi : \textbf{i-Mod}(I\mathbf{1},\mathbf{T}) \to \mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$. Moreover, if we have any arrow $\alpha : \mathbf{P} \to Q$ in $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$ with $\mathbf{P}$ in $\textbf{i-Mod}(I\mathbf{1},\mathbf{T})$ then we claim that α arises from a unique equivariant $\tau : \mathbf{P} \to Q$ as $\tau.\lambda$. To establish this claim observe that since $\lambda : \mathbf{TP} \to \mathbf{P}$ is the coequalizer of $\mu\mathbf{P}$ and $\mathbf{T}\lambda$, the second equation for α in Proposition 5.8 shows that $\alpha = \tau.\lambda$ for a unique $\tau : \mathbf{P} \to Q$. Then, since $T\lambda$ is also a coequalizer, the first equation for α in Proposition 5.8 shows that τ is equivariant. This claim shows in particular that $\Phi : \textbf{i-Mod}(I\mathbf{1},\mathbf{T}) \to \mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$ is fully faithful.

We define a functor $\Gamma : \mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set}) \to \textbf{i-Mod}(I\mathbf{1},\mathbf{T})$. For Q in $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$ with left action $\lambda : \mathbf{T}Q \to Q$, define $\mathbf{Q}$ to be the coequalizer of μQ and $\mathbf{T}\lambda$ and construct $k : \mathbf{Q} \to Q$ as below

$$\begin{tikzcd}
\mathbf{TT}Q \arrow[r, shift left, "\mu Q"] \arrow[r, shift right, "\mathbf{T}\lambda"'] & \mathbf{T}Q \arrow[r, "q"] \arrow[dr, "\lambda"'] & \mathbf{Q} \arrow[d, "k"] \\
& & Q
\end{tikzcd}$$

The following diagram shows that $\mathbf{Q}$ admits an i-module structure. Consider first the upper left square. The four serial equalities are easy to establish. For example,

$$left.top = top.left$$

follows from associativity of μ while the other three are instances of naturality. If one constructs the four coequalizers of the four parallel pairs of the upper left square then the left and middle vertical columns are the defining coequalizer (in $\mathbf{MAT}$) for taxon $\mathbf{T}$ applied to $\mathbf{T}Q$ and Q respectively. Note that the bottom left hand square shown satisfies the two serial equalities so that the bottom parallel pair is necessarily the induced pair and we have *defined* its coequalizer to be $q : \mathbf{T}Q \to \mathbf{Q}$. Thus the middle and upper rows are necessarily coequalizers as displayed. From the 9-Lemma the right column is the coequalizer of the induced parallel pair arising from the parallel pair in the middle column. By uniqueness, the right arrow of the induced pair is $\mu\mathbf{Q}$. For the moment, write x for the left arrow and $\bar{\lambda}$ for their coequalizer. The bottom right square commutes (and its diagonal is then the colimit of the upper left square). It follows at once that $\mathbf{T}\bar{\lambda}$, where we have x, gives the other serial commuativity. Thus $x = \mathbf{T}\bar{\lambda}$ by uniqueness.

$$\begin{tikzcd}
\mathbf{T}^4 Q \arrow[r, shift left, "\mathbf{T}^2\mu Q"] \arrow[r, shift right, "\mathbf{T}^3\lambda"'] \arrow[d, shift right, "\mathbf{T}\mu\mathbf{T}Q"'] \arrow[d, shift left, "\mu\mathbf{T}^2 Q"] & \mathbf{T}^3 Q \arrow[r, "\mathbf{T}^2 q"] \arrow[d, shift right, "\mathbf{T}\mu Q"'] \arrow[d, shift left, "\mu\mathbf{T}Q"] & \mathbf{T}^2\mathbf{Q} \arrow[d, shift right, "x"'] \arrow[d, shift left, "\mu\mathbf{Q}"] \\
\mathbf{T}^3 Q \arrow[r, shift left, "\mathbf{T}\mu Q"] \arrow[r, shift right, "\mathbf{T}^2\lambda"'] \arrow[d, "\mu\mathbf{T}Q"'] & \mathbf{T}^2 Q \arrow[r, "\mathbf{T}q"] \arrow[d, "\mu Q"] & \mathbf{T}\mathbf{Q} \arrow[d, "\bar{\lambda}"] \\
\mathbf{T}^2 Q \arrow[r, shift left, "\mu Q"] \arrow[r, shift right, "\mathbf{T}\lambda"'] & \mathbf{T}Q \arrow[r, "q"'] & \mathbf{Q}
\end{tikzcd}$$

It follows that $\mathbf{Q}$ together with $\bar{\lambda}$ is an i-module and, as a quotient of a small sum (since $\mathbf{T}$ is small) of smalls, $\mathbf{Q}$ is small so that we can define $\Gamma(Q, \lambda) = (\mathbf{Q}, \bar{\lambda})$. Next, we observe that the arrow $k : \mathbf{Q} \to Q$ is equivariant (by using that $\mathbf{TQ}$ is a coequalizer) and thus provides an arrow $k_Q : \Phi\Gamma Q \to Q$ in $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$.

We show that Γ is right adjoint to Φ, with counit given by $k : \mathbf{Q} \to Q$. For any $\mathbf{P}$ in $\mathbf{i\text{-}Mod}(I\mathbf{1}, \mathbf{T})$ and any (equivariant) $\tau : \mathbf{P} \to Q$ in $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$, there is a $\bar{\tau} : \mathbf{P} \to \mathbf{Q}$ unique with the property that

$$\begin{tikzcd}
\mathbf{TP} \arrow[r, "\lambda"] \arrow[d, "\mathbf{T}\tau"'] & \mathbf{P} \arrow[d, "\bar{\tau}"] \\
\mathbf{T}Q \arrow[r, "q"'] & \mathbf{Q}
\end{tikzcd}$$

in $\mathbf{MAT}$. It is obvious that $k\bar{\tau} = \tau$ and it is easy to check that $\bar{\tau}$ is equivariant, by using that $\mathbf{T}\lambda : \mathbf{T}^2\mathbf{P} \to \mathbf{TP}$ is a coequalizer. Assume now that $t : \mathbf{P} \to \mathbf{Q}$ is any equivariant arrow with $k.t = \tau$. An adjunction $\Phi \dashv \Gamma$ with counit k will be established if we can show that $t = \bar{\tau}$. It is convenient to show first that $\bar{\lambda}$, the i-action for $\mathbf{Q}$, is also given by $\bar{\lambda} = q.\mathbf{T}k$. From the square defining λ, it suffices to show $q.\mathbf{T}k.\mathbf{T}q = q.\mu Q$. But

$$q.\mathbf{T}k.\mathbf{T}q = q.\mathbf{T}\lambda = q.\mu Q$$

follows easily from the basic definitions (without reference to the hypothesized t). From the square defining $\bar{\tau}$ it follows that $t = \tau$ if $t.\lambda = q.\mathbf{T}\tau$. But

$$t.\lambda = \bar{\lambda}.\mathbf{T}t = q.\mathbf{T}k.\mathbf{T}t = q.\mathbf{T}\tau$$

using the equation for $\bar{\lambda}$ and the hypothesis on t.

To show that $\Phi \dashv \Gamma$ is an equivalence observe first that the unit which follows from our deduction can, and should, have its $\mathbf{P}$ component taken to be $1_{\mathbf{P}}$. So it suffices to show that $k:\mathbf{Q}\to Q$ is invertible in $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$. We have $k\bar{\lambda}:\mathbf{TQ}\to Q$, which using the description above for $\bar{\lambda}$ is more easily given as $\lambda.\mathbf{T}k:\mathbf{TQ}\to Q$, and $q:\mathbf{T}Q\to\mathbf{Q}$. (Note that $k\bar{\lambda}$ satisfies the 'α' equations of Proposition 5.8 because k is equivariant, while q satisfies the second of those equations as the coequalizer of $\mathbf{T}\lambda$ and μQ. The first of those equations for q is the defining square for $\bar{\lambda}$.) We consider the composites $q\star(\lambda.\mathbf{T}k):\mathbf{TQ}\to\mathbf{Q}$ and $(\lambda.\mathbf{T}k)\star q:\mathbf{T}Q\to Q$ as described in Proposition 5.8.

$$
\begin{tikzcd}
\mathbf{TTQ} \arrow[r, "\mu Q"] \arrow[d, "\mathbf{TT}k"'] & \mathbf{TQ} \arrow[d, "\mathbf{T}k"'] \arrow[dd, bend left, "\bar{\lambda}"] \\
\mathbf{TT}Q \arrow[r, "\mu Q"] \arrow[d, "\mathbf{T}\lambda"'] & \mathbf{T}Q \arrow[d, "q"'] \\
\mathbf{T}Q \arrow[r, "q"'] & \mathbf{Q}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{TT}Q \arrow[rr, "\mu Q"] \arrow[d, "\mathbf{T}q"'] \arrow[dr, "\mathbf{T}\lambda"] & & \mathbf{T}Q \arrow[d, "\lambda"] \\
\mathbf{TQ} \arrow[r, "\mathbf{T}k"'] & \mathbf{T}Q \arrow[r, "\lambda"'] & Q
\end{tikzcd}
$$

The first diagram shows, by uniqueness, that $q\star(\lambda.\mathbf{T}k)=\bar{\lambda}$, which represents $1_{\mathbf{Q}}$. The second diagram shows, again by uniqueness, that $(\lambda.\mathbf{T}k)\star q=\lambda$, which represents 1_Q. This completes the proof that $k:\mathbf{Q}\to Q$ is invertible in $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$.

Finally, note that $\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T})(\mathbf{P},\mathbf{Q})$ is a subset of $\prod_{T\in|\mathbf{T}|}\mathbf{set}(\mathbf{P}(T),\mathbf{Q}(T))$ which, being a small product of small sets, is small. ■

5.10. Remark. It is worth noting that for Q in $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$ the corresponding i-module ΓQ is given by $\Gamma Q=\mathbf{Q}=\mathbf{T}\bullet Q$.

5.11 For every T in a locally small taxon $\mathbf{T}$, there is a taxon functor $\mathbf{T}(-,T):\mathbf{T}^{\mathrm{op}}\to I\mathbf{set}$ and, for every $u:T\to U$ in $\mathbf{T}$, there is a taxon natural transformation $\mathbf{T}(-,u):\mathbf{T}(-,T)\to\mathbf{T}(-,U)$. For $f:A\to B$ in $\mathbf{T}$, we have $\mathbf{T}(-,u)_f:\mathbf{T}(B,T)\to\mathbf{T}(A,U)$, in the notation of 5.2, given by $\mathbf{T}(f,u):\mathbf{T}(B,T)\to\mathbf{T}(A,U)$ where, for $h:B\to T$, $\mathbf{T}(f,u)(h)=uhf:A\to U$. It follows that we have a taxon functor $y_{\mathbf{T}}:\mathbf{T}\to I\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$. But the taxon functors $\mathbf{T}(-,T)$ seen as modules are obviously i-modules so that, for $\mathbf{T}$ small, we can consider the assignment $T\mapsto\mathbf{T}(-,T)$ as taking values in the equivalent subtaxon $I\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T})$. We write $Y_{\mathbf{T}}:\mathbf{T}\to I\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T})$ and refer to both $y_{\mathbf{T}}$ and $Y_{\mathbf{T}}$ as *Yoneda taxon functors.*

For T in $\mathbf{T}$, a locally small taxon, and P in $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$, we have the sets $P(T)$ and $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})(\mathbf{T}(-,T),P)$ but if $\mathbf{T}$ is not a category we cannot guarantee that these sets are isomorphic. However, we have a taxon functor

$$\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})(\mathbf{T}(-,\Box),P):\mathbf{T}^{\mathrm{op}}\to\mathbf{SET}$$

whose value at T is $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})(\mathbf{T}(-,T),P)$

As before, we will write $p:T\twoheadrightarrow P$ for a typical element of $P(T)$. We note that any $f:U\to T$ together with an element $p:T\twoheadrightarrow P$ determine a natural transformation

$\tau_{f,p} : \mathbf{T}(-,U) \to P$ between taxon functors. Explicitly, for $g : W \to V$ in $\mathbf{T}$, the g'th component of $\tau_{f,p}$ is the function $(\tau_{f,p})_g : \mathbf{T}(V,U) \to P(W)$ whose value at $h : V \to U$ is

$$W \xrightarrow{g} V \xrightarrow{h} U \xrightarrow{f} T \xrightarrow{p} P$$

The following result should surely be considered a Yoneda lemma for taxons

5.12. Lemma. *For $\mathbf{T}$ a locally small taxon and P in $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$, there is an isomorphism of taxon functors*

$$iP \cong \mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})(\mathbf{T}(-,\square), P) : \mathbf{T}^{\mathrm{op}} \to \mathbf{SET}$$

Proof. For brevity, write $Q = \mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})(\mathbf{T}(-,\square), P) : \mathbf{T}^{\mathrm{op}} \to \mathbf{SET}$ in this proof. Define $\alpha : P \to Q$ so that, for $f : U \to T$ in $\mathbf{T}$, $\alpha_f : P(T) \to Q(U)$ at $p : T \to P$ is given by

$$\alpha_f(p) = \tau_{f,p} : \mathbf{T}(-,U) \to P$$

where $\tau_{f,p}$ is as described prior to the statement. Next, define $\beta : Q \to P$ so that for $f : U \to T$ and $\tau : \mathbf{T}(-,T) \to P$ in $Q(T)$ we have $\beta_f(\tau) = \tau_g(h)$ in $P(U)$, where $f = hg$ is any factorization of f. We leave as an exercise that β_f is well-defined. To show that α and β are inverse isomorphisms we follow the template provided earlier in Remark 5.3. Let $A \xrightarrow{g} B \xrightarrow{h} C$ be a composable pair in $\mathbf{T}$ and consider

$$p \in P(C) \xrightarrow{\alpha_h} Q(B) \xrightarrow{\beta_g} P(A)$$

together with some factorization $A \xrightarrow{u} M \xrightarrow{v} B$ of g. Assembling the definitions we have

$$\beta_g(\alpha_h(p)) = \beta_g(\tau_{h,p}) = (\tau_{h,p})_u(v) = phvu = phg = P(hg)(p)$$

Finally, consider also a factorization $B \xrightarrow{x} N \xrightarrow{y} C$ of h and

$$\sigma \in Q(C) \xrightarrow{\beta_h} P(B) \xrightarrow{\alpha_g} Q(A)$$

to get

$$\alpha_g(\beta_h(\sigma)) = \alpha_g(\sigma_x(y)) = \tau_{g,\sigma_x(y)} : \mathbf{T}(-,A) \to P$$

Since $\sigma : \mathbf{T}(-,C) \to P$ we have $\sigma_x(y) \in P(B)$ and it follows that $\tau_{g,\sigma_x(y)}$, where $A \xrightarrow{g} B$ and $B \xrightarrow{\sigma_x(y)} P$ is the natural transformation for which, given $W \xrightarrow{k} V \xrightarrow{l} A$, has $(\tau_{g,\sigma_x(y)})_k(l) = \sigma_x(y).glk = P(glk)\sigma_x(y) = \sigma_{xglk}(y)$, the last equation by naturality. On the other hand, we note that $Q(hg)(\sigma)$ is the composite natural transformation

$$\mathbf{T}(-,A) \xrightarrow{\mathbf{T}(-,hg)} \mathbf{T}(-,C) \xrightarrow{\sigma} P$$

For $W \xrightarrow{k} V \xrightarrow{l} A$, and a factorization $k = (W \xrightarrow{i} X \xrightarrow{j} V)$ we have

$$(\sigma.\mathbf{T}(-,hg))_k(l) = \sigma_i(\mathbf{T}(-,hg)_j(l)) = \sigma_i(hglj) = \sigma_i(yxglj) = \sigma_i(\mathbf{T}(xglj,C)(y))$$

and this last is $\sigma_{xglji}(y)$ by naturality. But $\sigma_{xglk}(y) = \sigma_{xglji}(y)$, thus $\alpha_g(\beta_h(\sigma)) = Q(hg)(\sigma)$ as required. ∎

5.13. Remark. Since the fully faithful $\Phi : \mathbf{i\text{-}Mod}(I\mathbf{1}, \mathbf{T}) \to \mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$ is an equivalence that identifies $Y_{\mathbf{T}}$ and $y_{\mathbf{T}}$ the Taxon Yoneda Lemma can be restated in terms of i-modules. For $\mathbf{T}$ a locally small taxon and $\mathbf{P}$ in $\mathbf{i\text{-}Mod}(I\mathbf{1}, \mathbf{T})$, $\mathbf{i\text{-}Mod}(I\mathbf{1}, \mathbf{T})(\mathbf{T}(-, T), \mathbf{P})$ can be regarded as taxon functor in T and hence as a module. It is not in general an i-module but applying Γ, the left adjoint of Φ, to $\mathbf{i\text{-}Mod}(I\mathbf{1}, \mathbf{T})(\mathbf{T}(-, \Box), \mathbf{P})$ gives us an i-module $\mathbf{T} \bullet \mathbf{i\text{-}Mod}(I\mathbf{1}, \mathbf{T})(\mathbf{T}(-, \Box), \mathbf{P})$ of which a typical T-element is of the form $\tau \otimes f$ for a pair

$$(T \xrightarrow{f} U, \mathbf{T}(-, U) \xrightarrow{\tau} \mathbf{P})$$

There is an equivariant

$$\epsilon : \mathbf{T} \bullet \mathbf{i\text{-}Mod}(I\mathbf{1}, \mathbf{T})(\mathbf{T}(-, \Box), \mathbf{P}) \longrightarrow i\mathbf{P}$$

given by $\epsilon(\tau \otimes f) = \tau_T(f) \in \mathbf{P}(T)$. In this form the Taxon Yoneda Lemma says "ϵ is an isomorphism".

5.14. Theorem. *For $\mathbf{T}$ a locally small taxon, $\mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$ is prototal.*

Proof. Write $\mathscr{T} = \mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$. Following the comment after Lemma 2.3, it suffices to exhibit a functor $X : \widehat{\mathscr{T}} \to \mathscr{T}$ and a natural transformation

$$\begin{tikzcd} & \widehat{\mathscr{T}} \arrow[dl, "X"'] \arrow[dr, "i^{\mathscr{T}^{\mathrm{op}}}"] \arrow[d, phantom, "\overset{\eta}{\longleftarrow}"] & \\ \mathscr{T} \arrow[rr, "Y"'] & {} & \mathbf{SET}^{\mathscr{T}^{\mathrm{op}}} \end{tikzcd}$$

that exhibits X as an absolute left lifting of $i^{\mathscr{T}^{\mathrm{op}}}$ through Y, in a suitable 2-category of categories, functors, and natural transformations. Explicitly this means that, for all P in $\widehat{\mathscr{T}}$ and for all F in $\mathscr{T}$, pasting η at X provides a bijection between $\mathscr{T}(X(P), F)$ and $\mathbf{SET}^{\mathscr{T}^{\mathrm{op}}}(iP, \mathscr{T}(-, F))$. We begin by defining $X(P)$ to be the composite

$$\mathbf{T}^{\mathrm{op}} \xrightarrow{y_{\mathbf{T}}^{\mathrm{op}}} I\mathscr{T}^{\mathrm{op}} \xrightarrow{IP} I\mathbf{set}$$

To give $\eta : i^{\mathscr{T}^{\mathrm{op}}} \to YX$ is to give, for all P in $\widehat{\mathscr{T}}$, an arrow $\eta_P : iP \to \mathscr{T}(-, IP.y_{\mathbf{T}}^{\mathrm{op}})$ in $\mathbf{SET}^{\mathscr{T}^{\mathrm{op}}}$. To give η_P is to give, for each object G in $\mathscr{T}$, a G-component, $\eta_{PG} : iP(G) \to \mathscr{T}(G, IP.y_{\mathbf{T}}^{\mathrm{op}})$, natural in G. Let $g \in iP(G)$. By the Yoneda Lemma for categories we can regard such g as a natural transformation

$$\begin{tikzcd} \mathscr{T}^{\mathrm{op}} \arrow[r, bend left=40, "\mathscr{T}(-,G)", ""{name=U, below}] \arrow[r, bend right=40, "iP"', ""{name=D, above}] & \mathbf{SET} \arrow[Rightarrow, from=U, to=D, "\bar{g}"] \end{tikzcd}$$

Thus to give η is to give, for such g, a natural transformation between taxon arrows

$$\begin{tikzcd} \mathbf{T}^{\mathrm{op}} \arrow[rr, bend left, "G"] \arrow[dr, bend right, "y_{\mathbf{T}^{\mathrm{op}}}"'] \arrow[rr, phantom, "\Downarrow \eta_{PG}(g)"] & & I\mathbf{set} \\ & I\mathscr{T}^{\mathrm{op}} \arrow[ur, bend right, "IP"'] & \end{tikzcd}$$

To give such $\eta_{PG}(g)$ is to give, for all $f : T \to U$ in $\mathbf{T}$,

$$G(U) \xrightarrow{(\eta_{PG}(g))_f} IP.y_{\mathbf{T}^{\mathrm{op}}}(T) = IP(\mathbf{T}(-, T))$$

To give such $(\eta_{PG}(g))_f$ is to give, for all $u \in G(U)$, $(\eta_{PG}(g))_f(u) \in IP(\mathbf{T}(-, T))$. Again, by the Yoneda Lemma for categories, such an element, call it σ, can be seen as a natural transformation $\bar{\sigma} : \mathscr{T}(-, \mathbf{T}(-, T)) \to iP$. From the data $f : T \to U$ in $\mathbf{T}$ and $u \in G(U)$ under consideration, we have the natural transformation between taxon functors $\tau_{f,u}$: $\mathbf{T}(-, T) \to G$, as described prior to Lemma 5.12. We define the required $\bar{\sigma}$ to be the composite

$$\begin{tikzcd} \mathscr{T}^{\mathrm{op}} \arrow[r, bend left=60, "\mathscr{T}(-{,}\mathbf{T}(-{,}T))"] \arrow[r, phantom, bend left=30, "\Downarrow \mathscr{T}(-{,}\tau_{f,u})"] \arrow[r, "\mathscr{T}(-{,}G)"] \arrow[r, phantom, bend right=30, "\Downarrow \bar{g}"] \arrow[r, bend right=60, "iP"'] & \mathbf{SET} \end{tikzcd}$$

We now construct, from any $\alpha : iP \to \mathscr{T}(-, F)$, a natural transformation between taxon functors $\beta : X(P) \to F$. To give such a β is to give, for all $f : T \to U$ in $\mathbf{T}$, a function $\beta_f : X(P)(U) \to F(T)$ which requires a definition of $\beta_f(p)$ for each $p \in X(P)(U) = P(\mathbf{T}(-, U))$. However, from α we have $\alpha_{\mathbf{T}(-,U)} : iP(\mathbf{T}(-, U)) \to \mathscr{T}(\mathbf{T}(-, U), F)$ and hence a natural transformation between taxon functors $\alpha_{\mathbf{T}(-,U)}(p) : \mathbf{T}(-, U) \to F$. Factor f : $T \to U$ as

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "v"'] & & U \\ & V \arrow[ur, "w"'] & \end{tikzcd}$$

and define $\beta_f(p) = (\alpha_{\mathbf{T}(-,U)}(p))_v(w)$. It remains to be shown that β is well-defined and unique with the property that $\mathscr{T}(-, \beta).\eta_P = \alpha$.

The first of these is a generality about natural transformations between taxon functors of the form $\tau : \mathbf{T}(-, U) \to F$; namely that for any

$$\begin{tikzcd} & V \arrow[dr, "y"] \arrow[dd, "a"] & \\ T \arrow[ur, "x"] \arrow[dr, "v"'] & & U \\ & X \arrow[ur, "w"'] & \end{tikzcd}$$

$\tau_v(w) = \tau_x(y)$ is a simple consequence of naturality.

To show that $\mathscr{T}(-,\beta).\eta_P = \alpha$ we must show that, for all $G \in \mathscr{T}$, for all $g \in P(G)$,

$$\beta.\eta_{PG}(g) = \alpha_G(g) : G \to F : \mathbf{T}^{\mathrm{op}} \to I\mathbf{set}$$

as natural transformations between taxon functors. To do this, we must show that, for any

$$\begin{tikzcd} S \arrow[rr, "k"] \arrow[dr, "h"'] & & U \\ & T \arrow[ur, "f"'] & \end{tikzcd}$$

in $\mathbf{T}$, for any $u \in G(U)$,

$$\beta_h((\eta_{PG}(g))_f(u)) = (\alpha_G(g))_k(u)$$

To establish this let $h = h_2h_1$ and $h_1 = h_{12}h_{11}$ and calculate

$$\begin{aligned}
\beta_h((\eta_{PG}(g))_f(u)) &= (\alpha_{\mathbf{T}(-,T)}((\eta_{PG}(g))_f(u)))_{h_1}(h_2) \\
&= (\alpha_{\mathbf{T}(-,T)}(\bar{g}.\mathscr{T}(-,\tau_{f,u})))_{h_1}(h_2) \\
&= (\alpha_{\mathbf{T}(-,T)}P(\tau_{f,u})(g))_{h_1}(h_2) \\
&= (\mathscr{T}(\tau_{f,u},F)\alpha_G)(g))_{h_1}(h_2) \\
&= (\alpha_G(g).\tau_{f,u})_{h_1}(h_2) \\
&= (\alpha_G(g))_{h_{11}}((\tau_{f,u})_{h_{12}}(h_2)) \\
&= (\alpha_G(g))_{h_{11}}(ufh_2h_{12}) \\
&= (\alpha_G(g))_{fh_2h_{12}h_{11}}(u) \\
&= (\alpha_G(g))_k(u)
\end{aligned}$$

Since the β we described is of the form $B(\alpha)$ we can establish its required uniqueness by showing that $B(\mathscr{T}(-,\beta).\eta_P) = \beta$. So with $(B(\alpha))_f(p) = (\alpha_{\mathbf{T}(-,U)}(p))_v(w)$, $f : T \to U$ in $\mathbf{T}$, $p \in X(P)(U) = P(\mathbf{T}(-,U))$, $f = w_Vv$ (obvious notation), and $v = v_{2W}v_1$, we calculate

$$\begin{aligned}
(B(\mathscr{T}(-,\beta).\eta_P))_f(p) &= ((\mathscr{T}(-,\beta).\eta_P)_{\mathbf{T}(-,U)}(p))_v(w) \\
&= (\beta.\eta_{P\mathbf{T}(-,U)}(p))_v(w) \\
&= \beta_{v_1}((\eta_{P\mathbf{T}(-,U)}(p))_{v_2}(w)) \\
&= \beta_{v_1}(\bar{p}.\mathscr{T}(-,\tau_{v_2,w})) \\
&= \beta_{v_1}(\bar{p}.\mathscr{T}(-,wv_2)) \\
&= \beta_{v_1}(X(P)(wv_2)(p)) \\
&= \beta_{wv_2v_1}(p) \\
&= \beta_f(p)
\end{aligned}$$

∎

5.15. THEOREM. *For $\mathbf{T}$ a small taxon,* $\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T})$ *is totally distributive.*

PROOF. If $\mathbf{T}$ is small then we have the equivalence $\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T}) \xrightarrow{\sim} \mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$ provided by Theorem 5.9 and now, since $\mathbf{T}$ small implies that $\mathbf{T}$ is also locally small, Theorem 5.14 tell us that $\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T})$ is prototal. Since $\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T})$ is locally small, also by Theorem 5.9, we have $\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T})$ total by Lemma 2.3.

The taxon functor $Y_{\mathbf{T}} : \mathbf{T} \to I\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T})$ gives us the adjunction

$$\begin{tikzcd} \mathbf{T} \arrow[r, shift left=2, "Y_{\mathbf{T}*}"] \arrow[r, phantom, "\perp"] & I\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T}) \arrow[l, shift left=2, "Y_{\mathbf{T}}{}^*"] \end{tikzcd}$$

in $\mathbf{i\text{-}Mod}$.

Applying $\mathbf{i\text{-}Mod}(I\mathbf{1},-)$, sends this adjunction to an adjunction in $\mathbf{CAT}$ where the right adjoint now has a further right adjoint because $\mathbf{i\text{-}MOD}$ has right liftings and the lifting in question takes small i-modules to small i-modules because $\mathbf{T}$ is small. These observations account for the adjoint string

$$\begin{tikzcd} \mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T}) \arrow[r, shift left=4, "\mathbf{i\text{-}Mod}(I\mathbf{1},Y_{\mathbf{T}*})"] \arrow[r, shift right=4, "Y_{\mathbf{T}}{}^* \Rightarrow (-)"'] & \mathbf{i\text{-}Mod}(I\mathbf{1}, I\mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T})) \arrow[l, "\mathbf{i\text{-}Mod}(I\mathbf{1},Y_{\mathbf{T}}{}^*)" description] \end{tikzcd}$$

As before, write $\mathscr{T} = \mathbf{Tax}(\mathbf{T}^{\mathrm{op}}, I\mathbf{set})$ and here also write $\mathscr{M} = \mathbf{i\text{-}Mod}(I\mathbf{1},\mathbf{T})$. Now we have the equivalence $\Phi : \mathscr{M} \to \mathscr{T}$ of Theorem 5.9 and the isomorphism $\widehat{\mathscr{T}} = \mathbf{Prof}(\mathbf{1},\mathscr{T}) \cong \mathbf{i\text{-}Mod}(I1, I\mathscr{T})$ of Proposition 5.6. Consider the diagram

$$\begin{tikzcd} \mathscr{M} \arrow[d, "\Phi"'] & i\mathbf{i\text{-}Mod}(I1, I\mathscr{M}) \arrow[l, "\mathbf{i\text{-}Mod}(I1,Y_{\mathbf{T}}{}^*)"'] \arrow[d, "\mathbf{i\text{-}Mod}(I1,I\Phi)"] \\ \mathscr{T} & \widehat{\mathscr{T}} = \mathbf{Prof}(\mathbf{1},\mathscr{T}) \cong \mathbf{i\text{-}Mod}(I1, I\mathscr{T}) \arrow[l, "X"] \end{tikzcd}$$

where X witnesses the totality of $\mathscr{T}$ as in Theorem 5.14. To show that $\mathscr{M}$ is totally distributive it suffices to show that this diagram commutes to within isomorphism, for if Φ identifies $\mathbf{i\text{-}Mod}(I\mathbf{1}, Y_{\mathbf{T}}{}^*)$ with X, then by the uniqueness of adjoints it also identifies $Y_{\mathbf{T}}{}^* \Rightarrow (-)$ with the Yoneda functor for $\mathscr{T}$. However, this last requirement is clear because $\mathbf{i\text{-}Mod}(I\mathbf{1}, Y_{\mathbf{T}}{}^*)$ is essentially given by precomposition with $Y_{\mathbf{T}}{}^{\mathrm{op}}$ and X is precisely given by precomposition with $y_{\mathbf{T}}{}^{\mathrm{op}}$ and Φ identifies $Y_{\mathbf{T}}$ and $y_{\mathbf{T}}$. ■

The bicategory $\mathbf{i\text{-}mod}$, like $\mathbf{prof}$, is a compact monoidal bicategory with monoidal structure given by cartesian product and dualization given by $(-)^{\mathrm{op}}$. Thus $\mathbf{i\text{-}mod}(\mathbf{T},\mathbf{S}) \simeq \mathbf{i\text{-}mod}(\mathbf{1}, \mathbf{T}^{\mathrm{op}} \times \mathbf{S})$ and hence

5.16. COROLLARY. *The hom-categories of both* $\mathbf{i\text{-}mod}$ *and* $\mathbf{prof}$ *are totally distributive.* ■

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